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CODES FROM k-RESOLVING SETS FOR STACKED PRISM GRAPHS

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Abstract. A k-resolving set S is a set of vertices $\{v_1, v_2, ...v_l\}$ of a graph $G(V,E)$ if for distinct vertices $u, w \in V$, the lists of distances $(d_G(u,v_1), d_G(u,v_2), ..., d_G(u,v_l))$ and $(d_G(w,v_1), d_G(w,v_2), ..., d_G(w,v_l))$ differ in at least k-positions. The least size of a k-resolving set is called the k-metric basis of G and its cardinality is called the k-metric dimension, denoted by $dim_k(G)$. We determine error correcting codes for stacked prism graphs namely $P_m \square C_n$ using k-resolving sets. We have also constructed an infinite family of stacked prisms of k-dimension. In this paper we have studied the k-metric dimension of $P_m \square C_n$. An explicit formula for $dim_k(P_m \square C_n)$ is determined and the codes arising from k-resolving sets of $P_m \square C_n$ are developed.

Keywords: error-correcting codes; k-resolving sets; k-metric dimension; k-metric basis; stacked prism graphs.
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1. Introduction

1.1. Error-correcting codes. Algebraic coding theory also known as theory of error-correcting codes, is an area of applied mathematics, inspired by the fundamental paper of Claude Shannon (1948) [28] along with results by Marcel Golay (1949) [13] and Richard Hamming (1950) [15]. Since then it has become an area of great interest where algebraic structures, probability and combinatorics all play a role [14].

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Coding theory is the study of methods for efficient and accurate transfer of information from one place to another. It deals with the problem of detecting and correcting transmission errors caused by a noise on the channel. It has diverse applications from transmission of financial information across telephone lines, data transfer from one computer to another, to information transmission from a distant source such as a weather or communications satellite [17].

Although the problems in coding theory often arise from engineering applications, it is indeed fascinating to know that mathematics plays a very crucial role in the development of codes [25].

For an introductory exposition one can refer [14, 17, 19].

Formally, an error-correcting code(or simply a code) is a collection $\mathscr{C}$ of vectors, called codewords, of given length l over a fixed alphabet. The Hamming distance between two codewords $\mathbf{x} = (x_1, ..., x_l)$, $\mathbf{y} = (y_1, ..., y_l)$ is the number of positions where they differ, that is, $|\{i : x_i \neq y_i\}|$. The minimum distance of $\mathscr{C}$ is the least Hamming distance between any two distinct codewords; If the minimum distance is D, then the correction capability of $\mathscr{C}$ is $r = \lfloor (D-1)/2 \rfloor$. Suppose that a codeword $\mathbf{x}$ is transmitted via a noisy channel causes errors to appear. Often the codes with highest error correction capabilities are sought for, so that at the decoding point efforts are less, nevertheless a decoding algorithm is always essential to get back the original message. Usage of group theory, combinatorics is not new in encoding and decoding [6, 19, 25].

1.2. ***k*-metric dimension.** Metric dimension is an effective tool to study different distance-based problems in the field of robot navigation [22], geographical routing protocols [26], combinatorial optimization [27], network discovery and verification [4], sensor networks [18] and chemistry [7].

Metric dimension was introduced separately by Slater [29] in 1975 and by Harary and Melter [16] in 1976. There are several variations of metric dimension in graphs such as strong metric dimension, local metric dimension, adjacency dimension, k-metric dimension, partition dimension and so on. For more information see the survey by Kuziak and Yero [24].

Estrada - Moreno et al. [10] introduced the k-metric dimension problem. In their paper [10] the k-metric dimension of path graphs, cycle graphs and trees were studied. The k-metric dimension of lexicographic products of graphs, corona product of graphs, unicyclic graphs was studied by Estrada et al. [11, 12, 9]. Yero et al. [31] showed that the decision problem regarding whether the k-metric dimension of a graph does not exceed a positive integer is NP-complete, which also shows the NP-hardness of computing $dim_k(G)$ for any graph G. Further, an algorithm is provided to compute the k-metric dimension and k-metric basis of any tree. Corregidor et al. [8] determined some bounds for k for which a graph is k-metric dimensional. Klavžar et al. [23] studied the k-metric dimension of hierarchical product of graphs, splice and link product of graphs. An integer linear programming model for finding the k-metric dimension and k-metric basis for a graph was developed. Bailey et al. [1] studied the k-metric dimension of grid graphs, that is the Cartesian product of path graphs.

Bailey et al. [1] have started a new trend with the usage of graph parameters in developing codes. Bailey et al. [1] used the metric property of a graph in developing effective codes. In their interesting paper [1], the authors have used k-resolving sets arising in graphs and simple graph product namely the grid graphs. In [20], the present authors extending this technique to cartesian products of different graphs not just involving paths as done by Bailey and Yero [1]. In this paper we extend this to another class of cartesian products, namely the stacked prisms, which are the cartesian products involving paths and cycles to obtain codes using k-resolving sets in these graphs. Decoding algorithms are developed, along with their complexity analysis.

We first give some important definitions and results about k-resolving sets.

2. Preliminaries

2.1. ***k*-resolving sets.** We consider finite, simple, connected, undirected graphs. The distance between two vertices u and v of a graph G is the length of a shortest path between u and v, and we denote this by $d_G(u,v)$.

Definition 2.1. [16, 29] *Let $G=(V,E)$ be a graph. Given a set $S=\{v_1,v_2,...,v_d\}\subseteq V(G)$ and a vertex $u\in V(G)$, the vector $r(u|S)=(d_G(u,v_1),d_G(u,v_2),...,d_G(u,v_d))$ is called the metric representation of u with respect to S. The set S is called a resolving set for G if the metric representations of all vertices of G are pairwise different. This means that for every pair of distinct vertices of G, their metric representations differ in at least one position.*

Definition 2.2. [16, 29] *A resolving set with the smallest possible cardinality is called a metric basis of G.*

Definition 2.3. [16, 29] *The cardinality of a metric basis of G is called metric dimension of G, denoted by* $dim(G)$.

Definition 2.4. [10] *Let* $G=(V,E)$ *be a graph. An ordered set of vertices* $\{v_1,v_2,...,v_l\}$ *is a k-resolving set for G if, for any distinct vertices* $u,w\in V$*, the lists of distances* $(d_G(u,v_1),d_G(u,v_2),...,d_G(u,v_l))$ *and* $(d_G(w,v_1),d_G(w,v_2),...,d_G(w,v_l))$ *differ in at least k-positions.*

Definition 2.5. [10] *A k-resolving set with minimum cardinality is called a k-metric basis of G.*

Definition 2.6. [10] *The cardinality of a k-metric basis of G is called k-metric dimension of G, denoted by* $dim_k(G)$.

Definition 2.7. [10] *If k is the largest integer for which G has a k-resolving set, then G is called a k-metric dimensional graph.*

Definition 2.8. [10] *Given two vertices* $x,y\in V(G)$*, the set of distinctive vertices of* x,y *is denoted by* $\mathscr{D}_G(x,y)$ *and is defined as* $\mathscr{D}_G(x,y)=\{z\in V(G): d_G(x,z)\neq d_G(y,z)\}$.

Many results were derived using this definition.

Theorem 2.9. [10] *A connected graph G is k-metric dimensional if and only if* $k=min_{x,y\in V(G)}|\mathscr{D}_G(x,y)|$.

Definition 2.10. [30] *The Cartesian product of G and H, written,* $G\Box H$*, is the graph with vertex set* $V(G)\times V(H)$ *specified by putting* (u,v) *adjacent to* (u',v') *if and only if (i)* $u=u'$ *and* $vv'\in E(H)$*, or (ii)* $v=v'$ *and* $uu'\in E(G)$.

3. Main Results

3.1. The k-metric dimension of $P_m\Box C_n$. A stacked prism is obtained by choosing $G=P_m$ and $H=C_n$ in Definition 2.10 [30]. As we will show, for any graph $P_m\Box C_n$ of order nm and any $k\in\{1,...,\lceil\frac{n}{2}\rceil m\}$,

$$dim_k(G)=\begin{cases} 2k-\left\lfloor\dfrac{k}{\lceil\frac{n}{2}\rceil}\right\rfloor, & \text{when } n \text{ is odd,}\\ 2k, & \text{if } k\geq 2 \text{ and } \ 2k+1, \quad \text{if } k=1, \text{ when } n \text{ is even and } n\geq 6.\end{cases}$$

This goes on to provide an interesting infinite family of $k-$ metric dimensional graphs namely, stacked prisms.

Suppose that G is the graph $P_m \square C_n$, and $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ are the vertex sets of P_m and C_n respectively. It is clear that, if G is a k-metric dimensional graph, then for every natural number $k' \leq k$, G also has a k'-metric basis. A characterization of k-metric dimensional graphs obtained in Theorem 2.9 [10], will be useful in our work.

To compute the k-metric dimension of stacked prism graphs, we need to first determine for which values of k there exists a k-metric basis. This is answered by our first result.

Theorem 3.1. *The stacked prism graph $G = P_m \square C_n$ is $\left\lceil \frac{n}{2} \right\rceil m$-metric dimensional for any integers $m \geq 2$ and $n \geq 3$.*

Proof. We will first show that $k \leq \left\lceil \frac{n}{2} \right\rceil m$. For this, let us consider the vertices (u_1, v_1) and (u_2, v_2) .

When n is odd, $\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big) = \big(\{u_1\} \times \{v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_{\lceil \frac{n}{2} \rceil+2}, ...v_{n-1}, v_n\}\big)$
$\cup\big(\{u_2,u_3,...,u_m\} \times \{v_2, v_3, ..., v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}\}\big)$.

Thus,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil + (m-1)(\lceil \frac{n}{2} \rceil + 1 - 1)$,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil (1 + m - 1)$,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil m.$

Similarly, when n is even, $\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big) = \big(\{u_1\} \times \{v_1, v_{\lceil \frac{n}{2} \rceil+2}, v_{\lceil \frac{n}{2} \rceil+3}, ...v_{n-1}, v_n\}\big)$
$\cup\big(\{u_2,u_3,...,u_m\} \times \{v_2, v_3, ..., v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}\}\big)$.

Thus,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil + (m-1)(\lceil \frac{n}{2} \rceil + 1 - 1)$,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil (1 + m - 1)$,

$\big|\mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)\big| = \lceil \frac{n}{2} \rceil m.$

We see that $\lceil \frac{n}{2} \rceil m$ is the largest minimum value possible (since we consider the full vertex set).

Hence G is k-metric dimensional for some $k \leq \lceil \frac{n}{2} \rceil m$.

Next we will show that $k \geq \lceil \frac{n}{2} \rceil m$. Let (u_i, v_j) and (u_s, v_t) be any two distinct vertices of G. Let us consider the following cases to determine the number of distinctive vertices.

Case 1. $i = s$.

Hence $j \neq t$ and it follows that

$$\{u_i\} \times \{v_j, v_t\} \subseteq \mathscr{D}_G\big((u_i, v_j), (u_s, v_t) \big)$$

for $i = 1, 2, .., m$.

Also $\big| \big(\{u_i\} \times V \big) \cap \mathscr{D}_G\big((u_i, v_j), (u_s, v_t) \big) \big| \geq n-1$ when n is odd and
$\big| \big(\{u_i\} \times V \big) \cap \mathscr{D}_G\big((u_i, v_j), (u_s, v_t) \big) \big| \geq n-2$ when n is even, for $i = 1, 2, ..., m$.

Thus, for odd n, we have

$\big|\mathscr{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| \geq \big|\{u_i\} \times \{v_j, v_t\}\big| + |(\{u_i\} \times V) \cap D_G\big((u_i, v_j), (u_s, v_t)\big)| - 2$, for $i = 1, 2, .., m$

$$\geq 2 + n - 1 - 2$$
$$\geq n-1, \text{ for } i = 1, 2, ..., m.$$

Hence in general,

$\big|\mathscr{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| \geq (n-1) + (n-1) + (n-1) + ... + (n-1)$ (m times)

$$\geq m(n-1) = mn - m$$

$$\geq mn - n \quad \text{if } n \geq m$$

$$\geq n, \text{ since } m \geq 2.$$

We have $mn - n \geq n$. Dividing by 2 on both sides and taking ceil function on both sides, we get, $\lceil \frac{mn-n}{2} \rceil \geq \lceil \frac{n}{2} \rceil$. Multiplying on both sides by m, we get $\lceil \frac{mn-n}{2} \rceil m \geq \lceil \frac{n}{2} \rceil m$. Hence $\big|\mathscr{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| \geq \lceil \frac{n}{2} \rceil m$ if $n \geq m$.

If $n \leq m$ then $mn - m \geq n^2 - n = n(n-1)$.

$$mn - m \geq n \text{ and } n-1.$$

Consider $mn - m \geq n$. This implies that $\lceil \frac{mn-m}{2} \rceil \geq \lceil \frac{n}{2} \rceil$. Multiplying both sides by m, we get $\lceil \frac{mn-m}{2} \rceil m \geq \lceil \frac{n}{2} \rceil m$. Hence $\big|\mathscr{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| \geq \lceil \frac{n}{2} \rceil m$ if $n \leq m$.

Similarly, for even n, we have

$$\begin{aligned}\left|\mathscr{D}_G\big((u_i,v_j),(u_s,v_t)\big)\right| &\geq \left|\{u_i\}\times\{v_j,v_t\}\right| + |(\{u_i\}\times V)\cap D_G\big((u_i,v_j),(u_s,v_t)\big)| - 2\\ &\geq 2+n-2-2\\ &\geq n-2, \text{ for } i=1,2,\dots,m.\end{aligned}$$

Hence in general,

$$\begin{aligned}\left|\mathscr{D}_G\big((u_i,v_j),(u_s,v_t)\big)\right| &\geq (n-2)+(n-2)+(n-2)+\dots+(n-2)\ (m \text{ times})\\ &\geq m(n-2) = mn-2m\\ &\geq mn-2n \quad \text{if } n\geq m\\ &\geq 0, \text{ since } m\geq 2.\end{aligned}$$

We have $mn-2m \geq 0$. This implies $mn \geq 2m \geq \lceil\frac{n}{2}\rceil m$ since $n\geq 3$. Hence $\left|\mathscr{D}_G\big((u_i,v_j),(u_s,v_t)\big)\right| \geq \lceil\frac{n}{2}\rceil m$ if $n\geq m$.

If $n\leq m$ then consider $mn-2m \geq n^2-2n = n(n-2)$.

$$mn-2m\geq n \text{ and } n-2.$$

Consider $mn-2m\geq n$. This implies that $mn\geq 2m+n\geq 2m\geq\lceil\frac{n}{2}\rceil m$ since $n\geq 3$. Thus $\left|\mathscr{D}_G\big((u_i,v_j),(u_s,v_t)\big)\right| \geq \lceil\frac{n}{2}\rceil m$ if $n\leq m$.

Case 2. $j=t$.

Proof is similar to Case 1.

Case 3. $i\neq s$ and $j\neq t$.

We assume $i<s$. Depending on $j<t$ or $>t$, we have the following cases:

Case 3(i). $j<t$.

Then,

$$\begin{aligned}\left|\mathscr{D}_G\big(\left(u_i,v_j\right),(u_s,v_t)\big)\right| &\geq \left|\{u_1,\dots,u_s\}\times\{v_j\}\right| + |\{u_i,\dots,u_m\}\times\{v_t\}| + |\{u_i\}\times\{v_1,\dots,v_t\}|\\ &\quad + \left|\{u_s\}\times\{v_j,\dots,v_n\}\right| - 4.\\ &\geq s+(m-i+1)+t+(n-j+1)-4\\ &= s+t+m+n-i-j-2\\ &\geq i+j+m+n-i-j-2 \text{ (because } i<s, j<t)\end{aligned}$$

$$\geq m+n-2$$
$$\geq n, \text{ since } m \geq 2.$$

Now $m+n-2 \geq n$. This implies $m+n \geq n+2$. Dividing by 2 throughout and taking ceil function on both sides, we get $\lceil \frac{m+n}{2} \rceil \geq \lceil \frac{n+2}{2} \rceil$. Multiplying both sides by m, we get $\lceil \frac{m+n}{2} \rceil m \geq \lceil \frac{n+2}{2} \rceil m \geq \lceil \frac{n}{2} \rceil m$. Hence $\left|\mathscr{D}_G\big((u_i,v_j),(u_s,v_t)\big)\right| \geq \lceil \frac{n}{2} \rceil m$ if $n \leq m$.

Case 3(ii). $j > t$.

Proof is similar to Case 3(i).

As a result of all the cases, $k \geq \left\lceil \frac{n}{2} \right\rceil m$. Hence $k = \left\lceil \frac{n}{2} \right\rceil m$. By Theorem 2.9 [10], a graph is k-metric dimensional if and only if $k = min_{x,y \in V(G)} |\mathscr{D}_G(x,y)|$. Hence $P_m \square C_n$ is $\left\lceil \frac{n}{2} \right\rceil m$-metric dimensional since $k = \left\lceil \frac{n}{2} \right\rceil m$. □

As in case of $P_s \square P_t$ [1], $K_n \square P_m$ [20] and $K_n \square K_m$[21] the distinctive vertices play an important role here also. It is observed that for a pair x,y, if S is a k-resolving set for $G = P_m \square C_n$, then $|\mathscr{D}(x,y) \cap S| \geq k$. We use this fact in proving the existence of k-metric dimension for every $k \in \left\{1,2,...,\left\lceil \frac{n}{2} \right\rceil m\right\}$ for $P_m \square C_n$. The result given below establishes this fact.

Theorem 3.2. *For the graph $G = P_m \square C_n$,*

$$dim_k(G) = \begin{cases} 2k - \left\lfloor \dfrac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor, & \text{when } n \text{ is odd,} \\ 2k, & \text{if } k \geq 2 \text{ and } \ 2k+1, \quad \text{if } k=1, \text{ when } n \text{ is even and } n \geq 6. \end{cases}$$

for any integers $m \geq 2, n \geq 3$ and $k \in \left\{1,2,...,\left\lceil \frac{n}{2} \right\rceil m\right\}$.

Proof. When $k = 1$, from [5] we know that,

$$\beta\left(P_m \square C_n\right) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Thus,

$$dim_1(G) = dim(G) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

From now on we consider only $k \geq 2$.

Let (u_i, v_j) and (u_s, v_t) be two distinct vertices of G.

The proof is divided into two cases depending on n being odd or even.

Case 1. n is odd.

First we will show that, $dim_k(G) \leq 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$. The metric basis of $P_m \square C_n$ follows an iterative process given below:

$$S_I = \left(\{u_l\} \times \left\{ v_{n-2q}, v_{n-q} \right\} \right)$$

$$S_{II} = \left(\{u_l\} \times \left\{ v_{n-2q}, v_{n-(2q-1)}, v_{n-q}, v_{n-(q-1)} \right\} \right)$$

To this set S_{II}, add pairs $\left\{ v_{n-(2q-2)}, v_{n-(q-2)} \right\}, \ldots\ldots, \left\{ v_{n-(2q-r)}, v_{n-(q-r)} \right\}$ where $q = \left\lfloor \frac{n}{2} \right\rfloor$ and $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$. Each pair gives a new k-resolving set. Next, we get

$$S_{III} = \left(\{u_l\} \times \left\{ v_{n-2q}, v_{n-(2q-1)}, \ldots, v_{n-(2q-r)}, v_{n-q}, v_{n-(q-1)}, \ldots, v_{n-(q-r)}, v_n \right\} \right).$$

Once we reach $\{u_l\} \times V$, we move to the next l iteration. The old l value becomes the new p value.

Thus, we have

$$S_{IV} = \left(\left\{ u_p \right\} \times V \right) \cup \left(\{u_l\} \times \left\{ v_{n-2q}, v_{n-q} \right\} \right)$$

where $l = 1, 2, \ldots, m$ and $p = 1, 2, \ldots, m-1$. Again we apply S_{II}, S_{III} to S_{IV} till we reach $\{u_l\} \times V$, in which case the old l value is added to p and l goes to the next iteration. We repeat this process till we get the full vertex set of $P_m \square C_n$.Note that when $k = n+1$, the metric basis is $S = \left(\{u_1, u_m\} \times V \right)$ (It does not follow the iteration process). Consider the distance matrix of $P_m \square C_n$ with respect to all the bases S (given above). For each k, we see that the metric representations of each vertex is unique and each pair of vertices differ in at least k positions. Hence S is a k-resolving set. Thus, $dim_k(G) \leq 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$.

Next we will show that $dim_k(G) \geq 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$.

We will show that no smaller k-resolving set can exist. We shall prove this by contradiction. When $k = \left\lceil \frac{n}{2} \right\rceil m$, $dim_k(G) \leq 2k - m$. Suppose S' is a k-metric basis for $G = P_m \square C_n$ of cardinality $< 2k - m$, that is of cardinality $2k - (m+1)$.

Then $S' = \{(u_1,v_1),...,(u_1,v_n),(u_2,v_1),....,(u_2,v_n),.......,(u_m,v_1),...,(u_m,v_{n-1})\}$. We consider the vertices $\big(u_{m-1},v_n\big),\big(u_m,v_1\big)$, and $\big(u_m,v_{n-1}\big)$. Let $A = \mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_1)\big)\cap S'$ and $B = \mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_{n-1})\big)\cap S'$. Hence

$$\begin{aligned}|A| &= \big|\mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_1)\big)\cap S'\big|\\ &= \Big|\Big(\big(\{u_1,...,u_{m-1}\}\times\{v_{\lceil\frac{n}{2}\rceil},...,v_n\}\big)\cup\big(\{u_m\}\times\{v_1,v_{\lceil\frac{n}{2}\rceil-1},...,v_{\lceil\frac{n}{2}\rceil}\}\big)\Big)\cap S'\Big|\\ &\geq k \text{ (This holds for } m\geq 5)\end{aligned}$$

and

$$\begin{aligned}|B| &= \big|\mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_{n-1})\big)\cap S'\big|\\ &= \Big|\Big(\big(\{u_1,...,u_{m-1}\}\times\{v_1,...,v_{\lceil\frac{n}{2}\rceil-1},v_n\}\big)\bigcup\big(\{u_m\}\times\{v_{\lceil\frac{n}{2}\rceil-1},...,v_{n-1}\}\big)\Big)\cap S'\Big|\\ &\geq k \text{ (This holds for } m\geq 6)\end{aligned}$$

We have,

$|S'| \geq |A|+|B|-(m+1)$ where $m+1$ represents the common vertices.
$|S'| \geq 2k-(m+1)$. Note that $2k-(m+1)$ is same as $2(k-1)-\left\lfloor\dfrac{k-1}{\lceil\frac{n}{2}\rceil}\right\rfloor$.

Now suppose $|S'| = 2(k-1)-\left\lfloor\dfrac{k-1}{\lceil\frac{n}{2}\rceil}\right\rfloor$. Thus, it follows, $|A| = k$ and $|B| = k$. Consider $\big(u_m,v_1\big),\big(u_1,v_n\big)\in S$ and $S = A\cup B$. Let P_1 denote all the vertices except $\big(u_m,v_1\big)$ and P_2 denote all the vertices except $\big(u_1,v_n\big)$. That is, $P_1 = \Big(\big(\{u_m\}\times\{v_2,...,v_{n-1}\}\big)\cup\big(\{u_1,...,u_{m-1}\}\times V\big)\Big)$ and $P_2 = \Big(\big(\{u_1,u_{m-1}\}\times\{v_1,...,v_{n-1}\}\big)\cup\big(\{u_2,...,u_{m-1}\}\times V\big)\Big)$.
So,

$$\big|P_1\cap S'\big| = k-1$$

and

$$\big|P_2\cap S'\big| = k-1.$$

Let us consider the vertices along the diagonal namely $(u_1,v_1),(u_2,v_2),....,(u_m,v_n)$. For every pair of these vertices, we denote the set of distinctive vertices by Q, that is, $Q = \mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)$. We note that since S' is a k-metric basis,$\big|Q\cap S'\big|\geq k$.
Thus,

$$\big|Q\cap S'\big| \leq \big|P_1\cap S'\big| = k-1$$

and

$$\left|Q \cap S'\right| \leq \left|P_2 \cap S'\right| = k-1.$$

This is a contradiction to the fact that S' is a k-metric basis. Therefore, $|S'| \neq 2(k-1) - \left\lfloor \frac{k-1}{\lceil \frac{n}{2} \rceil} \right\rfloor$. Hence $|S'| \geq 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$. Applying the same process for all other k values, we obtain similar contradictions. Thus, $dim_k(P_m \square C_n) \geq 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$. Hence, $dim_k(P_m \square C_n) = 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$.

Case 2. n is even.

We need to prove that $dim_k(G) = 2k$. We will first show that $dim_k(G) \leq 2k$. The metric basis of $P_m \square C_n$ follows an iterative process as given below:

$$S_I = \left(\{u_l\} \times \left\{v_{n-(n-1)}, v_{n-(2q-2)}, v_{n-(q-1)}\right\}\right)$$

$$S_{II} = \left(\{u_l\} \times \left\{v_{n-(n-1)}, v_{n-(2q-2)}, v_{n-(q-1)}, v_{n-(q-2)}\right\}\right)$$

To this set S_{II}, add pairs $\left\{v_{n-(2q-3)}, v_{n-(q-3)}\right\}, \ldots.., \left\{v_{n-(2q-r)}, v_{n-(q-r)}\right\}$ where $q = \frac{n}{2}$ and $r = \frac{n}{2}$.

Each pair gives a new k-resolving set. Once we reach $\{u_l\} \times V$, we move to the next l iteration. The old l value becomes the new p value.

Thus, we have

$$S_{III} = \left(\left\{u_p\right\} \times V\right) \cup \left(\{u_l\} \times \left\{v_{n-(n-1)}, v_{n-(q-1)}\right\}\right)$$

where $l = 1, 2, ..., m$ and $p = 1, 2, ..., m-1$.

To S_{III}, we add the pairs $\left\{v_{n-(2q-2)}, v_{n-(q-2)}\right\}, \ldots.., \left\{v_{n-(2q-r)}, v_{n-(q-r)}\right\}$ till $\{u_l\} \times V$ is attained. The old l value is added to p and the next l iteration is applied on S_{III}. We repeat this process till we get the full vertex set of $P_m \square C_n$. Consider the distance matrix of $P_m \square C_n$ with respect to all the bases S (given above). For each k, we see that the metric representations of each vertex is unique and each pair of vertices differ in at least k positions. Hence S is a k-resolving set. Thus, $dim_k(G) \leq 2k$.

Next we show that $dim_k(G) \geq 2k$. We will prove that no smaller k-resolving set can exist. We will prove this by method of contradiction. Let S' be a k-metric basis for $G = P_m \square C_n$ of cardinality $< 2k$, that is of cardinality $2k-2$. Then,

$S' = \{(u_1,v_1),...,(u_1,v_n),(u_2,v_1),....,(u_2,v_n),.....,(u_m,v_1),...,(u_m,v_{n-2})\}$. Let us consider the following vertices $\big(u_{m-1},v_1\big),\big(u_m,v_n\big)$, $\big(u_{m-1},v_n\big)$ and $\big(u_m,v_1\big)$. Let $A = |\mathscr{D}_G\big((u_m,v_n),(u_{m-1},v_1)\big)\cap S'|$ and $B = \big|\mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_1)\big)\cap S'\big|$. Then

$|A| = \big|\mathscr{D}_G\big((u_m,v_n),(u_{m-1},v_1)\big)\cap S'\big|$

$\quad = \Big|\big(\big(\{u_1,...,u_{m-1}\}\times\{v_1,...,v_{\lceil\frac{n}{2}\rceil}\}\big)\cup\big(\{u_m\}\times\{v_{\lceil\frac{n}{2}\rceil+1},....,v_{n-2}\}\big)\big)\cap S'\Big| \geq k-2$

and

$|B| = \big|\mathscr{D}_G\big((u_{m-1},v_n),(u_m,v_1)\big)\cap S'\big|$

$\quad = \Big|\big(\big(\{u_1,...,u_{m-1}\}\times\{v_{\lceil\frac{n}{2}\rceil+1},....,v_n\}\big)\cup\big(\{u_m\}\times\{v_1,....,v_{\lceil\frac{n}{2}\rceil}\}\big)\big)\cap S'\Big| \geq k.$

We have,

$|S'| \geq |A|+|B|$

$|S'| \geq 2k-2$

Now suppose $|S'| = 2k-2$. Thus, it follows, $|A| = k-2$ and $|B| = k$. Consider $\big(u_m,v_{\left\lceil\frac{n}{2}\right\rceil+1}\big),\big(u_1,v_n\big) \in S$ and $S = A\cup B$. Let P_1 denote all the vertices of G except $\big(u_m,v_{\left\lceil\frac{n}{2}\right\rceil+1}\big)$ and P_2 denote all the vertices of G except $\big(u_1,v_n\big)$. That is, $P_1 = \big(\big(\{u_m\}\times\{v_1,...,v_{\left\lceil\frac{n}{2}\right\rceil},v_{\left\lceil\frac{n}{2}\right\rceil+2},....,v_{n-2}\}\big)\cup\big(\{u_1,...,u_{m-1}\}\times V\big)\big)$ [This holds for $n > 6$, and when $n = 6$, P_1 is $\big(\big(\{u_m\}\times\{v_1,...,v_{\lceil\frac{n}{2}\rceil}\}\big)\cup\big(\{u_1,u_2,...,u_{m-1}\}\times V\big)\big)$]and $P_2 = \big(\big(\{u_1\}\times\{v_1,..,v_{n-1}\}\big)\cup\big(\{u_2,...,u_{m-1}\}\times V\big)\cup\big(\{u_m\}\times\{v_1,...,v_{n-2}\}\big)\big)$. So,

$$\big|P_1\cap S'\big| = k-3$$

and

$$\big|P_2\cap S'\big| = k-1.$$

Let us consider the vertices along the diagonal namely, $(u_1,v_1),(u_2,v_2),....,(u_m,v_m)$. For every pair of these vertices, let us denote the set of distinctive vertices by Q, that is, $Q = \mathscr{D}_G\big((u_1,v_1),(u_2,v_2)\big)$. We note that since S' is a k-metric basis, $|Q\cap S'| \geq k-2$ and $|Q\cap S'| \geq k$. Thus,

$$\big|Q\cap S'\big| \leq \big|P_1\cap S'\big| = k-3$$

and

$$\big|Q\cap S'\big| \leq \big|P_2\cap S'\big| = k-1.$$

This is a contradiction to the fact that S' is a k-metric basis.

Therefore, $|S'| \neq 2k-2$. That is, $|S'| \geq 2k-1$.

Suppose $|S'| = 2k-1$. Then $|A| = k-1$ and $|B| = k$. Applying the same procedure as above, we arrive at a similar contradiction. Therefore, $|S'| \neq 2k-1$. Hence we have that, $|S'| \geq 2k$.

Applying the same process as above for all other k values, we obtain similar contradictions. Thus, $dim_k(P_m \square C_n) \geq 2k$. Hence $dim_k(P_m \square C_n) = 2k$. □

3.2. Codes from stacked prisms $P_m \square C_n$.

In this section, we use the family of stacked prism graphs, that is, the cartesian product of the path graph P_m with cycle C_n, to obtain $(P_m \square C_n, k)$-codes.

The interesting application of purely graph theoretic concept of k-resolving sets comes in the form of generating codes from these. These two were nicely merged by Bailey and Yero [1], wherein they derived codes from k-resolving sets. For ready reference we list some important definitions and results that help us in getting our main results.

Definition 3.3. [1] *Let G be a graph with n vertices and diameter d, and let $S = \{v_1, v_2, ..., v_l\}$ be a k-resolving set for G of size l. Then the set $\mathscr{C}(G,S) = \{(d_G(u,v_1), d_G(u,v_2), ..., d_G(u,v_l)) : u \in V\}$ is called a (G,k)-code.*

Remark 3.4.

1. *$\mathscr{C}(G,S)$ is an error-correcting code of length l, size n and with minimum hamming distance at least k, over the alphabet $\{0, ..., d\}$, which can correct $r = \lfloor (k-1)/2 \rfloor$ errors. Also $k \geq 3$ for $r > 0$.*
2. *To get Hamming distance between codewords, label each codeword or row of the matrix A as $c_1, c_2, ...c_{rank(A)}$. Then pairwise Hamming distance is calculated and written as an array to form a new matrix $HD(\mathscr{C})$ in which the distances from c_i to all other codewords makes the i^{th} row. Therefore, $h_{ij} = d_H(c_i, c_j)$. Note that $HD(\mathscr{C})$ has order $rank(A) \times (rank(A)-1)$.*
3. *For $\mathscr{C}(G,S)$ to be used for error correction, we need a decoding algorithm see section* 3.3.
4. *Uncoverings was introduced by Bailey in* [2, 3] *where they were applied to decoding permutation codes.*

Uncovering design is defined as [1] *: Let* ν,κ,τ *be integers such that* $\nu \geq \kappa \geq \tau \geq 0$. *A* $(\nu,\nu-\kappa,\tau)$*-uncovering is a collection* $\mathscr{U}$ *of* $(\nu-\kappa)$*-subsets of* $\{1,...,\nu\}$ *with the property that any* τ*-subset of* $\{1,...,\nu\}$ *is disjoint from at least one member of* $\mathscr{U}$*. Taking the complements of each* $(\nu-\kappa)$*-subset in* $\mathscr{U}$*, we obtain a* (ν,κ,τ)*-covering design. We use covering and uncovering designs in our decoding algorithm.*

Theorem 3.5. *A* $(P_m\Box C_n,k)$*-code has mn codewords of length* $l = dim_k(G) = 2k - \left\lfloor \frac{k}{\lceil \frac{n}{2} \rceil} \right\rfloor$ *over an alphabet of size* $diam(P_m\Box C_n) = \left\lceil \frac{n}{2} \right\rceil + (m-2)$ *and with minimum distance at least k and so can correct* $r = \left\lfloor \frac{(k-1)}{2} \right\rfloor$ *errors, whenever n is odd.*

Proof. Proof follows from the above two results Theorem 3.1 and Theorem 3.2. □

We illustrate using the following example.

Example 3.6. *Consider* $P_4\Box C_3$ *as in Figure* 1.
By Theorem 3.1*, we know that* $P_4\Box C_3$ *is* $\left\lceil \frac{3}{2} \right\rceil \times 4 = 8$*-metric dimensional.*
From Theorem 3.2*,* $dim_8(G) = 16 - \left\lfloor \frac{8}{\lceil \frac{3}{2} \rceil} \right\rfloor = 12$.
The metric basis of $P_4\Box C_3$ *is as follows:*

$S_1 = \big(\{u_1\} \times \{v_1,v_2\}\big)$

$S_2 = \big(\{u_1\} \times \{v_1,v_2,v_3\}\big)$

$S_3 = \big(\{u_1\} \times V\big) \cup \big(\{u_2\} \times \{v_1,v_2\}\big)$

$S_4 = \big(\{u_1,u_4\} \times V\big)$

$S_5 = \big(\{u_1,u_2\} \times V\big) \cup \big(\{u_3\} \times \{v_1,v_2\}\big)$

$S_6 = \big(\{u_1,u_2,u_3\} \times V\big)$

$S_7 = \big(\{u_1,u_2,u_3\} \times V\big) \cup \big(\{u_4\} \times \{v_1,v_2\}\big)$

$S_8 = \big(\{u_1,u_2,u_3,u_4\} \times V\big)$

Thus, $S = (\{u_1,u_2,u_3,u_4\} \times \{v_1,v_2,v_3\})$ *is a* 8*-resolving set.*

We find the distance of each vertex in S with every vertex of $P_4\Box C_3$ *which is nothing but its* *distance matrix, as shown below. Note that each row represents the distance of one vertex to all the vertices of $P_4\square C_3$.*

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\
1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 & 3 & 4 \\
1 & 1 & 0 & 2 & 2 & 1 & 3 & 3 & 2 & 4 & 4 & 3 \\
1 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\
2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 \\
2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 1 & 3 & 3 & 2 \\
2 & 3 & 3 & 1 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 \\
3 & 2 & 3 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\
3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 1 \\
3 & 4 & 4 & 2 & 3 & 3 & 1 & 2 & 2 & 0 & 1 & 1 \\
4 & 3 & 4 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 0 & 1 \\
4 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0
\end{bmatrix}$$

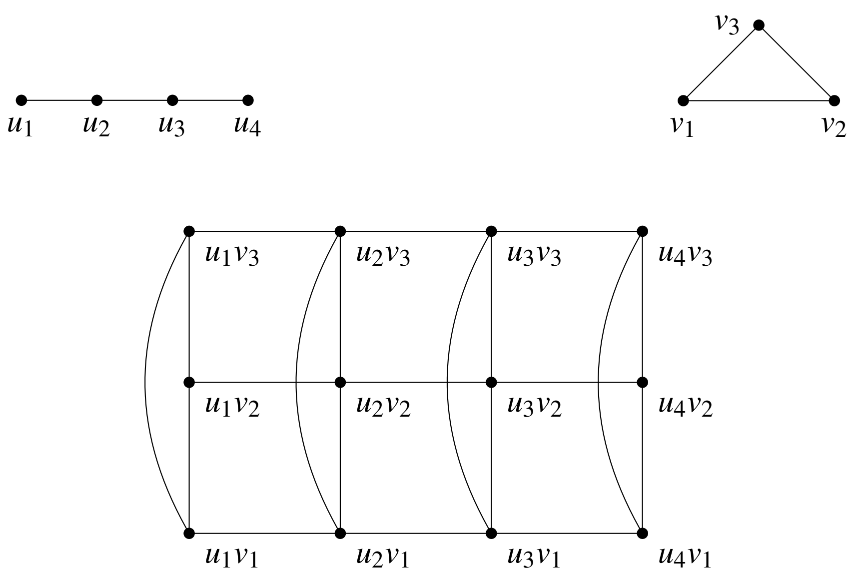

FIGURE 1. $P_4\square C_3$

Next we determine its minimum hamming distance. The hamming distance matrix $HD(\mathscr{C})$ is given below.

$$\begin{bmatrix} 8 & 8 & 12 & 8 & 8 & 9 & 11 & 11 & 12 & 10 & 10 \\ 8 & 8 & 8 & 12 & 8 & 11 & 9 & 11 & 10 & 12 & 10 \\ 8 & 8 & 8 & 8 & 12 & 11 & 11 & 9 & 10 & 10 & 12 \\ 12 & 8 & 8 & 8 & 8 & 12 & 8 & 8 & 9 & 11 & 11 \\ 8 & 12 & 8 & 8 & 8 & 8 & 12 & 8 & 11 & 9 & 11 \\ 8 & 8 & 12 & 8 & 8 & 8 & 8 & 12 & 11 & 11 & 9 \\ 9 & 11 & 11 & 12 & 8 & 8 & 8 & 8 & 12 & 8 & 8 \\ 11 & 9 & 11 & 8 & 12 & 8 & 8 & 8 & 8 & 12 & 9 \\ 11 & 11 & 9 & 8 & 8 & 12 & 8 & 8 & 8 & 8 & 12 \\ 12 & 10 & 10 & 9 & 11 & 11 & 12 & 8 & 8 & 8 & 8 \\ 10 & 12 & 10 & 11 & 9 & 11 & 8 & 12 & 8 & 8 & 8 \\ 10 & 10 & 12 & 11 & 11 & 9 & 8 & 9 & 12 & 8 & 8 \end{bmatrix}$$

We observe that the minimum hamming distance is 8*. Hence by Theorem* 3.5*, a* $(P_4\square C_3, 8)$*-code has* 12 *codewords over the alphabet* $\{0,1,2,3,4\}$ *and has minimum distance* 8*, so can correct* $r = \left\lfloor \frac{8-1}{2} \right\rfloor = 3$ *errors.*

The set of codewords obtained from distance matrix is as follows:

{011122233344, 101212323434, 110221332443, 122011122233, 212101212323, 221110221332, 233122011122, 323212101212, 332221110221, 344233122011, 434323212101, 443332221110}.

Theorem 3.7. *A* $(P_m\square C_n, k)$*-code has mn codewords of length*

$$l = dim_k(G) = \begin{cases} 2k, & \text{if } k \geq 2 \\ 2k+1, & \text{if } k = 1 \end{cases}$$

over an alphabet of size $diam\,(P_m\square C_n) = \left\lceil \frac{n}{2} \right\rceil + (m-1)$ *and with the minimum distance k and so can correct* $r = \left\lfloor \frac{(k-1)}{2} \right\rfloor$ *errors, whenever n is even.*

Proof. Proof follows from the two results Theorem 3.1 and Theorem 3.2. □

We illustrate using the following example.

Example 3.8. *Consider $P_3 \square C_6$ as in Figure 2.*
By Theorem 3.1, we know that $P_3 \square C_6$ is $\left\lceil \frac{6}{2} \right\rceil \times 3 = 9$-metric dimensional.
From Theorem 3.2, $dim_9(G) = 2(9) = 18$.
The metric basis of $P_3 \square C_6$ is as follows:

$S_1 = \big(\{u_1\} \times \{v_1, v_2, v_4\}\big)$

$S_2 = \big(\{u_1\} \times \{v_1, v_2, v_4, v_5\}\big)$

$S_3 = \big(\{u_1\} \times V\big)$

$S_4 = \big(\{u_1\} \times V\big) \cup \big(\{u_2\} \times \{v_1, v_4\}\big)$

$S_5 = \big(\{u_1\} \times V\big) \cup \big(\{u_2\} \times \{v_1, v_2, v_4, v_5\}\big)$

$S_6 = \big(\{u_1, u_2\} \times V\big)$

$S_7 = \big(\{u_1, u_2\} \times V\big) \cup \big(\{u_3\} \times \{v_1, v_4\}\big)$

$S_8 = \big(\{u_1, u_2\} \times V\big) \cup \big(\{u_3\} \times \{v_1, v_2, v_4, v_5\}\big)$

$S_9 = \big(\{u_1, u_2, u_3\} \times V\big)$

Thus, $S = (\{u_1, u_2, u_3\} \times \{v_1, v_2, v_3, v_4, v_5, v_6\})$ is a 9-resolving set.

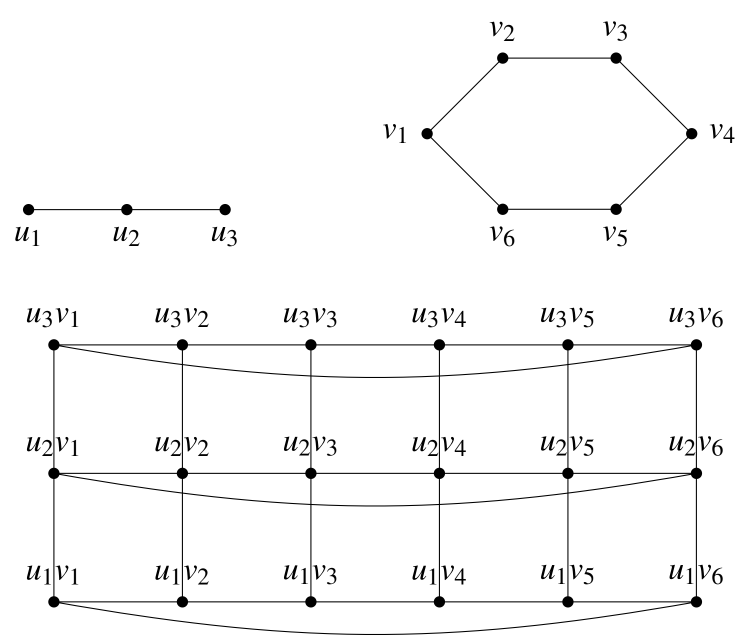

FIGURE 2. $P_3 \square C_6$

We find the distance of each vertex in S with every vertex of $P_3\square C_6$*. This is nothing but the distance matrix of* $P_3\square C_6$ *(shown below), where each row represents the distance of one vertex to all the vertices of* $P_3\square C_6$*.*

$$\begin{bmatrix}
0 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 4 & 3 & 2 & 2 & 3 & 4 & 5 & 4 & 3 \\
1 & 0 & 1 & 2 & 3 & 2 & 2 & 1 & 2 & 3 & 4 & 3 & 3 & 2 & 3 & 4 & 5 & 4 \\
2 & 1 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 3 & 4 & 5 \\
3 & 2 & 1 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 3 & 5 & 4 & 3 & 2 & 3 & 4 \\
2 & 3 & 2 & 1 & 0 & 1 & 3 & 4 & 3 & 2 & 1 & 2 & 4 & 5 & 4 & 3 & 2 & 3 \\
1 & 2 & 3 & 2 & 1 & 0 & 2 & 3 & 4 & 3 & 2 & 1 & 3 & 4 & 5 & 4 & 3 & 2 \\
1 & 2 & 3 & 4 & 3 & 2 & 0 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 4 & 3 & 2 \\
2 & 1 & 2 & 3 & 4 & 3 & 1 & 0 & 1 & 2 & 3 & 2 & 2 & 1 & 2 & 3 & 4 & 3 \\
3 & 2 & 1 & 2 & 3 & 4 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 3 \\
3 & 4 & 3 & 2 & 1 & 2 & 2 & 3 & 2 & 1 & 0 & 1 & 3 & 4 & 3 & 2 & 1 & 2 \\
2 & 3 & 4 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 0 & 2 & 3 & 4 & 3 & 2 & 1 \\
2 & 3 & 4 & 5 & 4 & 3 & 1 & 2 & 3 & 4 & 3 & 2 & 0 & 1 & 2 & 3 & 2 & 1 \\
3 & 2 & 3 & 4 & 5 & 4 & 2 & 1 & 2 & 3 & 4 & 3 & 1 & 0 & 1 & 2 & 3 & 2 \\
4 & 3 & 2 & 3 & 4 & 5 & 3 & 2 & 1 & 2 & 3 & 4 & 2 & 1 & 0 & 1 & 2 & 3 \\
5 & 4 & 3 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 2 \\
4 & 5 & 4 & 3 & 2 & 3 & 3 & 4 & 3 & 2 & 1 & 2 & 2 & 3 & 2 & 1 & 0 & 1 \\
3 & 4 & 5 & 4 & 3 & 2 & 2 & 3 & 4 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 0
\end{bmatrix}$$

Next we determine the minimum hamming distance for these. The hamming distance matrix $HD(\mathscr{C})$ *is given below.*

$$\begin{bmatrix}
18 & 12 & 18 & 12 & 18 & 18 & 9 & 18 & 12 & 18 & 9 & 12 & 18 & 12 & 18 & 12 & 18 \\
18 & 18 & 12 & 18 & 12 & 9 & 18 & 9 & 18 & 12 & 18 & 18 & 12 & 18 & 12 & 18 & 12 \\
12 & 18 & 18 & 12 & 18 & 18 & 9 & 18 & 9 & 18 & 12 & 12 & 18 & 12 & 18 & 12 & 18 \\
18 & 12 & 18 & 18 & 12 & 12 & 18 & 9 & 18 & 9 & 18 & 18 & 12 & 18 & 12 & 18 & 12 \\
12 & 18 & 12 & 18 & 18 & 18 & 12 & 18 & 9 & 18 & 9 & 12 & 18 & 12 & 18 & 12 & 18 \\
18 & 12 & 18 & 12 & 18 & 9 & 18 & 12 & 18 & 9 & 18 & 18 & 12 & 18 & 12 & 18 & 12 \\
18 & 9 & 18 & 12 & 18 & 9 & 18 & 12 & 18 & 12 & 18 & 18 & 9 & 18 & 12 & 18 & 9 \\
9 & 18 & 9 & 18 & 12 & 18 & 18 & 18 & 12 & 18 & 12 & 9 & 18 & 9 & 18 & 12 & 18 \\
18 & 9 & 18 & 9 & 18 & 12 & 12 & 18 & 18 & 12 & 18 & 18 & 9 & 18 & 9 & 18 & 12 \\
12 & 18 & 9 & 18 & 9 & 18 & 18 & 12 & 18 & 18 & 12 & 12 & 18 & 9 & 18 & 9 & 18 \\
18 & 12 & 18 & 9 & 18 & 9 & 12 & 18 & 12 & 18 & 18 & 18 & 12 & 18 & 9 & 18 & 9 \\
9 & 18 & 12 & 18 & 9 & 18 & 18 & 12 & 18 & 12 & 18 & 9 & 18 & 12 & 18 & 9 & 18 \\
12 & 18 & 12 & 18 & 12 & 18 & 18 & 9 & 18 & 12 & 18 & 9 & 18 & 12 & 18 & 12 & 18 \\
18 & 12 & 18 & 12 & 18 & 12 & 9 & 18 & 9 & 18 & 12 & 18 & 18 & 18 & 12 & 18 & 12 \\
12 & 18 & 12 & 18 & 12 & 18 & 18 & 9 & 18 & 9 & 18 & 12 & 12 & 18 & 18 & 12 & 18 \\
18 & 12 & 18 & 12 & 18 & 12 & 12 & 18 & 9 & 18 & 9 & 18 & 18 & 12 & 18 & 18 & 12 \\
12 & 18 & 12 & 18 & 12 & 18 & 18 & 12 & 18 & 9 & 18 & 9 & 12 & 18 & 12 & 18 & 18 \\
18 & 12 & 18 & 12 & 18 & 12 & 9 & 18 & 12 & 18 & 9 & 18 & 18 & 12 & 18 & 12 & 18
\end{bmatrix}$$

We observe that minimum hamming distance is 9*. Hence by Theorem* 3.7*, a* $(P_3\Box C_6, 9)$*-code has* 18 *codewords over the alphabet* $\{0,1,2,3,4,5\}$ *and has minimum distance* 9*, so can correct* $r = \left\lfloor \frac{9-1}{2} \right\rfloor = 4$ *errors.*

The set of codewords obtained from distance matrix is as follows:

{012321123432234543, 101232212343323454, 210123321234432345, 321012432123543234, 232101343212454323, 123210234321345432, 123432012321123432, 212343101232212343, 321234210123321234, 432123321012432123, 343212232101343212, 234321123210234321, 234543123432012321, 323454212343101232, 432345321234210123, 543234432123321012, 454323343212232101, 345432234321123210}.

In the next section, we obtain the decoding algorithm for $(P_m \Box C_n, k)$-codes.

3.3. Decoding Algorithm. In this section we give a decoding algorithm which is a modification of the decoding algorithm given by Bailey and Yero [1].

Our decoding algorithm is as follows: Suppose we have received the word $x = x_1, x_2, ..., x_{dim_k(P_m \Box C_n)}$ of length $dim_k(P_m \Box C_n)$. Consider a $(P_m \Box C_n, k)$-code having mn codewords of length $l = dim_k(P_m \Box C_n)$ with error-correction capability $r = \lfloor (k-1)/2 \rfloor$. We wish to correct r errors. Let $r' = 1 < r$. We determine the covering design $(dim_k(P_m \Box C_n), r+1, r' = 1)$. To obtain this, we partition $dim_k(P_m \Box C_n) = a(r+1) + b$ into p subsets including as many as possible sets of size t where $t = \lfloor r+1/r'+1 \rfloor = \lfloor r+1/2 \rfloor$ and $p = \lceil v/t \rceil$. For any combination r' of the sets in the partition $dim_k(P_m \Box C_n)$, take $r+1$ subsets of points which contain their union. Further, these $r+1$ subsets of points are chosen in such way that they are disjoint from other $r+1$ subsets of points(as much as possible). These form the blocks of $(dim_k(P_m \Box C_n), r+1, r' = 1)$. Taking the complement of these blocks we get $(dim_k(P_m \Box C_n), dim_k(P_m \Box C_n) - r - 1, r' = 1)$-uncovering design.

Next we partition the received word $x = x_1, x_2, ..., x_{dim_k(P_m \Box C_n)}$ and the blocks of the uncovering design into $r+1$ sets. We now compare each of $r+1$ bit of the received word with the distance matrix of $(P_m \Box C_n, k)$ at positions indexed by all the blocks of the uncovering design such that atleast r elements must be identical to the elements of x. The row belonging to $(P_m \Box C_n, k)$ having most number of common entries from all the blocks of the uncovering design is the original transmitted word. Thus on comparing the original received word with the common row (that we have obtained) we will obtain the error positions, which can easily be rectified.

Remark 3.9. *Our decoding is better than the decoding proposed by Bailey and Yero* [1]. *In our decoding we fix $r' = 1$. We consider covering designs $(dim_k(P_m \Box C_n), r+1, r')$ instead of $(dim_k(P_m \Box C_n), r, r')$ where $r' < r$. Also we consider uncovering designs $(dim_k(P_m \Box C_n), dim_k(P_m \Box C_n) - r - 1, r')$ instead of $(dim_k(P_m \Box C_n), dim_k(P_m \Box C_n) - r, r')$. We observe that in our decoding the number of blocks of uncovering design reduces. Further instead of checking each column corresponding to each symbol of the received word, we check* *only $r+1$ words such that atleast r words match with the codeword. This reduces the computation time.*

3.4. Complexity. Let $\mathscr{U}$ denote the uncovering design $(dim_k(P_m\Box C_n), dim_k(P_m\Box C_n) - r - 1, r^{'})$ and let the total number of blocks of $\mathscr{U}$ be denoted by T. For each block of $\mathscr{U}$, the distance matrix of $P_m\Box C_n$ is examined at most $|T|-1$ times and this repeats for every block of $\mathscr{U}$. Since we have T blocks, it follows that the complexity of the decoding algorithm is $O(|T|-1\cdot|T|)$.

4. Conclusion

The k-metric dimension is an extension of the classical metric dimension. While determining the k-metric dimension of any arbitrary graph is NP-hard, we have tried to solve the open case for Stacked prism graphs.

In this paper we have studied the k-metric dimension of a stacked prism $P_m\Box C_n$. An explicit formula for $dim_k(P_m\Box C_n)$ is determined and the codes arising from k-resolving sets of the stacked prisms $P_m\Box C_n$ are developed. Decoding in terms of covering and uncovering designs are given. In future communications this procedure is extended to obtain codes from other (Cartesian) products of some standard classes of graphs such as $K_n\Box P_m$ and Rook's graphs.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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