Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 824-829
ISSN: 1927-5307

CONVEXITY OF A CLASS OF MATRIX FUNCTIONS

XIANG GAO*

School of Mathematical Sciences, Ocean University of China, Lane 238, Songling Road, Laoshan District, Qingdao City, Shandong Province, People's Republic of China, 266100

Abstract. In this paper, we present some sufficient conditions for the convexity of the function $f(A) = g(\det A)$, where $g(x)$ is a monotonic increasing and convex function.

Keywords: convexity; determinant; Minkowski inequality.

2000 AMS Subject Classification: 15A15

1. Introduction and Main Results

The research of matrix inequalities which have well symmetry is very important and interesting in linear algebra and matrix theory. Among the matrix inequalities theory, the research of convexity of some particular matrix functions is also extremely valuable. In this paper, we deal with a class of interesting matrix functions $f(A) = g(\det A)$ and present some sufficient conditions for the convexity of these functions.

Firstly we recall some basic facts. A matrix $A \in M_n$ is said to be positive definite if $\mathrm{Re}\left(x^T Ax\right) > 0$ for all nonzero $x \in \mathbb{C}^n$. The convex set of positive definite matrices is denoted by M_n^+.

*Corresponding author

E-mail address: gaoxiangshuli@126.com(X. Gao)

Received December 16, 2011

Definition 1.1. A real valued function f defined on M_n^+ is said to be convex if

$$f(\alpha A+\beta B)\leq \alpha f(A)+\beta f(B),$$

and concave if

$$f(\alpha A+\beta B)\geq \alpha f(A)+\beta f(B)$$

for all $0<\alpha,\beta<1, \alpha+\beta=1$ and all $A,B\in M_n^+, A\neq B$.

It has been proved by Horn, Johnson [1] that the function $f(A)=\log(\det A)$ is strictly concave function on the convex set of positive definite Hermitian matrices M_n^+, that is

$$\log(\det(\alpha A+\beta B))\geq \alpha\log(\det A)+\beta\log(\det B) \tag{1}$$

for positive definite matrices $A,B\in M_n^+$ and $0<\alpha,\beta<1, \alpha+\beta=1$.

By the following famous Minkowski inequality which is widely used in linear algebra and matrix theory, we obtain that the function $f(A)=(\det A)^{\frac{1}{n}}$ is also concave on the set of positive definite Hermitian matrices which states that

$$(\det(\alpha A+\beta B))^{\frac{1}{n}}\geq \alpha(\det A)^{\frac{1}{n}}+\beta(\det B)^{\frac{1}{n}}. \tag{2}$$

Theorem 1.2.(Minkowski Inequality) *If $A,B\in M_n^+(R)$, then*

$$(\det(A+B))^{\frac{1}{n}}\geq (\det A)^{\frac{1}{n}}+(\det B)^{\frac{1}{n}}. \tag{3}$$

But in general, the function $f(A)=(\det A)^m$ is not concave for $m\neq\frac{1}{n}$, not to mention a general function $f(A)=g(\det A)$.

The purpose of this paper is to discuss this question, and we will present some sufficient conditions for the convexity of function $f(A)=g(\det A)$. In fact we will prove the following theorems.

Theorem 1.3. *Let A is a positive definite matrix, B is a symmetric matrix, $\lambda_i(A),\lambda_i(B)$, $i=1,\cdots,n$ be the eigenvalues of A and B, $0<\alpha,\beta<1$ with $\alpha+\beta=1$. Then for a monotonic increasing and convex function $g(x)$,*

$$g(\det(\alpha A+\beta B))\leq \alpha g(\det A)+\beta g(\det B) \tag{4}$$

holds true if one of the following conditions is satisfied

(i) $\lambda_i(B) \geq \max\limits_{1\leq i\leq n} \lambda_i(A)$ *for* $i = 1, \cdots, n$;

(ii) $\lambda_i(B) \leq \min\limits_{1\leq i\leq n} \lambda_i(A)$ *for* $i = 1, \cdots, n$.

Theorem 1.4. *Let A, B be symmetric matrices such that $AB = BA$, and let $\lambda_i(A), \lambda_i(B)$, $i = 1, \cdots, n$ be the eigenvalues of A and B, $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$. Then for a monotonic increasing and convex function $g(x)$,*

$$g(\det(\alpha A + \beta B)) \leq \alpha g(\det A) + \beta g(\det B) \tag{5}$$

holds true if one of the following conditions is satisfied

(i) $\lambda_i(B) \geq \lambda_i(A)$ *for* $i = 1, \cdots, n$;

(ii) $\lambda_i(B) \leq \lambda_i(A)$ *for* $i = 1, \cdots, n$.

2. Proof of the Main Results

For the proof of the main results we prove the following lemma firstly.

Theorem 2.1. *If $0 < \alpha, \beta < 1$ satisfying $\alpha + \beta = 1$, and $\lambda_i \geq \mu_i$ for arbitrary $1 \leq i \leq n$ or $\lambda_i \leq \mu_i$ for arbitrary $1 \leq i \leq n$, then*

$$\prod_{i=1}^{n} (\lambda_i\alpha + \mu_i\beta) \leq \alpha \prod_{i=1}^{n} \lambda_i + \beta \prod_{i=1}^{n} \mu_i. \tag{6}$$

Proof. The approach we use is mathematical induction. Firstly we consider $n = 2$, since $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, we have

$$\begin{aligned}(\lambda_1\alpha + \mu_1\beta)(\lambda_2\alpha + \mu_2\beta) &= ((\lambda_1 - \mu_1)\alpha + \mu_1)(\lambda_2 - (\lambda_2 - \mu_2)\beta) \\ &= \lambda_2(\lambda_1 - \mu_1)\alpha - \mu_1(\lambda_2 - \mu_2)\beta - (\lambda_1 - \mu_1)(\lambda_2 - \mu_2)\alpha\beta \\ &\quad + \lambda_2\mu_1 \\ &= \lambda_1\lambda_2\alpha + \mu_1\mu_2\beta - (\lambda_1 - \mu_1)(\lambda_2 - \mu_2)\alpha\beta.\end{aligned} \tag{7}$$

Then it follows from the hypotheses that $(\lambda_1\alpha + \mu_1\beta)(\lambda_2\alpha + \mu_2\beta) \leq \lambda_1\lambda_2\alpha + \mu_1\mu_2\beta$.

Assume that (6) is true for $n = k$, we prove that it is also true for $n = k + 1$. S-ince (6) holds for $n = k$ we have

$$\begin{aligned}\prod_{i=1}^{k+1}(\lambda_i\alpha+\mu_i\beta) &= (\lambda_{k+1}\alpha+\mu_{k+1}\beta)\prod_{i=1}^{k}(\lambda_i\alpha+\mu_i\beta)\\ &\le (\lambda_{k+1}\alpha+\mu_{k+1}\beta)\left(\alpha\prod_{i=1}^{k}\lambda_i+\beta\prod_{i=1}^{k}\mu_i\right).\end{aligned}\tag{8}$$

As the proof of (7) we obtain

$$\begin{aligned}&(\lambda_{k+1}\alpha+\mu_{k+1}\beta)\left(\alpha\prod_{i=1}^{k}\lambda_i+\beta\prod_{i=1}^{k}\mu_i\right)\\ &=\left(\alpha\lambda_{k+1}\prod_{i=1}^{k}\lambda_i+\beta\mu_{k+1}\prod_{i=1}^{k}\mu_i\right)-\alpha\beta\left(\lambda_{k+1}-\mu_{k+1}\right)\left(\prod_{i=1}^{k}\lambda_i-\prod_{i=1}^{k}\mu_i\right).\end{aligned}\tag{9}$$

Since the second term in (9) is nonpositive for $\lambda_i \ge \mu_i$ or for $\lambda_i \le \mu_i$, it follows from (8) and (9) that

$$\prod_{i=1}^{k+1}(\lambda_i\alpha+\mu_i\beta)\le\alpha\prod_{i=1}^{k+1}\lambda_i+\beta\prod_{i=1}^{k+1}\mu_i,$$

and consequently that inequality (6) holds for $n = k+1$.

With the help Lemma 2.1, we now turn to prove our main theorems.

Proof of Theorem 1.3. Since $g(x)$ is a convex function, we have $g(\alpha x+\beta y)\le \alpha g(x)+\beta g(y)$ for arbitrary $0<\alpha,\beta<1, \alpha+\beta=1$. Putting $x=\det A$ and $y=\det B$ it follows that

$$g(\alpha\det A+\beta\det B)\le\alpha g(\det A)+\beta g(\det B)\tag{10}$$

On the other hand, it is known from [1] that for positive definite matrix A and symmetric matrix B there exists a nonsingular matrix C such that $A = C^TC$ and $B = C^T\Lambda C$, where $\Lambda = diag\{\lambda_1,\cdots,\lambda_n\}$. The inequality

$$\det(\alpha A+\beta B)\le\alpha\det A+\beta\det B\tag{11}$$

is then equivalent to $\det(\alpha+\beta\Lambda)\le\alpha+\beta\det\Lambda$, that is

$$\prod_{i=1}^{n}(\alpha+\lambda_i\beta)\le\alpha+\beta\prod_{i=1}^{n}\lambda_i.$$

It follows from Ostrowski Theorem [1] that for each $1 \le i \le n$ there exists $\theta_i > 0$ such that $\min\limits_{1\le i\le n} \lambda_i(A) \le \theta_i \le \max\limits_{1\le i\le n} \lambda_i(A)$ and $\lambda_i(B) = \theta_i \lambda_i$. Thus we conclude that

$$\frac{\lambda_i(B)}{\max\limits_{1\le i\le n} \lambda_i(A)} \le \lambda_i \le \frac{\lambda_i(B)}{\min\limits_{1\le i\le n} \lambda_i(A)} \tag{12}$$

Therefore if the condition (i) satisfies, we have $\lambda_i \ge 1$ for arbitrary $1 \le i \le n$, and if the condition (ii) is satisfied, we have $\lambda_i \le 1$ for arbitrary $1 \le i \le n$. It follows from Lemma 2.1 that $\prod\limits_{i=1}^{n} (\alpha + \lambda_i \beta) \le \alpha + \beta \prod\limits_{i=1}^{n} \lambda_i$. Since $g(x)$ is a monotonic increasing function, together with (10) we complete the proof of Theorem 1.3.

Proof of Theorem 1.4. As the proof of Theorem 1.3, we only need to prove (11) is also satisfied under the hypotheses of Theorem 1.4. Indeed, since $AB = BA$, it is known from [1] that there exists a orthogonal matrix C such that $A = C^T \Lambda_A C$, $B = C^T \Lambda_B C$, where

$$\Lambda_A = diag\{\lambda_1(A), \cdots, \lambda_n(A)\}, \qquad \Lambda_B = diag\{\lambda_1(B), \cdots, \lambda_n(B)\}.$$

Thus (11) is equivalent to

$$\det(\alpha\Lambda_A + \beta\Lambda_B) \le \alpha \det \Lambda_A + \beta \det \Lambda_B,$$

that is

$$\prod_{i=1}^{n} (\alpha\lambda_i(A) + \beta\lambda_i(B)) \le \alpha \prod_{i=1}^{n} \lambda_i(A) + \beta \prod_{i=1}^{n} \lambda_i(B).$$

Therefore if the condition (i) satisfies, we have $\lambda_i = \lambda_i(A) \le \lambda_i(B) = \mu_i$ for arbitrary $1 \le i \le n$, and if the condition (ii) is satisfied, we have $\lambda_i = \lambda_i(A) \ge \lambda_i(B) = \mu_i$ for arbitrary $1 \le i \le n$. It follows from Lemma 2.1 that

$$\prod_{i=1}^{n} (\alpha\lambda_i(A) + \beta\lambda_i(B)) \le \alpha \prod_{i=1}^{n} \lambda_i(A) + \beta \prod_{i=1}^{n} \lambda_i(B).$$

Since $g(x)$ is a monotonic increasing function, together with (10) we complete the proof of Theorem 1.4.

Remark 2.2 In fact the key point of our proofs is (11), hence we have the following corollary.

Corollary 2.3. *Let A is a positive definite matrix, B is a symmetric matrix, $\lambda_i(A), \lambda_i(B)$, $i = 1, \cdots, n$ be the eigenvalues of A and B, $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$. Then for a monotonic decreasing and concave function $g(x)$,*

$$g(\det(\alpha A + \beta B)) \geq \alpha g(\det A) + \beta g(\det B)$$

holds true if one of the following conditions is satisfied

(i) $\lambda_i(B) \geq \max\limits_{1 \leq i \leq n} \lambda_i(A)$ *for* $i = 1, \cdots, n$;

(ii) $\lambda_i(B) \leq \min\limits_{1 \leq i \leq n} \lambda_i(A)$ *for* $i = 1, \cdots, n$.

Corollary 2.4. *Let A, B be symmetric matrices such that $AB = BA$, and let $\lambda_i(A), \lambda_i(B)$, $i = 1, \cdots, n$ be the eigenvalues of A and B, $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$. Then for a monotonic decreasing and concave function $g(x)$,*

$$g(\det(\alpha A + \beta B)) \geq \alpha g(\det A) + \beta g(\det B)$$

holds true if one of the following conditions is satisfied

(i) $\lambda_i(B) \geq \lambda_i(A)$ *for* $i = 1, \cdots, n$;

(ii) $\lambda_i(B) \leq \lambda_i(A)$ *for* $i = 1, \cdots, n$.

Acknowledgment. I would especially like to express my appreciation to my advisor professor Yu Zheng for longtime encouragement and meaningful discussions.

References

[1] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, 1999.