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MULTIPLE-SET SPLIT FEASIBILITY PROBLEMS FOR ASYMPTOTICALLY PSEUDO-NONSPREADING MAPPINGS IN HILBERT SPACES

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Abstract. We introduce a new class of asymptotically pseudononspreading mappings and demonstrate its relationship with the existing related families of mappings. Demiclosedness principle for the mappings as well as the convexity and closedness of its fixed point set are established. Furthermore, we propose and investigate a new iterative algorithm for solving multiple-set split feasibility problem for the new class of mappings. In particular, weak and strong convergence theorems for solving multiple-set split feasibility problem for a certain subclass of the new class of mappings in Hilbert spaces are proved and certain applications are given. The results presented in the paper extend and improve the results of Osilike and Isiogugu [1], Quan and Chang [2] and host of other corresponding related results in literature.

Keywords: multiple-set split feasibility; asymptotically pseudononspreading mappings; fixed point set; demiclosedness principle; weak convergence; strong convergence.

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1. Introduction

The split feasibility problem (SFP) in finite-dimensional spaces was first introduced by Censor and Elfving [3] for modeling inverse problems which arise from phase retrievals and in medical

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image reconstruction, see [4] and [3]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning (see [5] and references therein]. The multiple-set split feasibility problem (MSSFP) was studied in ([2] and references therein].

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $\{C_1, \cdots, C_N\}$ and $\{Q_1, \cdots, Q_M\}$ are nonempty closed convex subsets of H_1 and H_2, respectively. Then, the multiple set split feasibility problem is

$$to\ find\ x^* \in \bigcap_{i=1}^{N} C_i\ such\ that\ Ax^* \in \bigcap_{j=1}^{M} Q_j \qquad (1.1).$$

In the sequel, we use Γ to denote the set of solutions of the problem $(MSSFP)$ (1.1), that is,

$$\Gamma = \left\{ x \in \bigcap_{i=1}^{N} C_i : Ax \in \bigcap_{j=1}^{M} Q_j \right\}.$$

It is worthy of note that MSSFP (1.1) becomes Split Feasibility Problem (SFP) when $N = M = 1$. In this paper, we propose and investigate a new iterative algorithm for solving multiple-set SFP $(MSSFP)$ problem for a new class of asymptotically pseudononspreading mapping. Also, we prove the convergence of the presented algorithm as well as some salient properties of the fixed point set of the mapping under study.

2. PRELIMINARIES

Here, we recall some relevant definitions and lemmas which will be needed in the proof of our main result. In what shall follow, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$" respectively, the fixed point set of a mapping T by $F(T)$ and the solution set of (1.1) by Γ.

In what follows, H will denote a real Hilbert soace and C a nonempty subset of H.

Defiinition 2.1 [Demiclosedness principle] If $T : H \to H$ is a mapping, then $(I - T)$ is said to be demiclosed at zero if for any sequence, $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $(I - T)x_n \to 0$, we have $x^* = Tx^*$.

Defiinition 2.2 A single valued mapping $T : C \to C$ is said to be *semicompact*, if for any bounded sequence $\{x_n\} \subset C$ with $\|(I - T)x_n\| \to 0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in C$.

Defiinition 2.3 A mapping $T : C \to C$ is said to be asymptotically nonexpansive, see [6], if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \ \forall\, x, y \in C \textit{ and all } n \geq 1 \qquad (2.1).$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings.

Defiinition 2.4 A mapping $T : C \to C$ is said to be asymptotically pseudocontractive, see [7] if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \|x - T^n x - (y - T^n y)\|^2 \ \forall\, x, y \in C \textit{ and all } n \geq 1 \qquad (2.2).$$

It is clear that every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the converse is false. See Roades[8]

Defiinition 2.5 (see [9],[10], [11] and [12]) In 2008, Kohsaka and Takahashi [11] introduced nonspreading mapping, and obtained a fixed point theorem for a single nonspreading mapping, and a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2 \ \forall\, x, y \in C \qquad (2.3).$$

Inequality (2.3) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \ \forall\, x, y \in C,$$

and the class of nonspreading mappings contains the class of firmly nonexpansive mappings (i.e., mappings $T : C \to C$ satisfying $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$, $\forall\, x, y \in C$).

Defiinition 2.6 (see [7] A mapping $T : C \to C$ is said to be uniformly $L-$Lipschitzian if there exists a constant $L \geq 0$ such that for all $(x,y) \in H \times H$,

$$\|T^n x – T^n y\| \leq L\|x-y\|. \tag{2.4}$$

Defiinition 2.7 A mapping $T : C \to C$ is said to be pseudo- nonspreading if

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x-(I-T)y\|^2 + 2\langle x-Tx, y-Ty\rangle \ \forall \ x,y \in C, \qquad (2.5).$$

(See Osilike *et al* [13])

Defiinition 2.8 In 2011, Osilike and Isiogugu [1] introduced κ-strictly pseudo-nonspreading mappings in Hilbert spaces, and obtained weak mean convergence theorem of Baillon's type for such mappings. They presented it in the nomenclature of *nonspreading-type mapping* and showed that the class of nonspreading mappings is properly contained in the class of strictly pseudononspreading mappings. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be κ-strictly pseudo-nonspreading if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle \ \forall x,y \in C \qquad (2.6).$$

Defiinition 2.9 Later in 2014, Quan and Chang [2] introduced the class of κ-asymptotically strictly pseudo-nonspreading mappings in Hilbert space which contains the class of κ-strictly pseudo-nonspreading mappings. They studied multiple-set plit feasibility problems for the mappings. Let H be a real Hilbert space with $C \subset H$ being nonemptsy then a mapping $T : C \to C$ is said to be κ-asymptotically strictly pseudo-nonspreading if there exists a constant $\kappa \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 + k\|x-T^n x-(y-T^n y)\|^2 + 2\langle x-T^n x, y-T^n y\rangle \ \forall x,y \in C. \tag{2.7}$$

As far as we know, nothing has been reported about asymptotically pseudo-nonspreading mappings. Thus the following questions arise naturally.

question 1: Can one introduce the concept of asymptotically pseudo-nonspreading mappings using terminologies that are in conformity with the existing standard?

question 2: Can one display illustrative examples to show the relationship existing among the mapping and already existing related mappings?

question 3: Can one establish demiclosedness principle for the mapping as well as the convexity and closedness of its fixed point set?

question 4: Can one establish the solution of multiple-set split feasibility problem for the mapping?

question 5: Can one show some possible applications of such solutions established?

question 6: Is there any numerical example to demonstrate such solutions established?

Defiinition 2.10 Inspired and motivated by the above innovations as well as the questions, we introduce in this paper, a new family of mappings which is more general than the class of κ-asymptotically strictly pseudo-nonspreading mappings in Hilbert space. Illustrative examples given herein show that our new class of mappings is independent on the closely related class of asymptotically pseudo-contractive mappings.

Defiinition 2.11 Opial property, see [13]. A Banach space E is said to have the Opial property if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$, we have

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - x^*\| \tag{2.8}$$

for all $y \in E$ with $y \neq x^*$. It is known that each Hilbert space possess the Opial property.

Lemma 2.12 see Chidume and Ofoedu[14]. Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following relations $\alpha_{n+1} \leq (1+c_n)\alpha_n + \sigma_n, n \geq 0$, such that $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then, $\{\alpha_n\}$ is convergent.
If in addition that $\{\alpha_n\}$ has a subsequence, $\{\alpha_{n_k}\}$ that converges to 0, then $\{\alpha_n\}$ converges to 0 as $n \to \infty$.

Lemma 2.13 see [13]. Let H be a real Hilbert space. Then, the following results hold:

(i) For all $x, y \in H$ and for all $t \in [0,1]$, we have $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.

(ii) $\|x+y\|^2 \leq \|x\| + 2\langle y, x+y\rangle$.

(iii) If $\{x_n\}_{n=n}^{\infty}$ is a sequence in H which converges weakly to $z \in H$, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2.$$

3. Main Results

Defiinition 3.1 A mapping $T : C \to C$ is said to be asymptotically pseudo-nonspreading if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle \ \forall x,y \in C \quad (3.1),$$

or equivalently, a mapping $T : C \subset H \to C$ is said to be asymptotically pseudo-nonspreading if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x-y\|^2 + \|x - T^n x\|^2 + \|y - T^n y\|^2 \ \forall x,y \in C \qquad (3.2).$$

Remark 3.2 From, (2.1). (2.7) and (3.2), it can easily be deduced that the class of asymptotically nonexpansive mappings and the class of κ-asymptotically strictly pseudo-nonspreading are proper subclasses of the class of asymptotically pseudononspreading mappings.

Example 3.3 An example of an asymptotically pseudononspreading mapping which is not κ-asymptotically strictly pseudo-nonspreading. Let $C = \left[\frac{-3}{2}, \frac{3}{2}\right]$; $C_1 = \{x \in C : |x| < \frac{1}{6}\}$; $C_2 = \{x \in C : \frac{1}{6} \leq |x| \leq \frac{1}{2}$ and $C_3 = \{x \in C : \frac{1}{2} < |x| \leq \frac{3}{2}$. Define $T : C \to C$ by

$$Tx = \begin{cases} x, \ if \ x \in C_1; \\ -3x, \ if \ x \in C_2; \\ -\frac{1}{3}x, \ if \ x \in C_3. \end{cases} \qquad (3.3)$$

Case 1 Suppose that $x, y \in C_1$ then, $\forall n \in \mathbb{N}$, $T^n x = x$ while, $T^n y = y$. Hence,

$$\|T^n x - T^n y\|^2 = \|x-y\|^2 \leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.$$

Case 2 Suppose that $x, y \in C_2$, then, we show that T is asymptotically pseudononspreading as follows. If n is odd, then, $T^n x = -3x \in C_3$ while, $T^n y = -3y \in C_3$. Hence,

$$\|T^n x - T^n y\|^2 = 9\|x-y\|^2 \leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.$$

If n is even then, $T^n x = x \in C_2$ while, $T^n y = y \in C_2$,. and it follows from case 1.

Case 3: If $x, y \in C_3$ and n is odd. Then, we show that T is asymptotically pseudononspreading as follows.

$$\|T^n x - T^n y\|^2 = \frac{1}{9}\|x-y\|^2 \leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.$$

If $x, y \in C_3$ and n is even, then, $T^n x = x \in C_3$ while, $T^n y = y \in C_3$, and it follows from case 1.

Case 4 Suppose that $x \in C_1$ and $y \in C_2$ then, it is easy to see that $y^2 = \|y\|^2 > \|xy\| \geq xy$. Thus, $y^2 - xy > 0$. If n is odd, then, $T^n x = x \in C_1$ while, $T^n y = -3y \in C_3$. Hence,

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|x + 3y\|^2 \\
&= \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle - 8(y^2 - xy) \\
&\leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.
\end{aligned}$$

If n is even then, $T^n x = x \in C_1$ while, $T^n y = y \in C_2$, and it follows from case 1.

Case 5 Suppose that $x \in C_1$ and $y \in C_3$ then, it is easy to see that $y^2 = \|y\|^2 > \|xy\| \geq xy$. Thus, $y^2 - xy > 0$. If n is odd, then, $T^n x = x \in C_1$ while, $T^n y = -\frac{1}{3}y \in C_2$. Hence,

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|x + \frac{1}{3}y\|^2 \\
&= \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle - \frac{8}{3}(y^2 - xy) \\
&\leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.
\end{aligned}$$

If n is even then, $T^n x = x \in C_1$ while, $T^n y = y \in C_3$, and it follows from case 1.

Case 6: Suppose that $x \in C_2$, $y \in C_3$ and n is odd. Then, $T^n x = -3x \in C_3$ while, $T^n y = -\frac{1}{3}y \in C_2$. Hence,

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|3x - \frac{1}{3}y\|^2 \\
&= \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle - 8(x^2 + \frac{y^2}{3}) \\
&\leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.
\end{aligned}$$

If n is even, then, $T^n x = x \in C_2$ while, $T^n y = y \in C_3$, and it follows as in case 1. We thus conclude that for all $x, y \in C$" we have

$$\|T^n x - T^n y\|^2 \leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.$$

The mapping T is not λ-strictly asymptotically pseudononspreading because, if $x = \frac{1}{2}$, $y = -\frac{1}{2}$ and n odd, then $T^n x = -\frac{3}{2}$ and $T^n y = \frac{3}{2}$. We have that for an arbitrary $\{\kappa_n\}$ such that $\lim_{n\to\infty} \kappa_n = 1$ and $\lambda \in [0,1)$, take $\varepsilon = 1 - \lambda$. Since $\kappa_n \to 1$, it implies that there exist $n_\varepsilon \in \mathbb{N}$ such that $\kappa_n < \varepsilon + 1 \; \forall \; n \geq n_\varepsilon$. Hence, $\forall \; n \geq n_\varepsilon$, it follows that

$\|T^n x - T^n y\|^2 = 9,\ \kappa_n \|x-y\|^2 < (1+\varepsilon),$
$\lambda \|x - T^n x - (y - T^n y)\|^2 = 16\lambda,\ 2\langle x - T^n x, y - T^n y\rangle = -8$, and

$$\kappa_n \|x-y\|^2 + \lambda_n \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle \le -6 + 15\lambda < 9 = \|T^n x - T^n y\|^2.$$

Next, we demonstrate that the family of asymptotically pseudocontractive mappings and the family of asymptotically pseudononspreading are independent even when their fixed point sets are nonempty.

Example 3.4 Example of an asymptotically pseudocontractive mapping which is not asymptotically pseudononspreading. Let us consider a bijective linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\bar{x} = A\bar{x},\ \forall\ \bar{x} \in \mathbb{R}^2,\ where\ A = \begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix}. \tag{3.6}$$

Then, $F(T) \neq \emptyset$ since $(0,0) \in F(T)$.

<u>Proof</u> Observe that $T^n\bar{x} = A^n\bar{x}$ and $A^2 = -I, A^3 = -A, A^4 = I, A^5 = A, A^6 = A^2, A^7 = A^3, \ldots A^n = A^{n-4}$, $n \ge 5$. Thus we need only prove that

$$\|T^n\bar{x} - T^n\bar{y}\|^2 \le \|\bar{x} - \bar{y}\|^2 + \|\bar{x} - T^n\bar{x} - (\bar{y} - T^n\bar{y})\|^2,\ \forall\ \bar{x}, \bar{y} \in \mathbb{R}^2,\ \text{for } n = 1,2,3,4.$$

For $n = 1$ let $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in \mathbb{R}^2$ be arbitrary. Then,
$T\bar{x} = ((-1+\sqrt{2})x_2, (-1-\sqrt{2})x_1)^T$ and $T\bar{y} = ((-1+\sqrt{2})y_2, (-1-\sqrt{2})y_1)^T$. Hence,

$$\begin{aligned}
& \|\bar{x} - \bar{y}\|^2 + \|\bar{x} - T\bar{x} - (\bar{y} - T\bar{y})\|^2 \\
= \ & \|T\bar{x} - T\bar{y}\|^2 + 2[\|\bar{x} - \bar{y}\|^2 - \langle \bar{x} - \bar{y}, T(\bar{x} - \bar{y})\rangle] \\
= \ & \|T\bar{x} - T\bar{y}\|^2 + 2[\|(x_1 - y_1, x_2 - y_2)\|^2 - \langle (x_1 - y_1, x_2 - y_2), T(x_1 - y_1, x_2 - y_2)\rangle] \\
= \ & \|T\bar{x} - T\bar{y}\|^2 + 2[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 \\
& -[(-1+\sqrt{2})(x_1 - y_1)(x_2 - y_2) + ((-1-\sqrt{2})(x_1 - y_1)(x_2 - y_2))]] \\
= \ & \|T\bar{x} - T\bar{y}\|^2 + 2[(x_1 - y_1) + (x_2 - y_2)]^2 \\
\ge \ & \|T\bar{x} - T\bar{y}\|^2.
\end{aligned}$$

This shows that $\|\bar{x} - \bar{y}\|^2 + \|\bar{x} - T\bar{x} - (\bar{y} - T\bar{y})\|^2 \ge \|T\bar{x} - T\bar{y}\|^2$.

For $n = 2$ let $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in \mathbb{R}^2$ be arbitrary. Then, $T^2\bar{x} = -\bar{x}$ and $T^2\bar{y} = -\bar{y}$.

Hence,

$$\|T^n\bar{x}-T^n\bar{y}\|^2 = |\bar{x}-\bar{y}\|^2 \leq |\bar{x}-\bar{y}\|^2 + \|\bar{x}-T^n\bar{x}-(\bar{y}-T^n\bar{y})\|^2,\ \forall\ \bar{x},\bar{y}\in\mathbb{R}^2.$$

For $n=3$ let $\bar{x}=(x_1,x_2),\bar{y}=(y_1,y_2)\in\mathbb{R}^2$ be arbitrary. Then, $T^3\bar{x}=((1-\sqrt{2})x_2,(1+\sqrt{2})x_1)^T$ and $=T^3\bar{y}=((1-\sqrt{2})y_2,(1+\sqrt{2})y_1)^T$. Hence,

$$\begin{aligned}
& \|\bar{x}-\bar{y}\|^2+\|\bar{x}-T\bar{x}-(\bar{y}-T\bar{y})\|^2 \\
= \;& \|T\bar{x}-T\bar{y}\|^2+2[\|\bar{x}-\bar{y}\|^2-\langle\bar{x}-\bar{y},T(\bar{x}-\bar{y})\rangle] \\
= \;& \|T\bar{x}-T\bar{y}\|^2+2[\|(x_1-y_1,x_2-y_2)\|^2-\langle(x_1-y_1,x_2-y_2),T(x_1-y_1,x_2-y_2)\rangle] \\
= \;& \|T\bar{x}-T\bar{y}\|^2+2[\|x_1-y_1\|^2+\|x_2-y_2\|^2 \\
& -[(1-\sqrt{2})(x_1-y_1)(x_2-y_2)+((1+\sqrt{2})(x_1-y_1)(x_2-y_2))]] \\
= \;& \|T\bar{x}-T\bar{y}\|^2+2[(x_1-y_1)-(x_2-y_2)]^2 \\
\geq \;& \|T\bar{x}-T\bar{y}\|^2.
\end{aligned}$$

For $n=4$ let $\bar{x}=(x_1,x_2),\bar{y}=(y_1,y_2)\in\mathbb{R}^2$ be arbitrary. Then, $T^4\bar{x}=\bar{x}$ and $T^4\bar{y}=\bar{y}$. Hence,

$$\|T^n\bar{x}-T^n\bar{y}\|^2 = \|\bar{x}-\bar{y}\|^2 \leq |\bar{x}-\bar{y}\|^2 + \|\bar{x}-T^n\bar{x}-(\bar{y}-T^n\bar{y})\|^2,\ \forall\ \bar{x},\bar{y}\in\mathbb{R}^2.$$

However, the mapping T is not asymptotically pseudononspreading because, for $\bar{x}=(11,0)$ and $\bar{y}=(-11,0)$ with an arbitrary $\kappa_n\subseteq[1,\infty)$ and $\varepsilon=\frac{1}{10}$, we have that $\kappa_n\to 1$ implies that there exist $n_\varepsilon\in\mathbb{N}$ such that $\kappa_n<\varepsilon+1\ \forall n\geq n_\varepsilon$. Hence, for $n\in\{4n-3:n\in\mathbb{N}\}$, with $n\geq n_\varepsilon$ then, $\|T^n\bar{x}-T^n\bar{y}\|^2=\|(0,11+11\sqrt{2})-(0,-11-11\sqrt{2})\|^2=1452+968\sqrt{2}$, $\kappa_n\|\bar{x}-\bar{y}\|^2=\kappa_n\|\,(11,0)-(-11,0)\,\|^2<(1+\varepsilon)\,(484)=\left(\frac{2662}{5}\right)$; $\|\bar{x}-T^n\bar{x}\|^2=\|(11,0)-(0,11+11\sqrt{2})\|^2=484+242\sqrt{2}$; $\|\bar{y}-T^n\bar{y}\|^2=\|(-11,0)-(0,-11-11\sqrt{2})\|^2=484+242\sqrt{2}$. Hence,

$$\begin{aligned}
\kappa_n\|\bar{x}-\bar{y}\|^2+\|\bar{x}-T^n\bar{x}\|^2+\|\bar{y}-T^n\bar{y}\|^2 &< \frac{2662}{5}+(484+242\sqrt{2})+(484+242\sqrt{2}) \\
&= \frac{7502}{5}+484\sqrt{2} \\
&< 1452+968\sqrt{2} \\
&= \|T^nx-T^ny\|^2
\end{aligned}$$

Example 3.5 Example of an asymptotically pseudononspreading mapping which is not asymptotically pseudocontractive. Let $C = \left[\frac{1}{3}, \frac{1+\sqrt{5}}{2}\right]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{1-x}, x \in \left[\frac{1}{3}, \frac{3-\sqrt{5}}{2}\right] = C_1; \\ x, x \in \left(\frac{3-\sqrt{5}}{2}, \frac{3}{2}\right) = C_2; \\ \frac{x-1}{x}, x \in \left[\frac{3}{2}, \frac{1+\sqrt{5}}{2}\right] = C_3; \end{cases} \tag{3.7}$$

Then, $F(T) \neq \emptyset$ since $x \in F(T)\ \forall\ x \in C_2$. T is asymptotically pseudononspreading but T is not asymptotically pseudocontractive.

Proof

Case 1: Suppose that $x, y \in C_1$, then, $T(x) = \frac{1}{1-x} \in C_3$. Whereas, $T^2(x) = x \in C_1$. Hence, for each odd number, n and each $x, y \in C_1$, we have that $T^n(x) = \frac{1}{1-x}$ and $T^n(y) = \frac{1}{1-y}$. Thus, taking note of the fact that $x + y < 1\ \forall\ x, y \in C_1$, we have that

$$\begin{aligned} \|x-y\|^2 + \|x - T^n x\|^2 + \|y - T^n y\|^2 &= \|x-y\|^2 + \left\|x - \frac{1}{1-x}\right\|^2 + \left\|y - \frac{1}{1-y}\right\|^2 \\ &\geq x^2 - \frac{2x}{1-x} + \frac{1}{(1-x)^2} + y^2 - \frac{2y}{1-y} + \frac{1}{(1-y)^2} \\ &> \frac{1}{(1-x)^2} + \frac{1}{(1-y)^2} - 2\left(\frac{x}{1-x} + \frac{y}{1-y}\right) \\ &= \|T^n x - T^n y\|^2 + 2\left(\frac{2xy + 1 - (x+y)}{(1-x)(1-y)}\right) \\ &> \|T^n x - T^n y\|^2. \end{aligned}$$

On the other hand, for each even number, n and each $x, y \in C_1$, we have that $T^n(x) = x$ and $T^n(y) = y$. Hence,

$$\|T^n x - T^n y\|^2 = \|x-y\|^2 \leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle.$$

Case 2: Suppose that $x, y \in C_2$, then, $T^n(x) = x, T^n(y) = y\ \forall\ n \geq 1$, and the result follows as in case 1 when n is even.

Case 3: Suppose that $x, y \in C_3$, then, $T(x) = \frac{x-1}{x} \in C_1$. Whereas, $T^2(x) = x \in C_3$. Hence, for each odd number, n and each $x, y \in C_3$, we have that $T^n(x) = \frac{x-1}{x}$ and $T^n(y) = \frac{y-1}{y}$. Thus, we have that

$$
\begin{aligned}
\|T^n x - T^n y\|^2 &= \left\| \frac{x-1}{x} - \frac{y-1}{y} \right\|^2 \\
&= \left(\frac{1}{xy} \right)^2 \|x-y\|^2
\end{aligned}
$$

Since $x^2 - x - 1 \leq 0$, $\forall x \in C_3$, then, we have that

$$
\begin{aligned}
2\langle x - T^n x, y - T^n y \rangle &= 2\left\langle x - \frac{x-1}{x}, y - \frac{y-1}{y} \right\rangle \\
&= 2\left\langle \frac{x^2-x-1}{x}, \frac{y^2-y-1}{y} \right\rangle \\
&= \frac{2}{xy}\left\langle x^2-x-1, y^2-y-1 \right\rangle \\
&\geq 0
\end{aligned}
$$

On its own, $2 < xy$, $\forall x, y \in C_3$. Hence, $\left(1 - \frac{2}{xy}\right) > 0$ and so we have that

$$
\begin{aligned}
\|x - T^n x - (y - T^n y)\|^2 &= \left\| x - \frac{x-1}{x} - \left(y - \frac{y-1}{y} \right) \right\|^2 \\
&= \left\| (x-y) - \left(\frac{x-y}{xy} \right) \right\|^2 \\
&= \left(1 - \frac{1}{xy} \right)^2 \|x-y\|^2 \\
&= \left(1 - \frac{2}{xy} \right) \|x-y\|^2 + \left(\frac{1}{xy} \right)^2 \|x-y\|^2
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|T^n x - T^n y\|^2 &= \left(\frac{1}{xy} \right)^2 \|x-y\|^2 \\
&\leq \|x-y\|^2 + \left(1 - \frac{2}{xy} \right) \|x-y\|^2 + \left(\frac{1}{xy} \right)^2 \|x-y\|^2 \\
&= \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 \\
&\leq \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y \rangle
\end{aligned}
$$

On the other hand, for each even number, n and each $x, y \in C_3$, we have that $T^n(x) = x$ and $T^n(y) = y$, and the result follows as in Case 1 when n is even.

Case 4: Suppose that $x \in C_1$ and $y \in C_2$, then, $T(x) = \frac{1}{1-x} \in C_3$; $T^2(x) = x \in C_1$ while, $T^n(y) =$

y, $\forall\ y \in C_2$ and $n \in \mathbb{N}$. Hence, for each odd number, $n \geq 1$, $x \in C_1$ and $y \in C_2$, we have that $T^n(x) = \frac{1}{1-x} \in C_3$ and $T^n(y) = y \in C_2$. Thus, $0 < x < y < T^n(x)$ which in turn implies that

$$-y < -x$$

Hence, on addition of $T^n(x)$ to both sides we have that

$$0 < T^n(x) - y < T^n(x) - x.$$

It follows that $\|T^n(x) - y\| < \|T^n(x) - x\|$. Hence,

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|T^n(x) - y\|^2 \\
&< \|T^n(x) - x\|^2 \\
&\leq \|x - y\|^2 + \|x - T^n x\|^2 + \|y - T^n y\|^2
\end{aligned}$$

On the other hand, for each even number, n and each $x \in C_1$ and $y \in C_2$, we have that $T^n(x) = x$ and $T^n(y) = y$, and the result follows as in Case 1 when n is even.

Case 5: Suppose that $x \in C_1$, $y \in C_3$, and n is odd, then $T^n(x) = \frac{1}{1-x} \in C_3$ and $T^n(y) = \frac{y-1}{y} \in C_1$.

$$\begin{aligned}
\|x - y\|^2 + \|x - T^n x\|^2 + \|y - T^n y\|^2 &\geq 3 \times min\{\|x - y\|^2, \|x - T^n x\|^2, \|y - T^n y\|^2\} \\
&\geq 3 \times \left(\frac{3}{2} - \frac{3-\sqrt{5}}{2}\right)^2 \\
&= \frac{15}{4} \\
&> \frac{46 + 6\sqrt{5}}{36} \\
&= \left(\frac{1+\sqrt{5}}{2} - \frac{1}{3}\right)^2 \\
&\geq \|T^n x - T^n y\|^2
\end{aligned}$$

Assuming that $x \in C_1$, $y \in C_3$ and n is even, we have that $T^n(x) = x \in C_1$ and $T^n(y) = y \in C_3$, and the result follows as in Case 1 when n is even.

Case 6: Suppose that $x \in C_2$ and $y \in C_3$, then, $T(y) = \frac{y-1}{y} \in C_1$ and $T^2(y) = y \in C_3$. While, $T^n(x) = x$, $\forall x \in C_2$ and $n \in \mathbb{N}$. Hence, for each odd number, $n \geq 1$, $x \in C_2$ and $y \in C_3$, we have that $T^n(x) = x \in C_2$ and $T^n(y) = \frac{y-1}{y} \in C_1$. Thus, $0 < T^n(y) < x < y$ which in turn implies that

$$x < y.$$

Hence, on addition of $-T^n(y)$ to both sides we have that

$$0 < x - T^n(y) < y - T^n(y).$$

It follows that $\|x - T^n(y)\| < \|y - T^n(y)\|$. Hence,

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|x - T^n(y)\|^2 \\
&< \|y - T^n(y)\|^2 \\
&\leq \|x - y\|^2 + \|x - T^n x\|^2 + \|y - T^n y\|^2
\end{aligned}$$

On the other hand, for each even number, n and each $x \in C_2$ and $y \in C_3$, we have that $T^n(x) = x$ and $T^n(y) = y$, and the result follows as in Case 1 when n is even.

However, T is not asymptotically pseudocontractive because for $x = \frac{1}{3}, y = \frac{38}{100} \in D(T)$, we have that for $\varepsilon = 0.1$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\kappa_{n_\varepsilon} < 1 + \varepsilon$. Hence, if $n \geq n_\varepsilon$ is odd then

$$\begin{aligned}
\|x - T^n x - (y - T^n y)\|^2 + \kappa_n \|x - y\|^2 &< \left[\left(\frac{3}{2} + 2\right) - \left(\frac{16}{10} + \frac{10}{6}\right)\right]^2 + (1+\varepsilon)\left[\frac{3}{2} - \frac{16}{10}\right]^2 \\
&= \left[\frac{45}{30} + \frac{60}{30} - \frac{48}{30} - \frac{50}{30}\right]^2 + (1+\varepsilon)\left[\frac{15}{10} - \frac{16}{10}\right]^2 \\
&= \left[\frac{7}{30}\right]^2 + (1+\varepsilon)\left[\frac{1}{10}\right]^2 \\
&= \frac{29}{450} + \frac{\varepsilon}{100} \\
&= \|T^n x - T^n y\|^2
\end{aligned}$$

Lemma 3.6 Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly $L-$ Lipschitzean asymptotically pseudo-nonspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.

Proof. Suppose that $F(T) = \emptyset$, then its closedness is trivially assured. Let $\{p_n\}_{n \geq 1} \subset F(T)$ such that $p_n \to p$ be arbitrary. We prove that $p \in F(T)$.

$$\|T^n p - p\| = \|T^n p - T^n p_n + T^n p_n - p\|$$

$$
\begin{aligned}
&\leq \|T^n p - T^n p_n\| + \|T^n p_n - p\| \\
&\leq L\|p_n - p\| + \|p_n - p\| \\
&= (1+L)\|p_n - p\| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus, $p \in F(T)$ and $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Suppose that $F(T) = \emptyset$ or singleton, then its convexity is trivially assured. If $F(T)$ is neither empty nor singleton, then define $p = \lambda p_1 + (1-\lambda)p_2$ where $p_1, p_2 \in F(T)$ and $\lambda \in [0,1]$ are arbitrary. We show that $p \in F(T)$. To this end, we set $G_{n_\beta}(x) := T^n[(1-\beta)x + \beta T^n x]$ where $\beta \in \left(0, \frac{1}{\sqrt{L^2+1}+1}\right)$. Clearly,

$$
G_{n_\beta}(p_1) = p_1;\ G_{n_\beta}(p_2) = p_2;\ \|p - p_1\| = (1-\lambda)\|p_1 - p_2\| \text{ and } \|p - p_2\| = \lambda\|p_1 - p_2\|. \tag{3.8}
$$

Observe that,

$$
\begin{aligned}
\|G_{n_\beta}p - p\|^2 &= \|G_{n_\beta}p - \lambda p_1 + (1-\lambda)p_2\|^2 \\
&= \|\lambda(p_1 - G_{n_\beta}p) + (1-\lambda)(p_2 - G_{n_\beta}p)\|^2 \\
&= \lambda\|p_1 - G_{n_\beta}p\|^2 + (1-\lambda)\|p_2 - G_{n_\beta}p\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2 \qquad (3.9)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\|G_{n_\beta}p - p_1\|^2 \\
&= \|T^n[(1-\beta)p + \beta T^n p] - T^n p_1\|^2 \\
&\leq \kappa_n\|(1-\beta)p + \beta T^n p - p_1\|^2 + \|(1-\beta)p + \beta T^n p - G_{n_\beta}p - (p_1 - T^n p_1)\|^2 \\
&\quad 2\langle (1-\beta)p + \beta T^n p - G_{n_\beta}p, p_1 - T^n p_1\rangle \\
&= \kappa_n\|(1-\beta)(p - p_1) + \beta(T^n p - p_1)\|^2 + \|(1-\beta)(p - G_{n_\beta}p) + \beta(T^n p - G_{n_\beta}p)\|^2 \\
&= \kappa_n(1-\beta)\|p - p_1\|^2 + \beta\kappa_n\|T^n p - p_1\|^2 - \beta(1-\beta)\kappa_n\|T^n p - p\|^2 \\
&\quad +(1-\beta)\|p - G_{n_\beta}p\|^2 + \beta\|T^n p - T^n[(1-\beta)p + \beta T^n p]\|^2 - \beta(1-\beta)\|T^n p - p\|^2 \\
&\leq \kappa_n(1-\beta)\|p - p_1\|^2 + \beta\kappa_n[\kappa_n\|p - p_1\|^2 + \|T^n p - p\|^2] - \beta(1-\beta)\kappa_n\|T^n p - p\|^2 \\
&\quad +(1-\beta)\|p - G_{n_\beta}p\|^2 + \beta^3 L^2\|T^n p - p\|^2 - \beta(1-\beta)\|T^n p - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \kappa_n[1+\beta(\kappa_n-1)]\|p-p_1\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2-\beta[1-\beta(\kappa_n+1)-\beta^2L^2]\|T^np-p\|^2 \\
&= \kappa_n[1+\beta(\kappa_n-1)]\|p-p_1\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2-\beta[1-2\beta-\beta^2L^2]\|T^np-p\|^2
\end{aligned}
$$

Thus.

$$
\|G_{n_\beta}p-p_1\|^2 \leq \kappa_n[1+\beta(\kappa_n-1)]\|p-p_1\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2 \tag{3.10}
$$

Similarly,

$$
\|G_{n_\beta}p-p_2\|^2 \leq \kappa_n[1+\beta(\kappa_n-1)]\|p-p_2\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2 \tag{3.11}
$$

Substituting (3.10), (3.11) and (3.8) in (3.9), we have

$$
\begin{aligned}
&\|G_{n_\beta}p-p\|^2 \\
\leq\ & \lambda[\kappa_n[1+\beta(\kappa_n-1)]\|p-p_1\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2 \\
&+(1-\lambda)[\kappa_n[1+\beta(\kappa_n-1)]\|p-p_2\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2]-\lambda(1-\lambda)\|p_1-p_2\|^2 \\
=\ & \lambda[\kappa_n[1+\beta(\kappa_n-1)](1-\lambda)^2\|p_1-p_2\|^2+\lambda(1-\beta)\|p-G_{n_\beta}p\|^2 \\
&+(1-\lambda)[\kappa_n[1+\beta(\kappa_n-1)]\lambda^2\|p_1-p_2\|^2+(1-\beta)(1-\lambda)\|p-G_{n_\beta}p\|^2-\lambda(1-\lambda)\|p_1-p_2\|^2 \\
=\ & \lambda(1-\lambda)(\kappa_n-1)(1+\beta\kappa_n)\|p_1-p_2\|^2+(1-\beta)\|p-G_{n_\beta}p\|^2
\end{aligned}
$$

Thus,

$$
\beta\|G_{n_\beta}p-p\|^2 \leq \lambda(1-\lambda)(\kappa_n-1)(1+\beta\kappa_n)\|p_1-p_2\|^2 \tag{3.12}
$$

Since $k_n \to 1$ as $n \to \infty$, we obtain from (3.12) that

$$
\lim_{n\to\infty}\|G_{n_\beta}p-p\|=0 \tag{3.13}
$$

On its own,

$$
\begin{aligned}
\|T^np-p\| &= \|T^np-G_{n_\beta}p+G_{n_\beta}p-p\| \\
&\leq \|T^np-G_{n_\beta}p\|+\|G_{n_\beta}p-p\| \\
&= \|T^np-T^n[(1-\beta)p+\beta T^np]\|+\|G_{n_\beta}p-p\| \\
&\leq L\|p-[(1-\beta)p+\beta T^np]\|+\|G_{n_\beta}p-p\| \\
&\leq L\beta\|T^np-p\|+\|G_{n_\beta}p-p\|
\end{aligned}
$$

Thus, $(1-L\beta)\|T^n p-p\| \leq \|G_{n_\beta} p-p\|$, which implies that

$$\lim_{n\to\infty} \|T^n p-p\| = 0 \tag{3.14}$$

Therefore, $T^n p \to p$ as $n \to \infty$. This in turn implies that

$$p = \lim_{n\to\infty} T^n p = T \lim_{n\to\infty} (T^{n-1} p) = Tp.$$

Hence, $p \in F(T)$, which means that $F(T)$ is convex. $\square$

Lemma 3.7 Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly $L-$ Lipschitzean asymptotically pseudo-nonspreading mapping. Then for any sequence $\{x_n\}$ in C converging weakly to a point p and $\{\|x_n - Tx_n\|\}$ converging strongly to 0, we have $p = Tp$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. We show that $p - Tp = 0$. Since, $\{x_n\}_{n=1}^{\infty}$ is weak convergent, it is bounded. Then, we define $f : H \to \mathbb{R}^+$ for each $x \in H$ by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|^2 \ \forall x \in H \tag{3.15}$$

From lemma (2.13), we have

$$f(x) = \limsup_{n\to\infty} \|x_n - p\|^2 + \|x-p\|^2 \ \forall x \in H \tag{3.16}$$

It follows that

$$f(x) = f(p) + \|p - x\|^2 \ \forall x \in H \tag{3.17}$$

Observe that

$$\begin{aligned}
\|G_{n_\beta} x_n - x_n\| &\leq \|G_{n_\beta} x_n - T^n x_n\| + \|x_n - T^n x_n\| \\
&\leq L\|(1-\beta_n)x_n + \beta_n T^n x_n - x_n\| + \|T^n x_n - x_n\| \\
&= (L\beta_n + 1)\|T^n x_n - x_n\| \\
&\to 0 \ as \ n \to \infty \qquad (3.18)
\end{aligned}$$

Moreover,

$$\|[(1-\beta)x_n + \beta T^n x_n] - G_{n_\beta} x_n\|^2$$

$$
\begin{aligned}
&= \|(1-\beta)(x_n - G_{n_\beta}x_n) + \beta(T^n x_n - G_{n_\beta}x_n)\|^2 \\
&= (1-\beta)\|x_n - G_{n_\beta}x_n\|^2 + \beta\|T^n x_n - G_{n_\beta}x_n\|^2 - \beta(1-\beta)\|x_n - T^n x_n\|^2 \\
&= (1-\beta)\|x_n - G_{n_\beta}x_n\|^2 + \beta^3 L^2\|x_n - T^n x_n\|^2 - \beta(1-\beta)\|x_n - T^n x_n\|^2 \\
&= (1-\beta)\|x_n - G_{n_\beta}x_n\|^2 - \beta[1-\beta-\beta^2 L^2]\|x_n - T^n x_n\|^2 - \beta(1-\beta)\|x_n - T^n x_n\|^2 \\
&\leq (1-\beta)\|x_n - G_{n_\beta}x_n\|^2 \\
&\to 0 \ as\ n \to \infty \qquad (3.19)
\end{aligned}
$$

In particular, for arbitrary but fixed $m \geq 1$, we have from (3.17)

$$f(G_{m_\beta}p) = f(p) + \|p - G_{m_\beta}p\|^2 \tag{3.20}$$

Thus ,from (3.15), we have

$$
\begin{aligned}
f(G_{m_\beta}p) &= \limsup_{n\to\infty}\|x_n - G_{m_\beta}p\|^2 \\
&= \limsup_{n\to\infty}\|x_n - G_{m_\beta}x_n + G_{m_\beta}x_n - G_{m_\beta}p\|^2 \\
&= \limsup_{n\to\infty}(\|x_n - G_{m_\beta}x_n\|^2 + 2\langle x_n - G_{m_\beta}x_n, G_{m_\beta}x_n - G_{m_\beta}p\rangle + \|G_{m_\beta}x_n - G_{m_\beta}p\|^2) \\
&= \limsup_{n\to\infty}\|G_{m_\beta}x_n - G_{m_\beta}p\|^2
\end{aligned}
$$

$$Taking \limsup_{m\to\infty} of\ both\ sides$$

$$
\begin{aligned}
&\limsup_{m\to\infty} f(G_{m_\beta}p) \\
&= \limsup_{m\to\infty}\limsup_{n\to\infty}\|G_{m_\beta}x_n - G_{m_\beta}p\|^2 \\
&= \limsup_{m\to\infty}\limsup_{n\to\infty}\|T^m[(1-\beta)x_n + \beta T^m x_n] - T^m[(1-\beta)p + \beta T^m p]\|^2 \\
&\leq \limsup_{m\to\infty}\limsup_{n\to\infty}(\kappa_m\|(1-\beta)(x_n - p) + \beta(T^m x_n - T^m p)\|^2 \\
&\quad + \|[(1-\beta)x_n + \beta T^m x_n] - G_{m_\beta}x_n - ([(1-\beta)p + \beta T^m p] - G_{m_\beta}p)\|^2 \\
&\quad + 2\langle[(1-\beta)x_n + \beta T^m x_n]G_{m_\beta}x_n, [(1-\beta)p + \beta T^m p] - G_{m_\beta}p\rangle) \\
&\leq \limsup_{m\to\infty}\limsup_{n\to\infty}(\kappa_m(1-\beta)\|x_n - p\|^2 + \beta\kappa_m\|T^m x_n - T^m p\|^2 \\
&\quad - \beta(1-\beta)\kappa_m\|(x_n - T^m x_n) + T^m p - p)\|^2 + \|[(1-\beta)p + \beta T^m p] - G_{m_\beta}p)\|^2) \\
&\leq \limsup_{m\to\infty}\limsup_{n\to\infty}(\kappa_m(1-\beta)\|x_n - p\|^2 + \beta\kappa_m[\kappa_m\|x_n - p\|^2 + \|x_n - T^m x_n - (p - T^m p)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\langle x_n - T^m x_n, p - T^m p\rangle] - \beta(1-\beta)\kappa_m\|T^m p - p\|^2 + \|(1-\beta)(p - G_{m_\beta}p) + \beta(T^m p - G_{m_\beta}p)\|^2) \\
= \;& \limsup_{m\to\infty}\limsup_{n\to\infty} \kappa_m[1+\beta(\kappa_m - 1)]\|x_n - p\|^2 + \beta\kappa_m\|p - T^m p\|^2 - \beta(1-\beta)\kappa_m\|T^m p - p\|^2 \\
&+(1-\beta)\|p - G_{m_\beta}p\|^2 + \beta\|T^m p - G_{m_\beta}p\|^2 - \beta(1-\beta)\|T^m p - p\|^2) \\
= \;& \limsup_{m\to\infty}\limsup_{n\to\infty} \kappa_m[1+\beta(\kappa_m - 1)]\|x_n - p\|^2 + \beta\kappa_m\|p - T^m p\|^2 - \beta(1-\beta)\kappa_m\|T^m p - p\|^2 \\
&+(1-\beta)\|p - G_{m_\beta}p\|^2 + \beta^3 L^2\|T^m p - p\|^2 - \beta(1-\beta)\|T^m p - p\|^2) \\
= \;& \limsup_{m\to\infty}\limsup_{n\to\infty} \kappa_m[1+\beta(\kappa_m - 1)]\|x_n - p\|^2 + (1-\beta)\|p - G_{m_\beta}p\|^2 - \beta(1-2\beta-\beta^2L^2)\|T^m p - p\|^2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\limsup_{m\to\infty} f(G_{m_\beta}p) &\leq \limsup_{n\to\infty}\|x_n - p\|^2 + (1-\beta)\limsup_{m\to\infty}\|p - G_{m_\beta}p\|^2) \\
&= f(p) + (1-\beta)\limsup_{m\to\infty}\|p - G_{m_\beta}p\|^2) \qquad (3.21)
\end{aligned}
$$

It follows from (3.20) that

$$
\limsup_{m\to\infty} f(G_{m_\beta}p) = f(p) + \limsup_{m\to\infty}\|p - G_{m_\beta}p\|^2 \qquad (3.22)
$$

Considering (3.21) and (3.22), we have $f(p) + \limsup_{m\to\infty}\|p - G_{m_\beta}p\|^2 \leq f(p) + (1-\beta)\limsup_{m\to\infty}\|p - G_{m_\beta}p\|^2$ which in turn implies that

$$
\limsup_{m\to\infty}\|p - G_{m_\beta}p\| = 0 \; and \; \lim_{m\to\infty}\|p - G_{m_\beta}p\| = 0
$$

Observe that,

$$
\begin{aligned}
\|T^m p - p\| &\leq \|T^m p - G_{m_\beta}p\| + \|G_{m_\beta}p - p\| \\
&\leq L\|G_{m_\beta}p - p\| + \|G_{m_\beta}p - p\| \\
&= \beta L\|T^m p - p\| + \|G_{m_\beta}p - p\|
\end{aligned}
$$

This implies that $(1-\beta L)\|T^m p - p\| \leq \|G_{m_\beta}p - p\| \to 0 \; as \; n \to \infty$. So, we have that

$$
\lim_{m\to\infty}\|T^m p - p\| = 0, \; and \; p = \lim_{m\to\infty} T^m p = T\lim_{m\to\infty}(T^{m-1}p) = Tp.
$$

Therefore, $p = Tp$ and $I - T$ is demiclosed at the origin. □

Theorem 3.8 Let $H_1,\ H_2,\ A,\ \{S_i\},\ \{T_i\},\ C$ and Q be the same as in multiple set split feasibility problem (1.1). For each $i = 1,2,\cdots,N$. Let $T_i : H_2 \to H_2$ be a family of uniformly $\bar{L}_i-$Lipschitzian and asymptotically pseudononspreading mapping with the sequence $\{\delta_n\} \subset$

$[1,+\infty)$ such that $\sum_{n=1}^{\infty}(\delta_n^2-1)<\infty,\ i=1,2,\cdots,N$ and $\bar{L}=max\{\bar{L}_i\}$. Let $S_i: H_1 \to H_1$ be a family of uniformly L_i-Lipschitzian and asymptotically pseudononspreading mapping with the sequence $\{\kappa_n\}\subset[1,+\infty)$ such that $\sum_{n=1}^{\infty}(\kappa_n^2-1)<\infty,\ i=1,2,\cdots,N$ and $L=max\{L_i\}$, . Suppose that $\{x_n\}$ is a sequence generated by

$$\begin{cases} \forall\, x_1\in H_1,\ chosen\ arbitrarily, \\ \\ y_n = x_n+\gamma A^*[\mu I+(1-\mu)T^n_{n(modN)}((1-\eta)I+\eta T^n_{n(modN)})-I]Ax_n, \\ \\ x_{n+1}=(1-\alpha_n)y_n+\alpha_n S^n_{n(modN)}[(1-\beta_n)y_n+\beta_n S^n_{n(modN)}y_n] \end{cases} \tag{3.23}$$

where $\gamma,\ \eta$ and μ are constants while $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in[0,1]satisfying the following control conditions:

C_1 $0<\gamma<\frac{1}{\lambda}$ where λ is the spectral radius of the operator A^*A;

C_2 $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$;

C_3 $0<1-\mu\le\eta<\frac{1}{\sqrt{1+\bar{L}^2}+1}$ and $0<1-\alpha_n\le\beta_n\le\limsup_{n\to\infty}\beta_n<\frac{1}{\sqrt{1+L^2}+1}$.

Suppose that $\Gamma=\{x\in\bigcap_{i=1}^N F(S_i): Ax\in\bigcap_{i=1}^N F(T_i)\}\neq\emptyset$, then the sequence $\{x_n\}$ converges weakly to a point $x^*\in\Gamma$

Proof. The proof is divided into five steps.

Step(I) Firstly, we prove that $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p\in\Gamma$. To this end, we take an arbitrary $p\in\Gamma$ and define $G_n(x):=S^n_{n(modN)}[(1-\beta_n)x_n+\beta_n S^n_{n(modN)}x_n]$. and $g_n(x):=T^n_{n(modN)}[(1-\eta)I+\eta T^n_{n(modN)}]x_n$. Then, we have that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
=\ & \|(1-\alpha_n)y_n+\alpha_n G_n y_n-p\|^2 \\
=\ & \|(1-\alpha_n)(y_n-p)+\alpha_n(G_n y_n-p)\|^2 \\
=\ & (1-\alpha_n)\|y_n-p\|^2+\alpha_n\|G_n y_n-p\|^2-\alpha_n(1-\alpha_n)\|y_n-G_n y_n\|^2 \\
\le\ & (1-\alpha_n)\|y_n-p\|^2+\alpha_n\kappa_n\|[(1-\beta_n)y_n+\beta_n S^n_{n(modN)}y_n]-p\|^2 \\
& +\alpha_n\|[(1-\beta_n)y_n+\beta_n S^n_{n(modN)}y_n-G_n y_n\|^2]-\alpha_n(1-\alpha_n)\|y_n-G_n y_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\|y_n-p\|^2+\alpha_n\kappa_n\|(1-\beta_n)(y_n-p)+\beta_n(S^n_{n(modN)}y_n-p)\|^2 \\
&\quad +\alpha_n\|(1-\beta_n)(y_n-G_ny_n)+\beta_n(S^n_{n(modN)}y_n-G_ny_n)\|^2-\alpha_n(1-\alpha_n)\|y_n-G_ny_n\|^2 \\
&= (1-\alpha_n)\|y_n-p\|^2+\alpha_n\kappa_n(1-\beta_n)\|y_n-p\|^2+\alpha_n\beta_n\kappa_n\|S^n_{n(modN)}y_n-p\|^2 \\
&\quad -\alpha_n\kappa_n\beta_n(1-\beta_n)\|y_n-S^n_{n(modN)}y_n\|^2+\alpha_n(1-\beta_n)\|y_n-G_ny_n\|^2+\alpha_n\beta_n\|S^n_{n(modN)}y_n-G_ny_n\|^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)\|y_n-S^n_{n(modN)}y_n\|^2-\alpha_n(1-\alpha_n)\|y_n-G_ny_n\|^2 \\
&\leq (1-\alpha_n)\|y_n-p\|^2+\alpha_n\kappa_n(1-\beta_n)\|y_n-p\|^2+\alpha_n\beta_n\kappa_n^2\|y_n-p\|^2+\alpha_n\beta_n\kappa_n\|y_n-S^n_{n(modN)}y_n\|^2 \\
&\quad -\alpha_n\kappa_n\beta_n(1-\beta_n)\|y_n-S^n_{n(modN)}y_n\|^2+\alpha_n(1-\beta_n)\|y_n-G_ny_n\|^2+\alpha_n\beta_n^3L^2\|y_n-S^n_{n(modN)}y_n\|^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)\|y_n-S^n_{n(modN)}y_n\|^2-\alpha_n(1-\alpha_n)\|y_n-G_ny_n\|^2 \\
&= [1+\alpha_n(\kappa_n-1)(1+\beta_n\kappa_n)]\|y_n-p\|^2-\alpha_n\beta_n[1-\beta_n(\kappa_n+1)-\beta_n^2L^2]\|y_n-S^n_{n(modN)}y_n\|^2 \\
&\quad -\alpha_n(\beta_n-\alpha_n)\|y_n-G_ny_n\|^2 \qquad (3.25)
\end{aligned}
$$

In another development,

$$
\begin{aligned}
&\|g_nAx_n-Ap\|^2 \\
&= \|T^n_{n(modN)}[(1-\eta)Ax_n+\eta T^n_{n(modN)}Ax_n]-Ap\|^2 \\
&\leq \delta_n\|[(1-\eta)Ax_n+\eta T^n_{n(modN)}Ax_n]-Ap\|^2+\|[(1-\eta)Ax_n+\eta T^n_{n(modN)}Ax_n]-g_nAx_n\|^2 \\
&= \delta_n\|(1-\eta)(Ax_n-Ap)+\eta(T^n_{n(modN)}Ax_n-Ap)\|^2 \\
&\quad +\|(1-\eta)(Ax_n-g_nAx_n)+\eta(T^n_{n(modN)}Ax_n-g_nAx_n)\|^2 \\
&= \delta_n(1-\eta)\|Ax_n-Ap\|^2+\delta_n\eta\|T^n_{n(modN)}Ax_n-Ap\|^2-\delta_n\eta(1-\eta)\|Ax_n-T^n_{n(modN)}Ax_n\|^2 \\
&\quad +(1-\eta)\|Ax_n-g_nAx_n\|^2+\eta\|T^n_{n(modN)}Ax_n-g_nAx_n\|^2-\eta(1-\eta)\|Ax_n-T^n_{n(modN)}Ax_n\|^2 \\
&\leq \delta_n(1-\eta)\|Ax_n-Ap\|^2+\delta_n^2\eta\|Ax_n-Ap\|^2+\delta_n\eta\|Ax_n-T^n_{n(modN)}Ax_n\|^2 \\
&\quad -\eta(1-\eta)(1+\delta_n)\|Ax_n-T^n_{n(modN)}Ax_n\|^2+(1-\eta)\|Ax_n-g_nAx_n\|^2+\eta^3L^2\|Ax_n-T^n_{n(modN)}Ax_n\|^2 \\
&\leq [\delta_n(1-\eta)+\delta_n^2\eta]\|Ax_n-Ap\|^2-\eta[1-\eta(1+\delta_n)-\eta^2L^2]\|Ax_n-T^n_{n(modN)}Ax_n\|^2 \\
&\quad +(1-\eta)\|Ax_n-g_nAx_n\|^2
\end{aligned}
$$

Therefore,

$$
\|g_nAx_n-Ap\|^2\leq[\delta_n(1-\eta)+\delta_n^2\eta]\|Ax_n-Ap\|^2+(1-\eta)\|Ax_n-g_nAx_n\|^2 \qquad (3.26)
$$

and

$$\|G_n y_n - p\|^2 \leq [\kappa_n(1-\beta_n) + \kappa_n^2 \beta_n]\|y_n - p\|^2 + (1-\beta_n)\|y_n - G_n y_n\|^2 \tag{3.27}$$

From, (3.26), we have that

$$\begin{aligned}
& \|\mu A x_n + (1-\mu) g_n A x_n - Ap\|^2 \\
= \;& \|\mu(Ax_n - Ap) + (1-\mu)(g_n A x_n - Ap)\|^2 \\
= \;& \mu\|Ax_n - Ap\|^2 + (1-\mu)\|g_n A x_n - Ap\|^2 - \mu(1-\mu)\|Ax_n - g_n A x_n\|^2 \\
\leq \;& \mu\|Ax_n - Ap\|^2 + (1-\mu)[\delta_n(1-\eta) + \delta_n^2 \eta]\|Ax_n - Ap\|^2 + (1-\eta)\|Ax_n - g_n A x_n\|^2] \\
& -\mu(1-\mu)\|Ax_n - g_n A x_n\|^2 \\
= \;& \mu\|Ax_n - Ap\|^2 + (1-\mu)[\delta_n(1-\eta) + \delta_n^2 \eta]\|Ax_n - Ap\|^2 \\
& +(1-\mu)(1-\eta)\|Ax_n - g_n A x_n\|^2 - \mu(1-\mu)\|Ax_n - g_n A x_n\|^2 \\
= \;& [\mu + (1-\mu)[\delta_n(1-\eta) + \delta_n^2 \eta]]\|Ax_n - Ap\|^2 + (1-\mu)(1-\eta-\mu)\|Ax_n - g_n A x_n\|^2 \\
\leq \;& \|Ax_n - Ap\|^2
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
& \|Ax_n - Ap\|^2 \\
= \;& \|Ax_n + \mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n - [\mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n] - Ap\|^2 \\
= \;& \|[\mu A x_n + (1-\mu) g_n(Ax_n) - Ap] - [\mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n]\|^2 \\
= \;& \|\mu A x_n + (1-\mu) g_n(Ax_n) - Ap\|^2 + \|\mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n\|^2 \\
& -2\langle \mu A x_n + (1-\mu) g_n(Ax_n) - Ap, \mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n \rangle \\
\leq \;& \|Ax_n - Ap\|^2 + \|\mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n\|^2 \\
& -2\langle \mu A x_n + (1-\mu) g_n(Ax_n) - Ap, \mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n \rangle
\end{aligned}$$

Thus,

$$\langle \mu A x_n + (1-\mu) g_n(Ax_n) - Ap, \mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n \rangle \leq \frac{1}{2}\|\mu A x_n + (1-\mu) g_n(Ax_n) - Ax_n\|^2 \tag{3.29}$$

While,

$$
\begin{aligned}
& \|y_n - p\|^2 \\
= \;& \|x_n + \gamma A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n] - p\|^2 \\
= \;& \|(x_n - p) + \gamma A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n]\|^2 \\
= \;& \|x_n - p\|^2 + \gamma^2\|A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n]\|^2 + 2\gamma\langle A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n], x_n - p\rangle \\
= \;& \|x_n - p\|^2 + \gamma^2\langle A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n], A^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n]\rangle \\
& +2\gamma\langle \mu Ax_n + (1-\mu)g_nAx_n - Ax_n, Ax_n - Ap + (\mu Ax_n + (1-\mu)g_nAx_n) - (\mu Ax_n + (1-\mu)g_nAx_n)\rangle \\
= \;& \|x_n - p\|^2 + \gamma^2\langle \mu Ax_n + (1-\mu)g_nAx_n - Ax_n], AA^*[\mu Ax_n + (1-\mu)g_nAx_n - Ax_n]\rangle \\
& +2\gamma\langle \mu Ax_n + (1-\mu)g_nAx_n - Ax_n, \mu Ax_n + (1-\mu)g_nAx_n - Ap + Ax_n - \mu Ax_n + (1-\mu)g_nAx_n\rangle \\
\leq \;& \|x_n - p\|^2 + \gamma^2\lambda\|\mu Ax_n + (1-\mu)g_nAx_n - Ax_n\|^2 \\
& +2\gamma\langle \mu Ax_n + (1-\mu)g_nAx_n - Ax_n, \mu Ax_n + (1-\mu)g_nAx_n - Ap\rangle \\
& +2\gamma\langle \mu Ax_n + (1-\mu)g_nAx_n - Ax_n, Ax_n - \mu Ax_n + (1-\mu)g_nAx_n\rangle \\
= \;& \|x_n - p\|^2 + \gamma^2\lambda\|\mu Ax_n + (1-\mu)g_nAx_n - Ax_n\|^2 \\
& +\gamma\|\mu Ax_n + (1-\mu)g_n(Ax_n) - Ax_n\|^2 - 2\gamma\|\mu Ax_n + (1-\mu)g_n(Ax_n) - Ax_n\|^2 \\
= \;& \|x_n - p\|^2 - \gamma(1-\gamma\lambda)(1-\mu)^2\|g_nAx_n - Ax_n\|^2 \qquad (3.30)
\end{aligned}
$$

From (3.25) and (3.30), we have

$$
\begin{aligned}
& \|x_{n+1} - p\|^2 \\
\leq \;& [1 + \alpha_n(\kappa_n - 1)(1 + \beta_n\kappa_n)][\|x_n - p\|^2 - \gamma(1-\gamma\lambda)(1-\mu)^2\|g_nAx_n - Ax_n\|^2] \\
& -\alpha_n\beta_n[1 - \beta_n(\kappa_n + 1) - \beta_n^2L^2]\|y_n - S^n_{n(modN)}y_n\|^2 - \alpha_n(\beta_n - \alpha_n)\|y_n - G_ny_n\|^2 \qquad (3.31)
\end{aligned}
$$

Consequently, we have that

$$\|x_{n+1} - p\|^2 \leq [1 + \alpha_n(\kappa_n - 1)(1 + \beta_n\kappa_n)]\|x_n - p\|^2 \qquad (3.32)$$

From the assumption that $\sum_{n=1}^{\infty}(\kappa_n^2 - 1) < \infty$, we have that

$$\sum_{n=1}^{\infty}[a_n(\kappa_n - 1)(1 + \beta_n\kappa_n)] \quad < \quad \sum_{n=1}^{\infty}(\kappa_n - 1)(1 + \kappa_n)$$

$$\begin{aligned} &= \sum_{n=1}^{\infty}(\kappa_n^2-1) \\ &< \infty \end{aligned}$$

Hence, from Lemma 2.12, $\{\|x_n-p\|\}$ converges, both $\{x_n\}$ and $\{y_n\}$ are bounded and

$$\lim_{n\to\infty}\|g_nAx_n-Ax_n\| = \lim_{n\to\infty}\|y_n-S^n_{n(modN)}y_n\| = \lim_{n\to\infty}\|y_n-G_ny_n\| = 0 \tag{3.33}$$

From, (3.33), we have that

$$\begin{aligned} \|T^n_{n(modN)}Ax_n-Ax_n\| &= \|T^n_{n(modN)}Ax_n-g_nAx_n\|+\|g_nAx_n-Ax_n\| \\ &\leq \bar{L}\|Ax_n-[(1-\eta)Ax_n+\eta T^n_{n(modN)}Ax_n]\|+\|g_nAx_n-Ax_n\| \\ &= \bar{L}\eta\|Ax_n-T^n_{n(modN)}Ax_n\|+\|g_nAx_n-Ax_n\| \end{aligned}$$

Hence, $\|Ax_n-T^n_{n(modN)}Ax_n\| \leq \frac{1}{(1-\bar{L}\eta)}\|g_nAx_n-Ax_n\| \to 0$ as $n\to\infty$. This implies that

$$\lim_{n\to\infty}\|Ax_n-T^n_{n(modN)}Ax_n\| = 0. \tag{3.34}$$

Step(II) Next, we show that $\lim_{n\to\infty}\|y_n-p\|$ exists. But, it is clear from (3.30) and (3.33)that

$$\lim_{n\to\infty}\|y_n-p\| = \lim_{n\to\infty}\|x_n-p\| \tag{3.35}$$

Step(III) Next, we show that

$$\lim_{n\to\infty}\|y_n-x_n\| = \lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0.$$

Obviously, (3.23) has it that $y_n-x_n = \gamma A^*[\mu I+(1-\mu)T^n_{n(modN)}((1-\eta)I+\eta T^n_{n(modN)})-I]Ax_n$ which on taking norm of both sides and simplifying, we have that

$$\begin{aligned} \|y_n-x_n\| &= \|\gamma A^*[\mu I+(1-\mu)T^n_{n(modN)}((1-\eta)I+\eta T^n_{n(modN)})-I]Ax_n\| \\ &= \|\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]\| \\ &= \|\gamma A^*[(1-\mu)(g_nAx_n-Ax_n)]\|. \end{aligned}$$

Hence, from (3.33), we have that

$$\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{3.36}$$

Similarly,

$$\begin{aligned}
& \|x_{n+1}-x_n\| \\
= \quad & \|(1-\alpha_n)y_n+\alpha_n G_n y_n - x_n\| \\
= \quad & \|(1-\alpha_n)(x_n+\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]+\alpha_n G_n y_n - x_n\| \\
= \quad & \|(1-\alpha_n)\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]+\alpha_n(G_n y_n - x_n)\| \\
= \quad & \|(1-\alpha_n)\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]+\alpha_n(G_n y_n - y_n)+\alpha_n(y_n-x_n)\| \\
= \quad & \|(1-\alpha_n)\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]+\alpha_n(G_n y_n - y_n)+\alpha_n\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]\| \\
= \quad & \|\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]+\alpha_n(G_n y_n - y_n)\| \\
\leq \quad & \|\gamma A^*[(1-\mu)(g_nAx_n-Ax_n)]\|+\|\alpha_n(G_n y_n - y_n)\|
\end{aligned}$$

From (3.33), we deduce that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.37}$$

Similarly, from (3.23), we have that

$$\begin{aligned}
& \|y_{n+1}-y_n\| \\
= \quad & \|x_{n+1}+\gamma A^*[\mu Ax_{n+1}+(1-\mu)g_{n+1}Ax_{n+1}-Ax_{n+1}]-(x_n+\gamma A^*[\mu Ax_n+(1-\mu)g_nAx_n-Ax_n]\| \\
= \quad & \|(x_{n+1}-x_n)+\gamma A^*[(1-\mu)(g_{n+1}Ax_{n+1}-Ax_{n+1})]+(-\gamma A^*[(1-\mu)(g_nAx_n-Ax_n)]\| \\
\leq \quad & \|x_{n+1}-x_n\|+\|\gamma A^*[(1-\mu)(g_{n+1}Ax_{n+1}-Ax_{n+1})]\|+\|\gamma A^*[(1-\mu)(g_nAx_n-Ax_n)]\|
\end{aligned}$$

From (3.33), we deduce that

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.38}$$

Step(IV) We prove that, for each $j=1,2,3,\cdots,N$

$$\|y_{iN+j}-S_jy_{iN+j}\|\to 0 \ and \ \|Ax_{iN+j}-T_jAx_{iN+j}\|\to 0 \ as \ i\to\infty.$$

Clearly, From (3.33), we can obtain that

$$\|y_{iN+j}-S_j^{iN+j}y_{iN+j}\|\to 0 \ as \ i\to\infty. \tag{3.39}$$

Since S_j is uniformly L_j-Lipschitzian, it follows from (3.38) and (3.39) that

$$\begin{aligned}
& \|y_{iN+j}-S_jy_{iN+j}\| \\
\leq\ & \|y_{iN+j}-S_j^{iN+j}y_{iN+j}\|+\|S_j^{iN+j}y_{iN+j}-S_jy_{iN+j}\| \\
\leq\ & \|y_{iN+j}-S_j^{iN+j}y_{iN+j}\|+L_j\|S_j^{iN+j-1}y_{iN+j}-y_{iN+j}\| \\
=\ & \|y_{iN+j}-S_j^{iN+j}y_{iN+j}\|+L_j[\|S_j^{iN+j-1}y_{iN+j}-S_j^{iN+j-1}y_{iN+j-1}\|+\|S_j^{iN+j-1}y_{iN+j-1}-y_{iN+j}\|] \\
\leq\ & \|y_{iN+j}-S_j^{iN+j}y_{iN+j}\|+L_j^2\|y_{iN+j}-y_{iN+j-1}\| \\
& +L_j[\|S_j^{iN+j-1}y_{iN+j-1}-y_{iN+j-1}\|+\|y_{iN+j-1}-y_{iN+j}\|] \\
\to\ & 0 \ as \ n\to\infty. \qquad (3.40)
\end{aligned}$$

Similarly, we can prove that for each $j=1,2,3,\cdots,N$, we can obtain that

$$\|Ax_{iN+j}-T_j^{iN+j}Ax_{iN+j}\|\to 0 \ as \ i\to\infty. \qquad (3.41)$$

Since T_j is uniformly $\bar{L}_j$-Lipschitzian, in the same way as above, we can also prove that

$$\|Ax_{iN+j}-T_jAx_{iN+j}\|\to 0 \ as \ i\to\infty. \qquad (3.42)$$

Step(V) Finally, we prove that $x_n\rightharpoonup x^*, y_n\rightharpoonup x^*$, and it is a solution of problem $(MSSFP)$. In fact, since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}\subset\{y_n\}$ such that $y_{n_j}\rightharpoonup x^*\in H_1$. Hence, for any positive integer $j=1,2,3,\cdots,N$, there exists a subsequence $n_i(j)\subset n_i$ with $n_i(j) modN=j$ such that $y_{n_i(j)}\rightharpoonup x^*$. From(3.40) we have that

$$\|y_{n_i(j)}–S_jy_{n_i(j)}\|\to 0,\ n_i(j)\to\infty. \qquad (3.43)$$

Since S_j is demiclosed at zero, it follows that $x^*\in F(S_j)$. By the arbitrariness of $j=1,2,3,\cdots,N$, we have

$$x^*\in C:=\bigcap_{i=1}^{N}F(S_i)$$

Moreover, from the algorithm and (3.34) we have $y_{n_i}=x_{n_i}+\gamma A^*[\mu I+(1-\mu)T^{n_i}_{n_i(modN)}((1-\eta)I+\eta T^{n_i}_{n_i(modN)})-I]Ax_{n_i}\rightharpoonup x^*$. Since A is a linear bounded operator, it follows that $Ax_{n_i}\rightharpoonup Ax^*$. For any positive integer $k=1,2,3,\cdots,N$, there exists a subsequence $x_{n_i(k)}\subset x_{n_i}$ with $n_i(k)(modN)=k$ such that $Ax_{n_i(k)}\rightharpoonup Ax^*$ and $\|Ax_{n_i(k)}-T_jAx_{n_i(k)}\|\to 0$. Since T_k is demiclosed at zero, we have $Ax^*\in F(T_k)$. By the arbitrariness of k, it follows that $Ax^*\in Q:=\bigcap_{k=1}^{N}F(T_k)$.

This together with $x^* \in C$ shows that $x^* \in \Gamma$, that is, x^* is a solution to the problem (MSSFP). Next we prove that $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup x^*$. In fact, assume that there exists another subsequence $y_{n_l} \subset y_n$ such that $y_{n_l} \rightharpoonup y^*$ with $y^* \neq x^*$. Consequently, by virtue of the existence of $\lim_{n\to\infty} \|x_n - p\|$ and the celebrated Opial property which is a trivial inheritance of a Hilbert space, we have

$$\begin{aligned}
\liminf_{n_i\to\infty} \|y_{n_i} - x^*\| &< \liminf_{n_i\to\infty} \|y_{n_i} - y^*\| \\
&= \liminf_{n\to\infty} \|y_n - y^*\| \\
&= \liminf_{n_j\to\infty} \|y_{n_j} - y^*\| \\
&< \liminf_{n_j\to\infty} \|y_{n_j} - x^*\| \\
&= \liminf_{n\to\infty} \|y_n - x^*\| \\
&= \liminf_{n_j\to\infty} \|y_{n_j} - x^*\|
\end{aligned}$$

This is a contradiction. Therefore, $y_n \rightharpoonup x^*$. By the (3.23) and (3.34), we have

$$x_n = y_n - \gamma A^*[\mu I + (1-\mu)T^n_{n(modN)}((1-\eta)I + \eta T^n_{n(modN)}) - I]Ax_n \rightharpoonup x^*.$$

This completes the proof of Theorem 3.8. □

Theorem 3.9:

Suppose that the conditions of theorem 3.8 are satisfied and suppose in addition that there exists a positive integer j such that S_j is semicompact, then the sequence $\{x_n\}$ generated by (3.23) converges strongly to a point $x^* \in \Gamma$.

Proof. Proof Without loss of generality, we can assume that S_1 is semicompact. It follows from (3.43) that

$$\|y_{n_i(j)} - S_j y_{n_i(j)}\| \to 0,\ n_i(j) \to \infty.$$

Therefore, there exists a subsequence of $\{y_{n_i(j)}\}$, which (for the sake of convenience) we still denote by $\{y_{n_i(j)}\}$, such that $y_{n_i(j)} \to y^* \in H_1$. Since $y_{n_i(j)} \rightharpoonup y^*$, $y^* = x^*$ and so$y_{n_i(j)} \to x^* \in \Gamma$. By virtue of $\lim_{n\to\infty} \|x_n - p\|$ exists, we know that

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|y_n - p\| = 0$$

that is, $\{x_n\}$ and $\{y_n\}$ both converge strongly to the point $x^* \in \Gamma$. This completes the proof of Theorem 3.9 $\square$

4. Applications

In this section we shall utilize the results presented in Section 3 to study the hierarchical variational inequality problem.

Theorem 4.1 Let H be a real Hilbert space, $S_i = 1, 2, \cdots, N$ be uniformly L_i-Lipschitzian and asymptotically pseudo-nonspreading mappings with $\mathbb{F} = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $T : H \to H$ be a nonspreading mapping. The so-called hierarchical variational inequality problem for a finite family of mappings $\{S_i\}$ with respect to the mapping T is to find an $x^* \in \mathbb{F}$ such that

$$\langle x^* - Tx^*, x^* - x\rangle \leq 0 \tag{4.1}$$

It is easy to see that (4.1) is equivalent to the following fixed point problem:

$$find\ x^* \in \mathbb{F}\ such\ that\ x^* = P_{\mathbb{F}}Tx^*. \tag{4.2}$$

where $P_{\mathbb{F}}$ is the metric projection from H onto $\mathbb{F}$. Letting $C = \mathbb{F}$ and $Q = F(P_{\mathbb{F}}T)$ (the fixed point set of $P_{\mathbb{F}}T$ and $A = I$ (the identity mapping on H), problem (4.2) is equivalent to the following multi-set split feasibility problem:

$$find\ x^* \in C\ such\ that\ x^* \in Q. \tag{4.3}$$

Hence from Theorem 3.8 we have the following theorem.

Theorem 4.1:

Let H, $\{S_i\}$, T, C and Q be the same as in multiple set split feasibility problem (1.1). Let $\{x_n\}$, $\{y_n\}$ be the sequences defined by

$$\begin{cases} \forall\ x_1 \in H_1,\ chosen\ arbitrarily, \\ \\ y_n = x_n + \gamma[\mu I + (1-\mu)T((1-\eta)I + \eta T^n_{n(modN)}) - I]x_n, \\ \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n S^n_{n(modN)}[(1-\beta_n)y_n + \beta_n S^n_{n(modN)}y_n] \end{cases} \tag{4.4}$$

where γ, η and μ are constants while $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]satisfying the following control conditions:

C_1 $0 < \gamma < \frac{1}{\lambda}$ where λ is the spectral radius of the operator A^*A;

C_2 $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

C_3 $0 < 1-\mu \leq \eta < \frac{1}{\sqrt{1+\bar{L}^2}+1}$ and $0 < 1-\alpha_n \leq \beta_n \leq \limsup_{n\to\infty} \beta_n < \frac{1}{\sqrt{1+L^2}+1}$.

Suppose that $L = max\{L_i\}$, and $\Gamma = \{x \in \bigcap_{i=1}^N F(S_i) : Ax \in \bigcap_{i=1}^N F(T_i)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of hierarchical variational inequality problem (4.1).

Proof. In fact, by the assumption that T is an asymptotically pseudononspreading mapping implies that T is pseudononspreading as well as nonspreading with $x - Tx = y - Ty$. Taking $N = 1$ and $A = I$ in Theorem 3.8, by the same method as that given in Theorem 3.8, we can prove that $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$, which is a solution of hierarchical variational inequality problem (4.1) immediately. $\square$

5. Numerical Examples

Let $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{R}^2$. Let the mapping $S_i : \mathbb{H}_1 \to \mathbb{H}_1$ be defined by

$$S_i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \cos(i+1)\frac{\pi}{2} & -sin(i+1)\frac{\pi}{2} \\ sin(i+1)\frac{\pi}{2} & \cos(i+1)\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} (x_1-5) \\ (x_2-5) \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

It follows that

$$S_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 10-x_1 \\ 10-x_2 \end{bmatrix}$$

with Lipschitz constant $L_1 = 1$ and $C_1 = F(S_1) = \{(5,5)\}$.

$$S_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_2 \\ 10-x_1 \end{bmatrix}$$

with Lipschitz constant $L_2 = 1$ and $C_2 = F(S_2) = \{(5,5)\}$.

$$S_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with Lipschitz constant $L_3 = 1$ and $C_3 = F(S_3) = \mathbb{H}_1$., where $L = \max_{i\geq 1}\{L_i\} = 1$. Clearly, $\bigcap_{i=1}^{3} F(S_i) = \{(5,5)\}$, and, $\frac{1}{1+\sqrt{L^2+1}} = -1+\sqrt{2} < 0.42$. Thus, we consider $\beta_n = \{\frac{1}{9.1}(\frac{n}{n+1})\}$

and $\alpha_n = \{\frac{81}{91} + \frac{1}{9.1}(\frac{1}{n+1})\}$.

Let $\mathbb{H}_2 = \mathbb{R}^2$ with $\bar{0} = (0,0) \in \mathbb{R}^2$ and $Y = \{\bar{x} : \|\bar{x}\| < \frac{1}{6}\}$. Let the mapping $T_i : \mathbb{H}_2 \to \mathbb{H}_2$ be defined by

$$T_i(x) = \begin{cases} \bar{x};\ if\ \|\bar{x}\| < \frac{1}{6} \\ [1-2(n\ mod\ 3)]\bar{x};\ if\ \frac{1}{6} \leq \|\bar{x}\| \leq \frac{1}{2} \\ \frac{1}{1-2(n\ mod\ 3)}\bar{x};\ otherwise \end{cases}$$

It follows that $T_1(\bar{x}) = \bar{x}\ \forall \bar{x} \in \mathbb{H}_2$ with Lipschitz constant $L_1 = 1$ and $Q_1 = F(T_1) = Y$.

$$T_2(\bar{x}) = \begin{cases} \bar{x};\ if\ \|\bar{x}\| < \frac{1}{6} \\ -\bar{x};\ otherwise \end{cases}$$

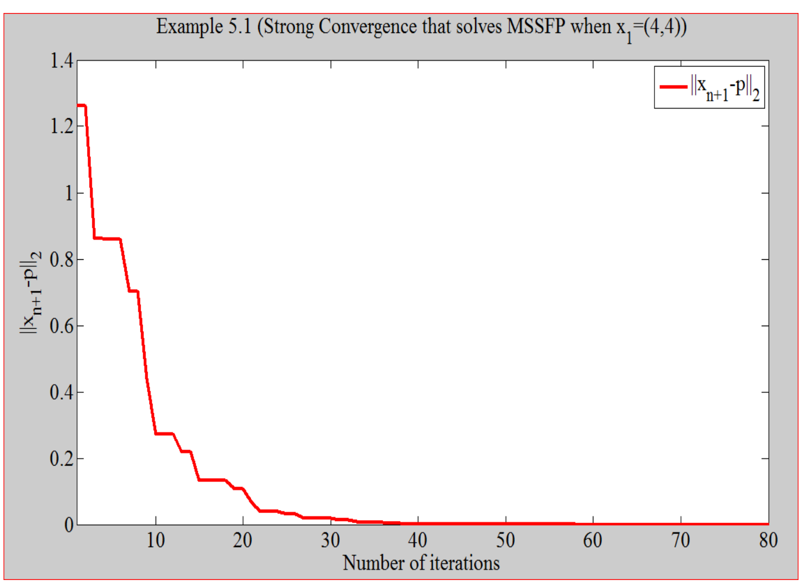
with Lipschitz constant $L_2 = 1$ and $Q_2 = F(T_2) = Y$.

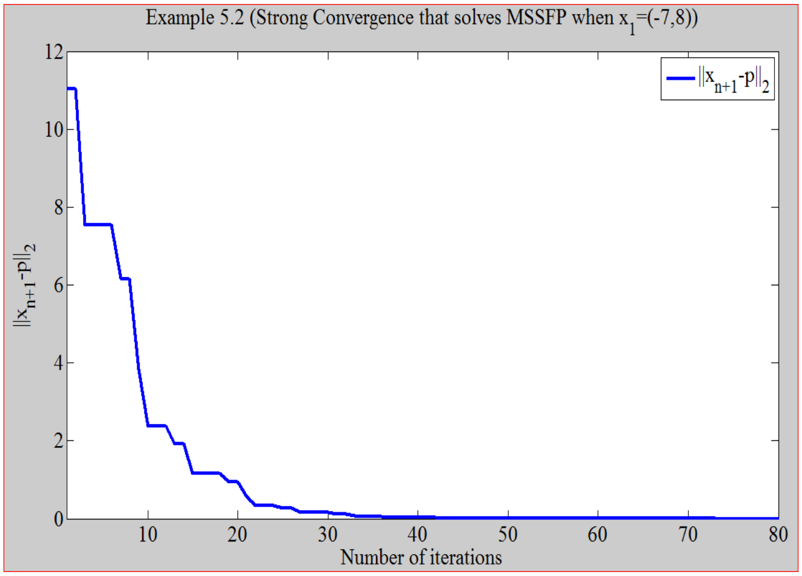
$$T_3(\bar{x}) = \begin{cases} \bar{x};\ if\ \|\bar{x}\| < \frac{1}{6} \\ -3\bar{x};\ if\ \frac{1}{6} \leq \|\bar{x}\| \leq \frac{1}{2} \\ -\frac{1}{3}\bar{x};\ otherwise \end{cases}$$

with Lipschitz constant $L_3 = 3$ and $Q_3 = F(T_3) = Y$, where $L = \underset{i\geq 1}{max}\{L_i\} = 3$. Clearly, $\bigcap_{i=1}^{3} F(T_i) = Y$, and, $\frac{1}{1+\sqrt{L^2+1}} = \frac{1}{9}(-1+\sqrt{10})$. Thus, we consider $\eta = 0.001$ and $\mu = 0.999$

Next, let us consider a bounded linear operator, $T : \mathbb{H}_1 \to \mathbb{H}_2$ defined by $T\bar{x} = A\bar{x}$ where $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Clearly, A is self-adjoint because $\langle Ax, y\rangle = \langle x, Ay\rangle\ \forall x, y \in \mathbb{H}_1$. Then, the spectral radius of AA^* is $\lambda = 4$. Thus, we consider $\gamma = 0.249 < 0.25 = \frac{1}{\lambda}$. By using stopping criterion of $\|x_{n+1} - p\| < 1e-4$, we have from the table and graphs that the solution of the MSSFP is $p = (5,5)$ indeed.

When $x_1 = (4,4)$				When $x_1 = (-7,8)$		
S/N	x_{n+1}	$\|x_{n+1} - p\|$		S/N	x_{n+1}	$\|x_{n+1} - p\|$
1	(4.1039,5.8901)	1.2631		1	(7.6432,15.7266)	11.0475
2	(4.1046,5.8893)	1.2620		2	(7.6467,15.7230)	11.0448
3	(5.6111,4.3930)	0.8614		3	(3.1844,-2.3281)	7.5497
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
32	(5.0105,5.0103)	0.0148		32	(4.9901,4.9899)	0.0142
33	(4.9937,4.9938)	0.0088		33	(5.0059,5.0060)	0.0084
34	(5.0037,5.0037)	0.0052		34	(4.9967,4.9962)	0.0050
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
57	(4.9999,4.9999)	0.0002		67	(4.9997,5.0001)	0.0003
58	(5.0001,5.0001)	0.0001		68	(4.9997,5.0001)	0.0003
59	(5.0001,5.0001)	0.0001		69	(5.0002,5.0000)	0.0002
60	(5.0001,5.0001)	0.0001		70	(4.9999,5.0000)	0.0001
61	(5.0001,4.9999)	0.0001		71	(4.9999,5.0000)	0.0001
62	(5.0001,4.9999)	0.0001		72	(4.9999,5.0000)	0.0001
63	(5.0000,5.0000)	0.0000		73	(5.0000,5.0001)	0.0001





Authors' Contributions

All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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