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THE STUDY OF RAY-KNIGHT COMPACTIFICATION ON TRANSFER FUNCTION

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Abstract: In order to construct a Markov chain of strong Markov, we need the state space of the compactification. The paper uses the properties of the resolvent operator to study some properties of Ray-Knight compactifications from a given transfer function on state space E.

Key words: transfer function; Ray resolvent; Ray-Knight Compactifications

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1 Introduction

Construction of the transfer function of the given E is an important part of research of Markov chain. The state space in the canonical chain on $E \bigcup \infty$ is very simple, but its orbit has only lower semicontinuity, which just keep part of the strong Markov. In order to construct a Markov chain of strong Markov, the State space of the compactification is needed. We will study the properties of Ray-Knight Compactifications from a given transfer function on state space E.

2 Preliminary knowledge

Let $E = \{1, 2, \cdots\}$, and the topology on E is discrete topology, then E is

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L.C.C.B(Locally compact and has a countable topological group). The elements of E are called state. ε is a Borel algebra on E, (E,ε) is a topological space, and $C_b(E)$ represents all bounded continuous functions. If E is a compact metric space, $C_b(E)$ abbreviated $C(E)$. Remove the dense subset $\{g_m\}_{m=1}^{\infty}$ of $\Re$, set $d(x,y)=\sum_{m=1}^{\infty}\frac{1}{2^m}\Lambda\left|g_m(x)-g_m(y)\right|,\forall x,y\in E$, $d(\cdot,\cdot)$ is the metric on E, completion of E under $d(\cdot,\cdot)$ written $\overline{E}$. It is obvious that $\overline{E}$ is a compact metric space. The metric on $\overline{E}$ still denoted as $d(\cdot,\cdot)$. $(U^{\alpha})_{\alpha>0}$ is the Ray resolvent operator on $\overline{E}$, D is a no branch point set.

Definition 1: $(\overline{E},d)$ is called Ray-Knight compactification on E.

Definition2: If $p_{ij}(t)\geq 0;\sum_{k=1}^{\infty}p_{ik}(t)\leq 1;p_{ij}(t+s)=\sum_{k=1}^{\infty}p_{ik}(t)p_{kj}(s)$ establish ,then the function $P(t)=(p_{ij}(t))_{i,j\in E}$ on $[0,\infty)$ is called the transfer function on E.

If $\lim_{t\to 0}p_{ij}(t)=p_{ij}(0)=\delta_{ij}$,then $P(t)(t\geq 0)$ is called the standard. If $\forall i\in E,\sum_{k=1}^{\infty}p_{ik}(t)=1$, Then $P(t)(t\geq 0)$ is said to be honest.

Definition 3:If $0\leq q_{ij}<\infty,0\leq q_i\equiv -q_{ii}\leq\infty,\sum_{k\neq i}q_{ik}\leq q_i,\forall i,j\in E$,then $Q=(q_{ij})_{i,j\in E}$ is called density matrix of $P(t)(t\geq 0)$.If $q_i<\infty$,then i is called stable state of $P(t)$ .Otherwise i is called instantaneous state of $P(t)$ .If $\sum_{k\neq i}q_{ik}=q_i$,then i is called conservative state of $P(t)$. If all States are stable, then $P(t)$ is said to be fully stabilized. If all States are conservative, then $P(t)$ is said to be fully conservative.

Definition 4: Let $P(t)(t \ge 0)$ is the transfer function on E, $R_{ij}(\lambda)$ is called resolvent of $P(t)(t \ge 0)$.Among then $R_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt, i, j \in E, \lambda > 0$.

Lemma 1: $R_{ij}(\lambda)$ is the resolvent of transfer function $P(t)(t \ge 0)$,if and only if $\lambda \sum_{k \in E} R_{ik}(\lambda) \le 1; R_{ij}(\lambda) - R_{ij}(\mu) + (\lambda - \mu) \sum_{k \in E} R_{ik}(\lambda) R_{kj}(\mu) = 0$;

$\lim_{\lambda \to \infty} \lambda R_{ij}(\lambda) = \delta_{ij}; \lim_{\lambda \to \infty} \lambda[\lambda R_{ij}(\lambda) - \delta_{ij}] = q_{ij}$.

Proof: Reference[2]

Lemma 2 If $Q = (q_{ij})_{i,j \in E}$ is density matrix of $P(t)(t \ge 0)$,and $P(t)$ is fully stabilized,then

$$p_{ij}^{/} \ge -q_i p_{ij}(t) + \sum_{k \ne i} q_{ik} p_{kj}(t)$$
$$p_{ij}^{/} \ge -p_{ij}(t) q_j + \sum_{k \ne j} p_{ik}(t) q_{kj}$$

Proof: From the definition of density matrix and the whole stability can be directly obtained the conclusion.

Note1 If the two formulas in lemma 2 an equality, we get two group of linear differential equation group, Are called backward equations and the forward equations.

Lemma 3 Let $Q = (q_{ij})_{i,j \in E}$ is the whole stability density matrix of $P(t)(t \ge 0)$, set

$$f_{ij}^{(0)}(t) = \delta_{ij} e^{-q_j t}, f_{ij}^{(n)}(t) = \sum_{k \ne i} \int_0^t e^{-q_i s} q_{ik} f_{kj}^{(n-1)}(t-s) ds, n = 1, 2, \cdots$$
$$f_{ij}^{(0)}(t) = \delta_{ij} e^{-q_i t}, f_{ij}^{(n)}(t) = \sum_{k \ne j} \int_0^t f_{ik}^{(n-1)}(s) q_{kj} e^{-q_j (t-s)} ds, n = 1, 2, \cdots$$

$$p_{ij}^{\min}(t) = \sum_{n=0}^{\infty} f_{ij}^{(n)}(t)$$

Then (1) $P^{\min}(t)=(p_{ij}^{\min}(t))_{i,j\in E}$ is transfer function on E,its density matrix is Q.

(2) $P^{\min}(t)$ is minimum, that is $p_{ij}(t)\geq p_{ij}^{\min}(t),\forall i,j\in E,t\geq 0$.

(3) $P^{\min}(t)$ satisfy the forward equations and backward equations.

Proof: Reference[1]

Note2: $P^{\min}(t)$ is called the minimum transfer function.

3 Main results

If the transfer function $P(t)$ isnot honest, set $\Delta\notin E$,and

$$E_{\Delta}=E\bigcup\{\Delta\},p_{\Delta\Delta}(t)=1,p_{\Delta i}(t)=0,p_{i\Delta}=1-\sum_{k\in E}p_{ik}(t),\forall i\in E$$

Then $P(t)=(p_{ij}(t))_{i,j\in E_{\Delta}}$ is the honest transfer function on E_{Δ},so the $P(t)$ can be transformed into $P(t)$ to discuss.

Let $P(t)=(p_{ij}(t))_{i,j\in E}$ is the honest transfer function on E. $\forall f\in M$,the function $i\mapsto\sum_{k\in E}R_{ik}(\lambda)f(k)$ on E is noted $R_{\lambda}f$.

Theorem 1: $(R_{\lambda})_{\lambda>0}$ is the Markov resolvent on E, and have:

(1)E is L.C.C.B topological space.

(2) $\{R_{\lambda}\}_{\lambda>0}$ is Markov,and $R_{\lambda}C_b(E)\subset C_b(E),\forall\lambda>0$.

(3)For any $f\in C_b(E)$,and $x\in E,\lim_{\lambda\to\infty}\lambda R_{\lambda}f(x)=f(x)$

Proof:(1)and (2) is obvious.we proof (3):

$$\lambda R_{\lambda}f(i)=\lambda\sum_{k\in E}R_{ik}(\lambda)f(k)=\lambda\int_0^{\infty}e^{-\lambda t}\sum_{k\in E}p_{ik}(t)f(k)dt$$
$$=\lambda\int_0^{\infty}e^{-\lambda t}p_{ii}(t)f(i)dt+\lambda\int_0^{\infty}e^{-\lambda t}\sum_{k\neq i}p_{ik}(t)f(k)dt$$

Because $\lim_{t\to 0}p_{ii}(t)=1$, when $\lambda\to\infty$, $\lambda\int_0^{\infty}e^{-\lambda t}p_{ii}(t)f(i)dt\to f(i)$;

$$\lambda\int_0^{\infty}e^{-\lambda t}\sum_{k\neq i}p_{ik}(t)f(k)dt\leq\|f\|\lambda\int_0^{\infty}e^{-\lambda t}(1-p_{ii}(t))dt\to 0$$

So $\lim_{\lambda\to\infty}\lambda R_\lambda f(i)=f(i)$。

Theorem 2: Let $(\overline{E},d(\cdot,\cdot))$ is Ray-Knight compactifications on E，then：

（1）$E\subseteq D$

（2）For any $i\in E,\alpha>0,U^\alpha(i,\overline{E}\setminus E)=0$，and for any $k\in E,$

$$U^\alpha(i,\{k\})=R_{ik}(\alpha)$$

(3)For any $i\in E,t>0,\ P_t(i,\overline{E}\setminus E)=0$，and for any $k\in E$，

$$P_t(i,\{k\})=p_{ik}(t)$$

Proof：（1）For any $i\in E,\alpha>0,f,g\in\Re$，because $U^\alpha f,U^\alpha g$ are $R_\alpha f,R_\alpha g$ expansion in the E，so

$$U^\alpha(f-g)(i)=R_\alpha f(i)-R_\alpha g(i)=\sum_{k\in E}R_{ik}(\alpha)[f(k)-g(k)],$$

For any $f\in C(\overline{E})$，$U^\alpha f(i)=\sum_{k\in E}R_{ik}(\alpha)f(k)=R_\alpha f(i)$，because R_α meet three conclusions of the theorem 1.so we have $\lim_{\alpha\to\infty}\alpha U^\alpha f(i)=f(i)$，so that $i\in D$。For i is optional nature，we have $E\subseteq D$.

（2）For bounded measurable function f on any $\overline{E}$,it is obvious:

$$U^\alpha f(i)=R_\alpha f(i)=\sum_{k\in E}R_{ik}(\alpha)f(k)$$

For any $k\in E$，Use the characteristic function $I_k(\cdot)$ of $\{k\}$ instead of f,we get $U^\alpha(i,\{k\})=R_{ik}(\alpha)$. Both ends of the sum of k,we get

$$U^\alpha(i,E)=\sum_{k\in E}U^\alpha(i,\{k\})=\sum_{k\in E}R_{ik}(\alpha)=\frac{1}{\alpha}=U^\alpha(i,\overline{E})$$

So that $U^{\alpha}(i,\overline{E}\setminus E)=0$.

(3) Beacuse $(U^{\alpha})_{\alpha>0}$ is Ray resolvent on $\overline{E}$， so for any $i,k\in E, f\in C(\overline{E}),\alpha>0$，

$$\int_0^{\infty} e^{-\alpha t}P_t f(i)dt = U^{\alpha} f(i) = \sum_{k\in E} R_{ik}(\alpha) f(k) = \int_0^{\infty} e^{-\alpha t}\sum_{k\in E} p_{ik}(t) f(k)dt$$

For any $t_0>0,t>0,h>0$，

$$\begin{aligned}
&\left|\sum_{k\in E} p_{ik}(t_0+t+h) f(k) - \sum_{k\in E} p_{ik}(t_0+t) f(k)\right| \\
&\le \|f\|\sum_{k\in E}\left|p_{ik}(t_0+t+h) - p_{ik}(t_0+t)\right| \\
&= \|f\|[\sum_{k\in E}\left|\sum_{m\in E} p_{im}(t_0)(\sum_{l\in E} p_{ml}(h)p_{lk}(t) - p_{mk}(t))\right|] \\
&\le \|f\|[\sum_{k\in E}\sum_{m\in E} p_{im}(t_0)\left|p_{mm}(h)-1\right| p_{mk}(t)] + \|f\|[\sum_{k\in E}\sum_{m\in E}\sum_{l\in E} p_{im}(t_0)p_{ml}(h)p_{lk}(t)] \\
&= 2\|f\|[\sum_{m\in E} p_{im}(t_0)[1-p_{mm}(h)]]
\end{aligned}$$

By the $p_{ij}(t)$ standard and control convergence theorem [5] to：

$$\lim_{h\to 0}[\sum_{k\in E} p_{ik}(t_0+t+h) f(k) - \sum_{k\in E} p_{ik}(t_0+t) f(k)] = 0$$

That is $t\mapsto \sum_{k\in E} p_{ik}(t) f(k)$ is continuous function on $(0,\infty)$， For α is optional nature，we have $:\forall t>0$， $\int_{\overline{E}} P_t(i,dy) f(y) = P_t f(i) = \sum_{k\in E} p_{ik}(t) f(k)$.By the monotone class theorem [5]， The formula for arbitrary bounded measurable function on $\overline{E}$ is also established.

$\forall k\in E$， let $f(\cdot)=I_k(\cdot)$， substitution $U^{\alpha} f(i) = \sum_{k\in E} R_{ik}(\alpha) f(k) = R_{\alpha} f(i)$，

we get $P_t(i,\{k\}) = p_{ik}(t)$， On both sides of the k sum,then $\sum_{k\in E} P_t(i,\{k\}) = 1$， so $P_t(i,\overline{E}\setminus E)=0$。 □

Note3： Let $E_R=\{x\,|\,x\in\overline{E}, U^1(x,E)=1\}$， it is clear that E_R is Borel subset on $\overline{E}$.

Theorem3: Let $x \in E_R$, then $\forall t > 0$, $P_t(x, \overline{E}) = P_t(x, E)$, and $\forall s \geq 0, k \in E,$

$$P_{t+s}(x, \{k\}) = \sum_{m \in E} P_t(x, \{m\}) p_{mk}(s) .$$

Proof: because $\forall t > 0, P_t(x, E) \leq P_t(x, \overline{E}) = 1$, so

$$U^1(x, E) = \int_0^{\infty} e^{-t} P_t(x, E) dt \leq \int_0^{\infty} e^{-t} P_t(x, \overline{E}) dt = 1$$

By $x \in E_R$, $1 = \int_0^{\infty} e^{-t} P_t(x, E) dt = \int_0^{\infty} e^{-t} P_t(x, \overline{E}) dt$, so $P_t(x, E) = P_t(x, \overline{E}) = 1$, then $\exists \{t_n\}_{n=1}^{\infty}$, so that when $t \to 0$ and $\forall t_n$, $P_{t_n}(x, E) = 1$.For any $t > 0$, Let $t_n < t$, by The properties [6] of semigroup of $(P_s)_{s \geq 0}$: $P_t(x, E) = \int_{\overline{E}} P_{t_n}(x, dy) P_{t-t_n}(y, E)$

$$= \int_E P_{t_n}(x, dy) P_{t-t_n}(y, E) = \sum_{k \in E} P_{t_n}(x, \{k\}) P_{t-t_n}(k, E)$$

$$= P_{t_n}(x, E) = 1$$

We get : $P_t(x, E) = P_t(x, \overline{E})$. $\forall t > 0, s \geq 0, k \in E$:

$$P_{t+s}(x, \{k\}) = \int_{\overline{E}} P_t(x, dy) P_s(y, \{k\}) = \int_E P_t(x, dy) P_s(y, \{k\}) = \sum_{m \in E} P_t(x, \{m\}) p_{mk}(s)$$

$\forall$ $x \in E_R, k \in E, t > 0, P_t(x, \{k\})$ is denoted $p_{xk}(t)$ 。

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