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COMPLEXITY OF STAR (m, n)-GON AND OTHER RELATED GRAPHS

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Abstract. In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we derive simple formulas of the complexity, number of spanning trees, of Star (m, n)-gon and other related Graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Keywords: Number of spanning trees; Star (m, n)-gon ; Book graphs; Chebyshev polynomials.

Mathematics Subject Classification: 05C05, 05C50.

1. INTRODUCTION

In this work we deal with simple and finite undirected graphs $G=(V,E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff, [17] can be used to determine the number of spanning trees for $G=(V,E)$. Let $V=\{v_1,v_2,.....,v_n\}$, then the Kirchhoff matrix H defined as $n\times n$ characteristic matrix $H=D-A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H=[a_{ij}]$ defined as follows: (i)

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$a_{ij} = -1\, v_i$ and v_j are adjacent and $i \neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i = j$, and (iii) $a_{ij} = 0$ otherwise. All of co-factors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq .... \geq \mu_p$ denote the eignvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [16] shown that, $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d - \mu_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, ..., \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}, n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1}, p, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [4, 19]. Another result is due to Sedlacek [20] who derived a formula for the wheel on $n+1$ vertices, W_{n+1} , he showed that $\tau(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2$, for $n \geq 3$. Sedlacek [21] also later derived a formula for the number of spanning trees in a Mobius ladder, M_n , $\tau(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et .al. [1,2].
Daoud et. al.,[5-15] later derived formulas for the number of spanning trees for many graphs.

Now, we can introduce the following lemma:

<u>Lemma 1.1</u> [5]

$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A})$ where $\bar{A}$, $\bar{D}$ are the adjacency and degree matrices of $\bar{G}$, the complement of G , respectively, and I is the $n \times n$ unit matrix.

The advantage of this formula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations.

We begin from their definitions,Yuanping, et. al. [22].

Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix}$$, where all other elements are zeros.

Further we recall that the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos x) \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{3}$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), n \geq 1 \tag{4}$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}], n \geq 1 \tag{5}$$

Where the identity is true for all complex x (except at $x = \pm 1$ where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\prod_{j=1}^{n-1}(x - \cos\frac{j\pi}{n}) \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1} U_{n-1}(x) \qquad (7)$$

These two results yield another formula for $U_n(x)$,

$$U_{n-1}^{\ 2}(x) = 4^{n-1} \prod_{j=1}^{n-1} (x^2 - \cos^2 \frac{j\pi}{n}) \qquad (8)$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}^{\ 2}(\sqrt{\frac{x+2}{4}}) = \prod_{j=1}^{n-1} (x - 2\cos \frac{2j\pi}{n}) \qquad (9)$$

Furthermore one can show that

$$U_{n-1}^{\ 2}(x) = \frac{1}{2(1-x^2)}[1 - T_{2n}] = \frac{1}{2(1-x^2)}[1 - T_n(2x^2 - 1)], \qquad (10)$$

and

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]. \qquad (11)$$

Now let $B_n(x)$, $C_n(x)$, $D_n(x)$ and $E_n(x)$ be $n \times n$ matrices

<u>Lemma 2.1</u>,[9]

$$(i)\ B_n(x) = \begin{pmatrix} x & -1 & 0 & & \\ -1 & x+1 & -1 & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & -1 & x+1 & -1 \\ & & 0 & -1 & x \end{pmatrix} \Rightarrow \det(B_n(x)) = (x-1) U_{n-1}(\frac{1+x}{2}).$$

$$(ii)\ C_n(x) = \begin{pmatrix} x & 0 & 1 & & \\ 0 & x+1 & 0 & \ddots & \\ 1 & 0 & \ddots & \ddots & 1 \\ & \ddots & \ddots & x+1 & 0 \\ & & 1 & 0 & x \end{pmatrix} \Rightarrow \det(C_n(x)) = (n + x - 2) U_{n-1}(\frac{x}{2}), n \geq 3, x > 2.$$

$$(iii)\ D_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 0 \\ 0 & x & 0 & \ddots & 1 \\ 1 & 0 & x & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 0 & x \end{pmatrix} \Rightarrow \det(D_n(x)) = \frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2}) - 1], \quad n \geq 3, x \geq 3.,$$

$$(iv)\ \ E_n(x)=\begin{pmatrix} x & 1 & 1 & \cdots & \cdots & 1 \\ 1 & x & 1 & \ddots & & \vdots \\ 1 & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & 1 & 1 & x \end{pmatrix} \Rightarrow \det(E_n(x))=(x+n-1)(x-1)^{n-1}$$

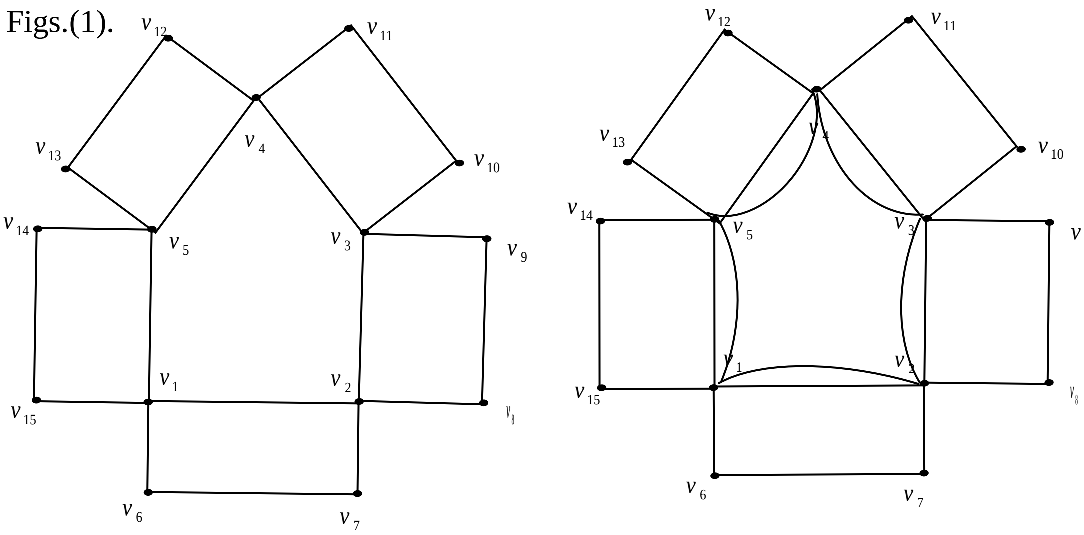
Lemma 2.2[18]:Let $A\in F^{n\times n}, B\in F^{n\times m}, C\in F^{m\times n}$ and $D\in F^{m\times m}$ and assume that D is nonsingular. Then: $\det\begin{pmatrix} A & B \\ C & D\end{pmatrix}=(-1)^{nm}\det(A-BD^{-1}C)\det D.$

This formula gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

3. COMPLEXITY OF STAR(n, m)- GON

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this section we derive the explicit formulas of the number of spanning trees of the new graph" star (m ,n) –gon".

Let m, $m,n\geq 3$.The star (m ,n)-gon , $S_n^{(m)}$ (resp.$\overline{S}_n^{(m)}$),is constructed from the cycle C_n (resp. cycle C_n with double edges) by adjoining the end vertices of the path P_{m-2} to the consecutive vertices of the cycle C_n (resp. cycle C_n with double edges) such that each of the end vertices of the path is connected to exactly one vertex of the cycle (resp. cycle C_n with double edges), so we obtain C_m as a subgraph of the resulting .See Figs.(1).



$S_n^{(m)}$ Fig(1) $\overline{S}_n^{(m)}$

Theorem3.1 $\tau(S_n^{(m)}) = n(m-1)m^{n-1}$

Proof:

Applying lemma (1.1), we have:

$$\tau(S_n^{(m)}) = \frac{1}{(n(m-1))^2}\det((n(m-1))I - \bar{D} + \bar{A})$$

$$\tau(S_n^{(m)}) = \frac{1}{(n(m-1))^2}\det\begin{pmatrix}
5 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & \ddots & \ddots & \ddots & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
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\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 3 & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 4 & \ddots & 1 \\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 4 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 3
\end{pmatrix}$$

$$= \frac{1}{(n(m-1))^2}\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \frac{1}{(n(m-1))^2}\det(A - BC^{-1}B^T)\det C = \frac{1}{(n(m-1))^2}\det(C - B^T A^{-1} B)\det A$$

Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial in lemma (2.1) , we have:

$$\tau(S_n^{(m)}) = n(m-1)m^{n-1}$$

Theorem 3.2

$$\tau(\bar{S}_n^{(m)}) = n(m-1)(2m-1)^{n-1}$$

Proof: Applying lemma (1.1)we have:

$$\tau(\bar{S}_n^{(m)}) = \frac{1}{(n(m-1))^2}\det((n(m-1))I - \bar{D} + \bar{A})$$

$$\tau(\overline{S}_n^{(m)})=\frac{1}{(n(m-1))^2}\det\begin{pmatrix}
7 & -1 & 1 & \cdots & -1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & \ddots & \ddots & \ddots & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
1 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
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-1 & 1 & \cdots & -1 & 7 & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
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\vdots & 1 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & 1 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & 0 & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & 1 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 3 & 1 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 3 & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 4 & \ddots & 1 \\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 4 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 3
\end{pmatrix}$$

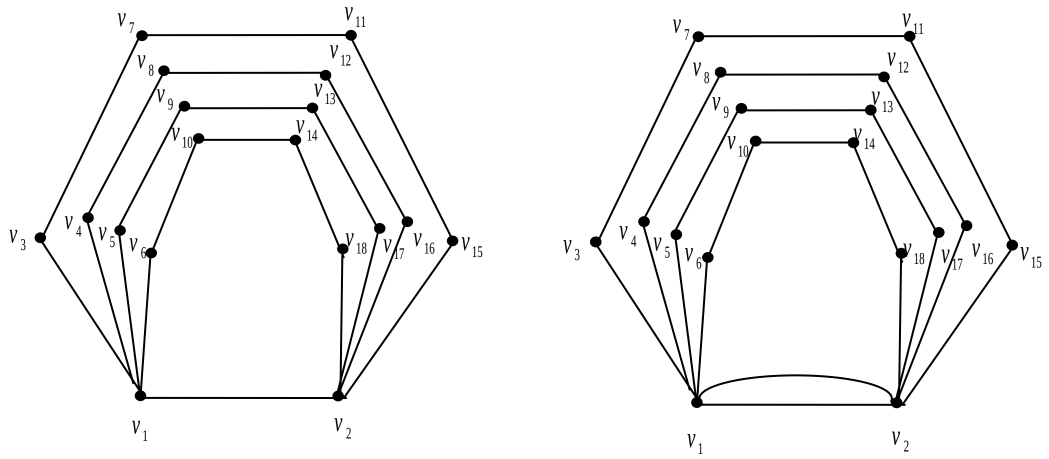
$$=\frac{1}{(n(m-1))^2}\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}=\frac{1}{(n(m-1))^2}\det(A-BC^{-1}B^T)\det C=\frac{1}{(n(m-1))^2}\det(C-B^TA^{-1}B)\det A$$

Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial in lemma (2.1) , we have:

$$\tau(\overline{S}_n^{(m)})=n(m-1)(2m-1)^{n-1}.$$

4. COMPLEXITY OF BOOK WITH DIFFERENT PAGES GRAPH

The book graph, $S_2\times S_n$,is a connected graph obtained by adding ' n ' number of C_4 with one edge. It has $2n$ vertices and $3n-2$ edges. In this section we derive the explicit formulas of the number of spanning trees of some graphs extension to the book graph (book graph with pages C_k , instead of C_4 only which obtained by adding 'n' number of C_k with one edge, $B_n^{(k)}$ (resp. double edges , $\overline{B}_n^{(k)}$)).See Fig.(2).



$B_4^{(6)}$ Fig(2) $\overline{B}_4^{(6)}$

<u>Lemma4.1</u>

Let C_n and C_m be two cycles have an edge in common ($C_n \vee C_m$), then

$$\tau(C_n \vee C_m) = nm - 1$$

<u>Theorem 4.2</u>

$$\tau(B_n^{(k)}) = (n + k - 1)(k - 1)^{n-1},$$

Where $n = \#$pages , $k = \#$vertices in the page.

<u>Proof</u>

Applying lemma (1.1), we have:

$$\tau(B_n^{(k)}) = \frac{1}{(n(m-1))^2} \det((n(m-1))I - \bar{D} + \bar{A})$$

$$\tau(B_n^{(k)})=\frac{1}{(n(k-2)+2)^2}\det\begin{pmatrix}
n+2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & n+2 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & 3 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \vdots & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 \\
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\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\
\vdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

$$=\frac{1}{(n(k-2)+2)^2}\det\begin{pmatrix} A & B \\ B^T & C\end{pmatrix}=\frac{1}{(n(k-2)+2)^2}\det(A-BC^{-1}B^T)\det C=\frac{1}{(n(k-2)+2)^2}\det(C-B^TA^{-1}B)\det A$$

Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial in lemma(2.1), we have:

$$\tau(B_n^{(k)})=(n+k-1)(k-1)^{n-1}$$

Lemma 4.3

Let C_n and C_m be two cycles have two vertices without edge in common ($C_n \wedge C_m$), then:

$$\tau(C_n \wedge C_m)=(2n-1)m-n$$

Theorem 4.4

$$\tau(\overline{B}_n^{(k)})=(n+k+1)(k-1)^{n-1},$$

Where $n=\#$pages $,k=\#$vertices in the page

Proof Applying lemma (1.1), we have:

$$\tau(\overline{B}_n^{(k)})=\frac{1}{(n(m-1))^2}\det((n(m-1))I-\overline{D}+\overline{A})$$

$$\tau(\overline{B}_n^{(k)}) = \frac{1}{(n(k-2)+2)^2} \det \begin{pmatrix}
n+3 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & n+3 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 3 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \vdots & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\
\vdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

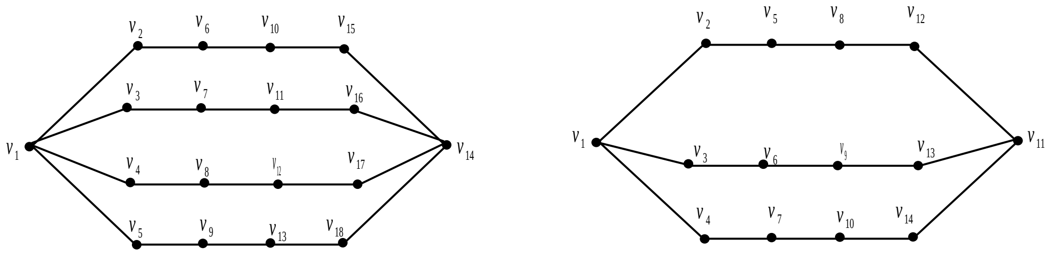
$$= \frac{1}{(n(k-2)+2)^2} \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \frac{1}{(n(k-2)+2)^2} \det(A - BC^{-1}B^T) \det C = \frac{1}{(n(k-2)+2)^2} \det(C - B^T A^{-1} B) \det A$$

Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial in lemma(2.1), we have**:**

$$\tau(\overline{B}_n^{(k)}) = (n+k+1)(k-1)^{n-1}$$

5. COMPLEXITY OF GATGAT GRAPH

The gadget graph, $G_n^{(k)}$, is a connected graph with $n(k-2)+2$ and $n(k-1)$ edges obtained by adjoining number n of k -pathsfrom their end points, See Fig.(3).



$G_4^{(6)}$ Fig(3) $G_3^{(6)}$

Theorem 5.1 $\tau(G_n^{(k)}) = n(k-1)^{n-1}$

Proof: Applying lemma (1.1), we have:

$$\tau(G_n^{(k)}) = \frac{1}{(n(k-2)+2)^2}\det(n(k-2)+2)I - \bar{D} + \bar{A})$$

$$= \frac{1}{(n(k-2)+2)^2}\det\begin{pmatrix}
n & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & 3 & 1 & \cdots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
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\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
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\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & n & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & 3 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

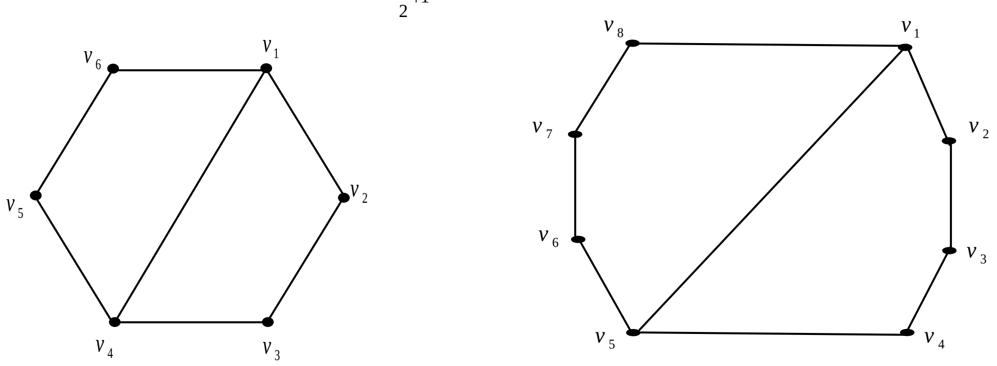
$$= \frac{1}{(n(k-2)+2)^2}\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \frac{1}{(n(k-2)+2)^2}\det(A - BC^{-1}B^T)\det C = \frac{1}{(n(k-2)+2)^2}\det(C - B^T A^{-1} B)\det A$$

Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial in lemma(2.1) , we have:

$$\tau(G_n^{(k)}) = n(k-1)^{n-1}$$

6. COMPLEXITY OF DIAMOND GRAPH

The diamond graph, D_n, is a connected graph with n and $n+1$ edges obtained by even n cycle with chord from v_1 to $v_{\frac{n}{2}+1}$. See Fig.(4).



D_6 Fig(4) D_8

Theorem 6.1 $\tau(D_n) = \frac{n}{4}(n+4)$

Proof:

Applying lemma (1.1), we have:

$$\tau(D_n) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A})$$

$$= \frac{1}{n^2}\det\begin{pmatrix} 4 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 \\ 0 & 3 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & 3 & 0 & 1 & \cdots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 4 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 3 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 3 \end{pmatrix} = \frac{1}{n^2}\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

$$= \frac{1}{n^2}\det(A - BC^{-1}B^T)\det C = \frac{1}{n^2}\det(C - B^T A^{-1} B)\det A$$

$$= \frac{1}{n^2}\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \frac{1}{n^2}\det(A - BC^{-1}B^T)\det C = \frac{1}{n^2}\det(C - B^T A^{-1} B)\det A = \frac{n}{4}(n+4)$$

7. CONCLUSION

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and lemmas and their proofs.

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