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EXACT TRAVELLING WAVE SOLUTIONS OF THE JEFFERY-HAMEL FLOW OF A NON-NEWTONIAN FLUID USING SINE-COSINE METHOD

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Abstract: In this study, we aim to determine the exact travelling wave solutions of the Jeffery-Hamel (JH) flow of a non-Newtonian fluid (NNF), specifically Casson fluid (CF). By employing the conformation transform, we successfully reduce the governing nonlinear partial differential equation (NPDE) to an ordinary differential equation (ODE). To ascertain the exact solutions of the JH flow of a NNF in the field of civil, chemical, mechanical, environmental and bio-mechanical engineering, among others, we utilized the sine-cosine method. The numerical simulations are performed by using Mathematica. Influence of emerging parameters on converging and diverging channels are established through graphical veil.

Keywords: Jeffery-Hamel flow; exact solutions; Casson fluid; sine-cosine method.

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1. INTRODUCTION

The Jeffery-Hamel (JH) flow is widely utilized as a significant model for exploring diverse characteristics of engineering applications, including chemical, mechanical, environmental, aerospace and biomechanical sciences. Examples of these applications include the flow in channels, rivers and blood circulation, where arteries as well as capillaries are interconnected. Recently, there has been increased attention to further elucidate the applications of the JH equations, mostly in chemical engineering, fluid mechanics, and aerospace engineering. Jeffery and Hamel [1,2] were the pioneers in mathematically formulating flows between nonparallel walls. That is the reason why this sort of flow is called the Jeffery-Hamel flow. This groundbreaking work has attracted much attention to solve problems relating to flow. The exact parallel solution of conservation equations has also been applied in a specific scenario, namely, 2D flow over a channel with sloped boundaries converging at an apex containing a source or sink. Eventually, in 1940 Rosenhead [3] found and extensively discussed the general solution. Numerous authors in their textbooks [4-6] have studied and widely discussed the Jeffery-Hamel flows.

A single model is inadequate to simulate the intricate rheological nature of non-Newtonian fluids. Consequently, researchers have introduced diverse non-Newtonian flow models over the years. Among these models, the Casson fluid stands out for its blood-like behavior [6,7], with further studies available in [8-10]. Equations describing physical problems inherently exhibit non-linearity in most cases, making it challenging to obtain exact solutions. Nevertheless, various approximation methods, such as Homotopy Perturbation Method (HPM), Adomian's Decomposition Method (ADM), Variational Iteration Method (VIM), and Homotopy Analysis Method (HAM) [11-19], have been created to address the complex equations. Despite their utility, these techniques often prove difficult to employ and demand extensive computational effort. Moreover, these techniques may exhibit a low level of accuracy, impacting both results and reliability in certain cases. Therefore, to address nonlinear evolution equations (NLEEs), numerous approaches, such as Homogeneous balance method, inverse scattering method, ADM, tanh-function method, extended tanh-function method, Exp-function method, Jacobi elliptic function expansion method, ansatz method, Hirota's bilinear transformation method, F-expansion method, generalized Riccati equation method, modified simple equation (MSE) method, sine-cosine method and others are used. In this study, we employed the well-known sine-cosine method to address this particular problem.

The utilization of the sine-cosine method offers several key advantages, including freedom from rounding-off errors, avoidance of the calculation of perturbation, Adomian's polynomials, discretization or linearization. Furthermore, it relies solely on initial conditions, making implementation more straightforward. Additionally, this method upholds a higher level of accuracy and diminishes computational workload when obtaining exact solutions for various types of nonlinear PDEs. The efficacy and accuracy of this technique have been demonstrated in numerous studies [20-23]. In this article, we have successfully applied the sine-cosine approach to resolve a complex problem governing the Casson fluid flow in converging and diverging channels.

2. Mathematical Formulation

Assume that, the emergence of a Casson fluid (CF) flow is attributed to a source or sink at the juncture of two stiff plane boundaries. Let 2α represent the angle between the boundaries and suppose the flow is both symmetrical and decently radial. The rheological equation that characterizes an incompressible CF [8-10] is as follows:

$$\tau_{ij} = 2\left(\mu_B + \frac{p_y}{\sqrt{2\pi}}\right) e_{ij}, \pi \neq \pi_c \tag{1}$$

where $\pi = e_{ij} e_{ij}$ with e_{ij} existence the (i, j)th element of distortion degree. The variable π is the result of multiplying the distortion degree by itself, and the terms π_c, μ_B, and p_y respectively signify the critical value according to a non-Newtonian model, the plastic dynamic viscosity of NNF, and the yield stress of the fluid.

$$\nabla . \bar{V} = 0, \tag{2}$$

$$\rho \left[\frac{\partial \bar{V}}{\partial t} + (\bar{V}.\nabla)\bar{V}\right] = -\nabla p + \nabla . \tau_{ij} \tag{3}$$

Considering the assumptions mentioned earlier, equations (1) to (3) simplify to

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0, \tag{4}$$

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(1+\frac{1}{\beta}\right)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2}\right], \tag{5}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \left(1+\frac{1}{\gamma}\right)\frac{2\nu}{r^2}\frac{\partial u}{\partial \theta} = 0 \tag{6}$$

where ν represents kinematic viscosity, p denotes pressure, $\beta = \mu_B \frac{\sqrt{2\pi}}{p_y}$ is the Casson fluid constraint and k denotes the permeability of the porous medium.

The followings are the supplementary conditions

$$u_r = U,\ \frac{\partial u}{\partial \theta} = 0 \text{ at } \theta = 0 \text{ and } u_r = 0 \text{ at } \theta = \alpha \tag{7}$$

From continuity equation (4), we may develop the following normalized boundary conditions for velocity and temperature profiles are:

$$f(\theta) = ru \tag{8}$$

The following transformation has been used to make the problem dimensionless

$$F(\xi) = \frac{f(\theta)}{ru}, \xi = \frac{\theta}{\alpha} \tag{9}$$

By eliminating p from equations (5) and (6) and subsequently applying equations (8) and (9), we derive a nonlinear ODE describing the dimensionless velocity profile $F(\xi)$ comply with:

$$\left(1+\frac{1}{\beta}\right)F'''(\xi) + 2\alpha\, Re\, F(\xi)F'(\xi) + 4\alpha^2\left(1+\frac{1}{\beta}\right)F'(\xi) = 0 \tag{10}$$

The boundary conditions transform to

$$F(0) = 1, F'(0) = 1, F(1) = 0 \tag{11}$$

Reynolds number (*Re*) considered by

$$Re = \frac{Ur\alpha}{\nu}\begin{pmatrix}\text{Divergent Channel: } \alpha > 0, U > 0\\ \text{Convergent Channel: } \alpha < 0, U < 0\end{pmatrix} \tag{12}$$

where U represents the velocity at the central axis of the channel, specifically at $r = 0$. It's relevant to note that, in the scenario of a straightforward viscous fluid, $\beta \to \infty$. The determination of skin friction coefficient values may be achieved by employing:

$$C_f = \frac{[\tau_{r\theta}]_{\xi=1}}{U^2} = \frac{1}{Re\left(1+\frac{1}{\beta}\right)}F'(1) \tag{13}$$

3. SINE-COSINE METHOD

Assume the nonlinear PDE, expressed within two independent variables, namely *x* and *t*, as in the following format

$$Q(u, u_t, u_x, u_{xx}, u_{tt}, \dots) = 0 \tag{14}$$

where $u = u(x,t)$ is an unknown function, Q is a polynomial involving $u(x,t)$ as well as its derivatives, encompassing the highest order derivatives (HOD) and nonlinear terms. The partial derivatives of *u* are denoted as $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Step 1: We use the variables x and t by the wave variable ξ, within the subsequent formula:

$$\xi = x \pm vt,\ u(x,t) = F(\xi) \tag{15}$$

where v denotes the velocity wave.

$$S(F, F', F'', F''', F^{(iv)}, F^{(v)}, \dots) = 0 \tag{16}$$

where S is a polynomial involving $F(\xi)$ and its derivatives, where the HOD and nonlinear terms are interconnected. The superscripts indicate the ordinary derivatives with regard to ξ, where $F'(\xi) = \frac{dF}{d\xi}$.

Step 2: We then integrate the ODE (16) for as long time as possible and then neglect the constants of integration.

Step 3: The solution of equation (16) can be shown as the subsequent form:

$$F(\xi) = a\{sin(\mu\xi)\}^b \tag{17}$$

or

$$F(\xi) = a\{cos(\mu\xi)\}^b \tag{18}$$

where a, μ and b are parameters which need to be find out.

Step 4: The derivatives of equation (17) become

$$\begin{gathered} F(\xi) = a\, sin^b(\mu\xi) \\ F^n(\xi) = a^n\, sin^{nb}(\mu\xi) \\ (F^n)_\xi = n\mu b\, cos(\mu\xi) a^n\, sin^{nb-1}(\mu\xi) \\ (F^n)_{\xi\xi} = -n^2\mu^2 b^2 a^n\, sin^{nb}(\mu\xi) + n\mu^2 a^n b(nb-1)\, sin^{nb-2}(\mu\xi) \end{gathered} \tag{19}$$

And the derivatives of equation (18) become

$$F(\xi) = a\, cos^{b}(\mu\xi)$$
$$F^{n}(\xi) = a^{n}\, cos^{nb}\,(\mu\xi)$$
$$(F^{n})_{\xi} = -n\mu b \sin(\mu\xi) a^{n}\, cos^{nb-1}\,(\mu\xi) \tag{20}$$
$$(F^{n})_{\xi\xi} = -n^{2}\mu^{2}b^{2}a^{n}\, cos^{nb}(\mu\xi) + n\mu^{2}a^{n}b(nb-1)\, cos^{nb-2}\,(\mu\xi)$$

and so on.

Step 5: Substitute (19) or (20) into the compact equation derived in (16), ensuring a balance the terms of the *sine* functions in the case of (19), or balance the terms of the *cosine* functions in the case of (20). Then accumulate all terms with matching indices in $sin^{k}(\mu\xi)$ or $cos^{k}(\mu\xi)$ and equate their coefficients to zero. This process yields a system of algebraic equations with unknowns a, μ and b. Computerized symbolic calculations are then employed to solve this system, resulting in the determination of all probable values for the parameters a, μ and b.

4. NUMERICAL APPROACH

We will now employ the sine-cosine approach to solve the nonlinear evolution equations (NLEEs) i.e., the JH flow of a NNF. Therefore, the solution of this method takes the following form:

$$F(\xi) = a\, sin^{b}(\mu\xi) \tag{21}$$

where a, μ and b are parameters to be ascertained.

Differentiating equation (21) with regard to ξ, first to third derivatives are as follows:

$$F'(\xi) = ab\mu\, cos(\mu\xi)\, sin^{b-1}(\mu\xi) \tag{22}$$
$$F''(\xi) = ab\mu^{2}(b-1)\, cos^{2}(\mu\xi)\, sin^{b-2}(\mu\xi) - ab\mu^{2}\, sin^{b}(\mu\xi) \tag{23}$$
$$F'''(\xi) = ab\mu^{3}(b-1)(b-2)\, cos^{3}(\mu\xi)\, sin^{b-3}(\mu\xi) - 2ab\mu^{3}(b-1)\, sin^{b-1}(\mu\xi)\, cos(\mu\xi) - ab^{2}\mu^{3}\, sin^{b-1}(\mu\xi)\, cos(\mu\xi) \tag{24}$$

where $F' = \frac{dF(\xi)}{d\xi}$, $F'' = \frac{d^{2}F}{d\xi^{2}}$ and $F''' = \frac{d^{3}F}{d\xi^{3}}$.

5. EXAMPLE

We need to employ the sine-cosine method to derive exact travelling wave solutions for the JH flow of a NNF. Let's now, consider a new ODE for the JH flow of a NNF, formulated through mathematical invention of the succeeding form:

$$\left(1+\frac{1}{\beta}\right)F'''(\xi)+2\alpha\, Re\, F(\xi)F'(\xi)+4\alpha^2\left(1+\frac{1}{\beta}\right)F'(\xi)=0 \tag{25}$$

So, the solution of (25) takes the form, which is equivalent to the equation (21).

By substituting the values of F, F' and F''' from equations (21), (22) and (24) into equation (25) and subsequently equating the exponents and coefficients of *sin* functions, we derive

$$-1+2b=-3+b \tag{26}$$

$$-8a\alpha^2\mu-\frac{8a\alpha^2\mu}{\beta}+8a\mu^3+\frac{8a\mu^3}{\beta}=0 \tag{27}$$

$$-4a^2\, Re\, \alpha\, \mu-24a\mu^3-\frac{24a\mu^3}{\beta}=0 \tag{28}$$

Utilize *Mathematica* software to solve the above system of differential and algebraic equations.

From equations (26) to (28), we derive the following results:

$$\left\{b=-2, a=-\frac{6\alpha(1+\beta)}{Re\,\beta}, \mu=\pm\alpha\right\} \tag{29}$$

Substituting (29) into equation (21), we get the required solutions

$$F(\xi)=-\frac{6\alpha(1+\beta)\,cosec^2(\alpha\xi)}{Re\,\beta} \tag{30}$$

6. GRAPHICAL REPRESENTATION

In this paper, we have drawn the graphs of the obtained solution to the JH flow of a NNF using the sine-cosine method. We have also shown the obtained solution in the following figures 1 to 6 using the symbolic computational software *Mathematica*.

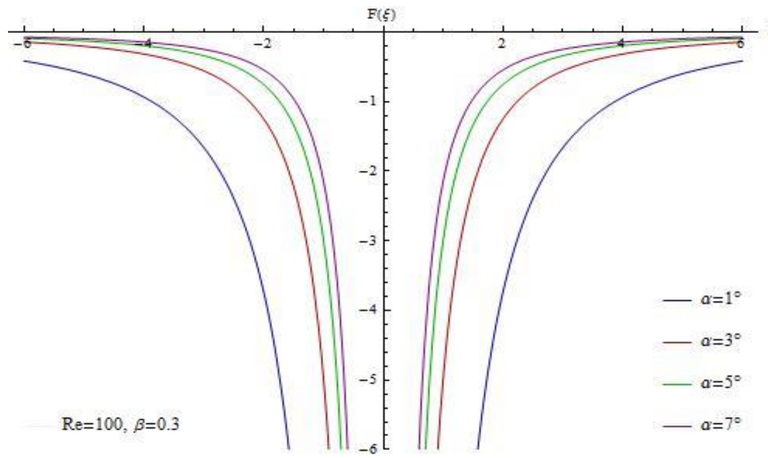

Fig. 1. Impacts of α on diverging channel for various values of α when Re = 100, β = 0.3.

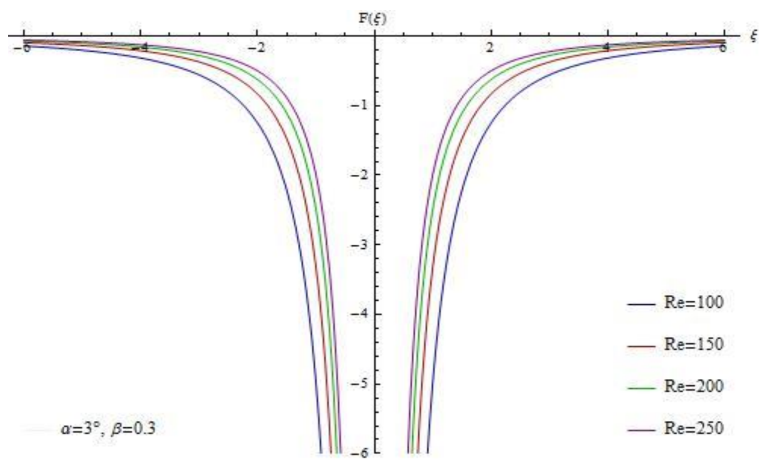

Fig. 2. Impacts of Re on diverging channel for various values of Re when α = 3°, β = 0.3.

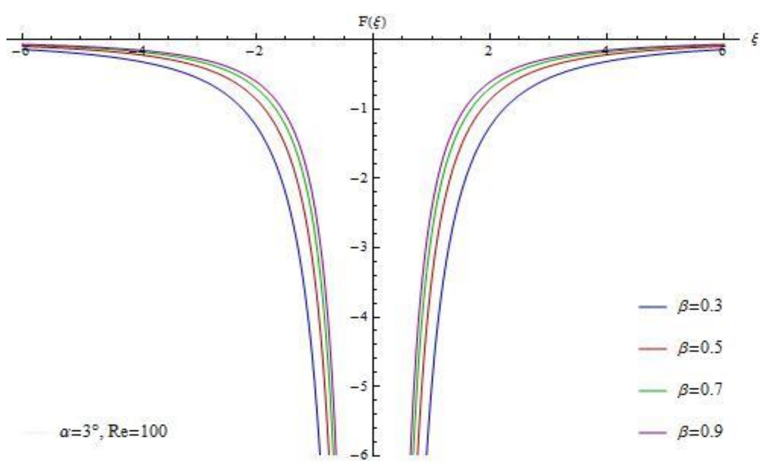

Fig. 3. Impacts of β on diverging channel for various values of β when α = 3°, Re = 100.

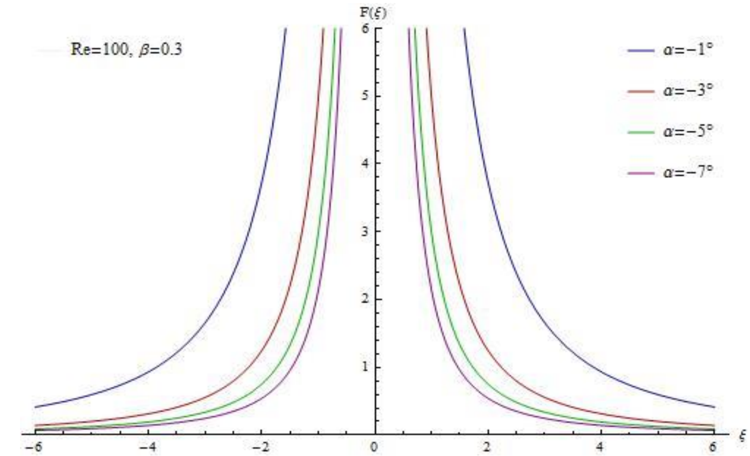

Fig. 4. Impacts of α on converging channel for different values of α when Re = 100, β = 0.3.

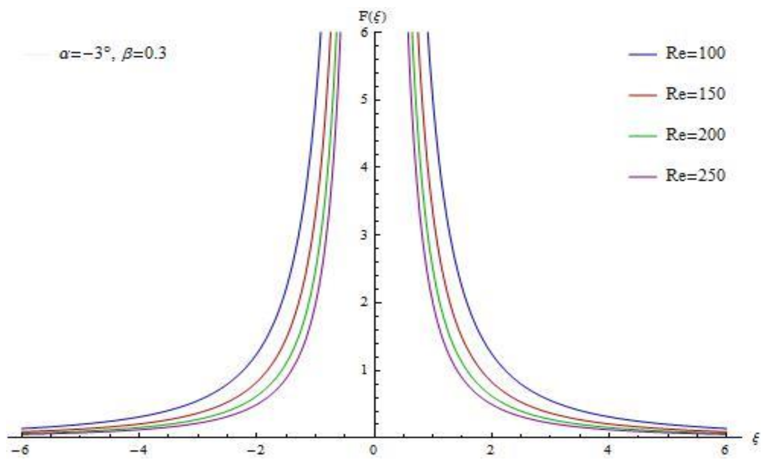

Fig. 5. Impacts of Re on converging channel for different values of Re when α = 3°, β = 0.3.

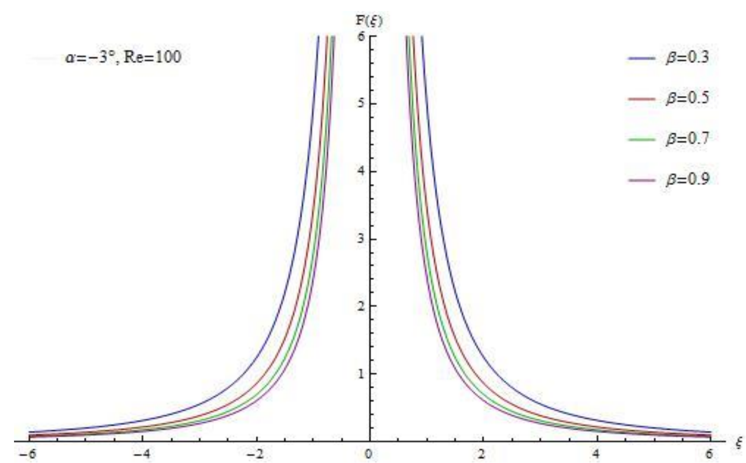

Fig. 6. Impacts of β on converging channel for different values of β when α = 3°, Re = 100.

7. RESULTS AND DISCUSSION

We conducted an analytical solution to the JH flow of a non-Newtonian fluid using the sine-cosine method. This article explores the graphical representation of the effect of various emerging constraints on the velocity profile of the Jeffery-Hamel flow of a non-Newtonian fluid. Figures 1 to 6 have been generated for this purpose. Figures 1 to 3, illustrate the impacts of α, Re and β on a diverging channel. These figures reveal a consistent impact on the velocity profile, indicating that an increase in α, Re and β results in a decrease in the velocity profile. The maximum velocity is observed near the center of channel for all these parameters. Figures 4 to 6 depicts the effects of same parameters α, Re and β for a converging channel, revealing an opposite behaviour where an increase in α, Re and β leads to increase in velocity. Through this method, $\beta \to \infty$ does not give the velocity for a simple non-Newtonian fluid.

8. CONCLUSIONS

In this paper, we have presented the exact traveling wave solution of the Jeffery-Hamel (JH) flow of a non-Newtonian fluid (NNF). Utilizing an efficient technique known as sine-cosine method, we successfully solved the governing nonlinear equation. Through the applicaton of this method, we explored the exact travelling wave solutions for the JH flow of a NNF. The obtained results are reliable and have implication for various applications involving nonlinear ordinary differential equations. Our findings indicate that, as the parameters increase, the velocity behaviour in a constricted channel is contrary to that in an expanding channel where the velocity decreases. Consequently, we propose that the applied approach offers an effective mathematical tool for adderssing nonlinear wave equations in both mathematical physics and engineering domain.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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