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COMMON FIXED POINT RESULTS IN CONE METRIC SPACES

K. PRUDHVI

Department of Mathematics, University College of Science, Saifabad, Osmania University,

Hyderabad, Andhra Pradesh, India

Abstract: In this work, we prove a unique common fixed point theorem in cone metric spaces without appealing to continuity and commutativity conditions. These results generalize several well-known comparable results in the literature.

Keywords: Common fixed Point; Cone metric space; Coincidence points.

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1. Introduction

In 2007 Huang and Zhang [3] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3], [5] and the references mentioned therein). Recently, Stojan Radenović [6] has obtained coincidence point result for two mappings in cone metric spaces which satisfies new contractive conditions. In this paper, we prove coincidence point results in cone metric spaces which satisfy generalized contractive condition without appealing the continuity and commutativity conditions.

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In all that follows B is a real Banach Space, and θ denotes the zero element of B. For the mapping f, g: X $\rightarrow$ X, let C (f, g) denote the set of coincidence points of f and g, that is C(f, g) = {z $\in$ X : fz = gz}.

2. Preliminaries

The following definitions are due to Huang and Zhang [3].

Definition 1.1. Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:

(a). P is closed , non –empty and P $\neq \{\theta\}$;

(b). a,b $\in R$, a,b ≥ 0 ,x,y $\in P$ implies ax+by $\in P$;

(c). x $\in$ P and -x $\in P$ implies x $= \theta$.

Definition 1.2. Let P be a cone in a Banach Space B, define partial ordering '$\leq$' with respect to P by x $\leq$ y if and only if y-x $\in P$ .We shall write x<y to indicate x $\leq y$ but

x $\neq y$ while X<<y will stand for y-x $\in$ Int P, where Int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3. Let B be a Banach Space and P $\subset$ B be an order cone .The order cone P is called normal if there exists K>0 such that for all x,y $\in$ B,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of B .Suppose that the map

d: X $\times$ X $\rightarrow$ B satisfies :

(d1). $\theta \leq d(x, y)$ for all x, y $\in X$ and

$d(x, y) = \theta$ if and only if $x = y$;

(d2).$d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3).$d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space .

The concept of a cone metric space is more general than that of a metric space.

Definition 1.5. Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c >> \theta$,there is N such that

for all n, m>N, $d(x_n, x_m) << c$;

(ii) a convergent sequence if for any $c >> \theta$,there is an N such that for all

n>N, $d(x_n, x) << c$, for some fixed x in X .We denote this $x_n \to x$ (as $n \to \infty$) .

Lemma 1.6. Let (X,d) be a cone metric space , and let P be a normal cone with normal constant K .Let $\{x_n\}$ be a sequence in X .Then

(i). $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ ($n \to \infty$).

(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ ($n,m \to \infty$).

3. Main results

In this section, we prove existence of coincidence point and a common fixed point theorem in cone metric spaces without appealing to continuity and commutativity conditions, which generalizes the results of Stojan Radenović [6].

The following theorem generalizes the Theorem 2.1 [6].

Theorem 2.1. Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that the mappings f, g: $X \to X$ are such that for some constant $\lambda \in (0,1)$ and for every x, $y \in X$ are two self-maps of X satisfying

$$\| d(fx, fy) \| \leq \lambda \| d(gx, gy) \| . \quad (1)$$

If the range of g contains the range of f and g(X) is a complete subspace of X, then f and g have coincidence point. Then, f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and let $x_1 \in X$ be chosen such that $y_0 = f(x_0) = g(x_1)$. Since $f(X) \subseteq g(X)$. Let $x_2 \in X$ be chosen such that $y_1 = f(x_1) = g(x_2)$. Continuing this process, having chosen $x_n \in X$, we chose $x_{n+1} \in X$
such that $y_n = f(x_n) = g(x_{n+1})$.

We first show that

$$\| d(y_n, y_{n-1}) \| \leq \lambda \| d(y_{n-1}, y_{n-2}) \| \quad \text{for} \quad n = 2,3,\ldots.. \quad (2)$$

Indeed,

$$\| d(y_n, y_{n-1}) \| = \| d(fx_n , fx_{n-1}) \| \leq \lambda \| d(gx_n , gx_{n-1}) \| .$$

(2) implies that

$$\| d(y_n, y_{n-1}) \| \leq \lambda \| d(y_{n-1}, y_{n-2}) \| \leq \ldots\ldots.. \leq \lambda^{n-1} \| d(y_1, y_0) \| . \quad (3)$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence. By the triangle inequality, for n>m we have

$$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots.. + d(y_{m+1}, y_m) .$$

Hence, as p is a normal cone,

$$\|d(y_n, y_m)\| \le K\|d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots + d(y_{m+1}, y_m)\| .$$

$$\le K(\|d(y_n, y_{n-1})\| + \|d(y_{n-1}, y_{n-2})\| + \ldots\ldots + \|d(y_{m+1}, y_m)\|) .$$

Now by (3),

$$\|d(y_n, y_m)\| \le K(\lambda^{n-1} + \lambda^{n-2} + \ldots\ldots + \lambda^m)\|d(y_1, y_0)\| .$$

$$\le \frac{K\lambda^m}{1-\lambda}\|d(y_1, y_0)\| \to 0 \text{ as } m\to\infty.$$

From ([3], Lemma 4) it follows that $\{y_n\}$ is a Cauchy sequence. Since g(X) is complete, there exists a q in g(X) such that $y_n \to q$ as $n\to\infty$. Consequently, we can find p in X such that g(p) = q. We shall show that f(p) = q. From (1)

$$\|d(gx_n, fp)\| = \|d(fx_{n-1}, fp)\| \le \lambda\|d(gx_{n-1}, gp)\| ,$$

$$\Rightarrow \quad \|d(gp, fp)\| \le \lambda\|d(gp, gp)\| = 0.$$

That is, $\|d(gp, fp)\| = 0$.

Hence,

gp =q= fp , p is a coincidence point of f and g. (4)

Now using (1),

$$d(p,gp) \le d(p, y_n) + d(y_n, gp) \quad \text{(by the triangle inequality)}$$

$$= d(p, y_n) + d(fx_n, fp)$$

$\leq d(p, y_n) + \lambda\, d(gx_n, gp)$

$\leq d(p, fx_n)+\lambda d(gx_n, gp)$ as $n\to\infty$

$\leq d(p, p) + \lambda\, d(p, gp)$

$\leq \lambda\, d(p,gp)$

$< \lambda\, d(p,gp)$ (since, $\lambda<1$))

$d(p, gp) < d(p, gp)$, which is a contradiction .

Therefore, $d(p, gp)=0$.

$\Rightarrow$ $p = gp$.

Now,

$d(fp,p) = d(fp, gp)$

$= d(fp, fp)$ (since $fp=gp$)

$\leq \lambda\, d(gp, gp)$

$= \lambda\, d(p, p) = 0$ (by (1))

$\Rightarrow$ $d(fp, p) = 0$

$\Rightarrow$ $fp = p$.

Since, fp = gp.

Therefore, fp = gp = p, f and g have a common fixed point.

Uniqueness, let p_1 be another common fixed point of f and g, then

$$\begin{aligned} d(p, p_1) &= d(fp, g\,p_1) \\ &= d(fp, f\,p_1) \\ &\leq d(gp, g\,p_1) \text{ (by (1))} \\ &\leq d(p, p_1) \leq 0. \end{aligned}$$

Therefore, $d(p, p_1) = 0$.

$\Rightarrow$ $p = p_1$.

Therefore, f and g have a unique common fixed point.

This completes the proof.

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