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ON THE OPTIMAL CONTROL PROBLEMS CONSTRAINED BY ORDINARY DIFFERENTIAL EQUATIONS USING CONJUGATE GRADIENT METHOD AND FICO XPRESS MOSEL

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Abstract. In this paper, the general quadratic continuous optimal control problem constrained by an ordinary differential equation is considered. For the analytical solution, the necessary conditions of optimality are applied to the Hamiltonian function. This results in a system of first-order ordinary differential equations that are solved to obtain the optimal state and optimal control variables. In order to obtain the numerical solution, the discretization of the objective function and the corresponding constraints are carried out using $\frac{1}{3}$ Simpson's rule and fifth-order Implicit method respectively. The discretized Optimal Control Problems (OCPs) are converted into unconstrained problems using Augmented Lagrangian Method. The Conjugate Gradient Method (CGM) and Fico Xpress Mosel are used to solve the resulting nonlinear programming problem. Convergence analyses are conducted to determine the effectiveness of the proposed scheme. Two examples are considered to illustrate the robustness of the proposed methods and compare the analysis of the solutions from the CGM and Fico Xpress Mosel. The results show that FICO XPress Mosel performs better than CGM for this class of problems.

Keywords: analytical solution; Halmitonian function; discretization; augmented Lagrangian method; conjugate gradient method; FICO Xpress Mosel; fundamental matrix.

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1. Introduction

Optimization is the process of finding the best decision for a given problem within a define set of goals and constraints [21]. Optimization can be divided into two major subfields which are discrete in which set of feasible solutions is typically a discrete or finite set and continuous optimization, this is when the feasible set is characterized by continuous value and the objective function is a continuous function . The knowledge gained from optimization can be applied in many fields such as engineering, science and technology, health, business, fraud and financial crimes etc.

Optimization can be classified into Constraints and Unconstrained Optimization. Constrained Optimization is the process of optimizing an objective function with respect to some variables subject to certain limitations usually called constraints. Unconstrained optimization on the other hand, optimize the objective function in the absence of any limitation or constraint [5].

A standard constrained optimization problem takes a perfomance index $\mathscr{J}(t,x(t),u(t))$ which is subject to a constraint $\dot{x}(t) = g(t,x(t),u(t))$ with an initial condition $x(t_0) = x_0$. In optimal control problems, we categorize variables into two groups: the state (or phase) variable $x(t)$ and the control variable $u(t)$. The behavior of the state variables is influenced by the control variables through a set of differential equations. Additionally, both the control and state variables typically have constraints associated with them. These constraints introduce complexity into many optimal control problems, as traditional calculus of variations struggles to handle problems involving path constraints effectively.

To address optimal control problem, a specific set of conditions, known as necessary conditions, must be met. These conditions are the essentially the requirements that need to be satisfied when solving the problem. It is widely acknowledged that the Maximum Principle was initially proven by L. Pontryagin and his collaborators during the late 1950s [1, 6].

Pontryagin introduced the concept of an adjoint function, which was attached to the differential equation of the objective function. This adjoint function serves a similar purpose to Lagrange multipliers in multivariable calculus. The necessary conditions essential for solving the basic problem are derived from what is known as the Hamiltonian function [5].

Over the years many researchers have worked on optimal control problems that involves a combination of equality and inequality constraints on both state and control variables. Moreover, this class of problems include fixed constraints on the initial and final states. The control variables are described as piecewise continuous functions, while the state variables are represented as piecewise smooth functions. The approach involves discretizing the continuous-time optimal control problem using various numerical techniques and then optimizing it through established iterative methods. This process effectively transforms the constrained problem into an unconstrained nonlinear programming problem, utilizing methods like penalty or multiplier techniques [15, 17, 18, 19].

[1] obtained the analytical solutions by applying the necessary optimality conditions of optimality to the Lagrangian function. The method of the fundamental matrix method was used to solve the resultant non-homogeneous system of first-order ordinary differential equations, resulting in analytical solutions for both the state and control variables, along with the value of the objective function.

Similarly, [2] presented the analytical and numerical solutions of optimal control problems with equality and inequality constraints. Two numerical methods - Simpson's Rule and Adams-Bashforth explicit method were employed for the numerical solutions of constrained optimal control problems. The resulting non-linear programming problem was then solved using the Exterior penalty function and the Conjugate Gradient Method. The convergence analysis established the reliability of this approach [16].

A fifth order implicit method for the solution of initial valued first order Ordinary differential equations (ODEs) was developed using sixth order Lagrangian Interpolation formula. The method yielded superior results compared to those obtained using implicit formula on Euler and Runge-Kutta methods [14]. A Romberg scheme was employed to enhance the accuracy of the results obtained [13]

[4] A quadratic cost functional having quasilinear systems of first order ordinary differential equations as its constraints was examined. The optimal control was characterized by employing the fixed-point theorem together with the associated Riccati equation. A novel numerical method was utilized to approximate the solution and this was validated within the context of a

quasilinear quadratic cancer therapy model. The methodology presented in this research work provided the frame work for obtaining the solutions of this class of optimal control problems across various fields.

Accuracy estimates for both first and second orders of an optimal control problem regulated by a system of ordinary differential equations featuring a bilinear control mechanism was presented. The finite element method, utilizing continuous piecewise linear functions, was applied for numerical time discretization. When considering box constraints on the control, first-order error estimated for the control function was derived, a piecewise constant approximation of the control was assumed. On the other hand, a continuous, piecewise polynomial approximation allows for achieving second-order accuracy. The presented numerical evidence substantiates the theoretical results. [11]

[12] introduced optimal control problems employing neural ordinary differential equations (neural ODEs) as a proficient approach for iteratively approximating continuous-time control functions in situations involving analytical challenging and computational intensive control tasks. The research addressed certain gaps in knowledge related to effective hyperparameter optimization. It involved an analysis of the impact of both truncated and non-truncated back propagation through time on both runtime performance and the capability of neural networks to learn optimal control functions. The investigation utilized analytical and numerical methods to explore the influence of parameter initializations, optimizers, and neural network architecture on the study's objectives.

Xpress Optimization Suite also called Xpresss MP is a mathematical modelling software which was developed by FICO, a leading analytical Software company. It include Xpress solutions which are being used by Business analysts and Xpress Technology which are being used by Operation Researchers, Data Scientists and Solution Developers. FICO designed Xpress MP to solve a wide range of mathematical optimization problems such as Linear and Non-Linear programming, Quadratic programming, Mixed-integer programming etc. These are being used to solve business problems in different industries all over the world such as Banking and Finance Services, Marketing, Deposit Pricing, Impairment management, Supply chain, Energy, Transportation and Logistics etc.

The Xpress mosel and Xpress Optimizer are both major components of Fico Xpress Optimization but they serve different purposes. Mosel is a solution technique and modeling environment. It is a language that may be used for both modeling and programming. The Xpress Optimizer uses the models that are created in the mosel to find optimal solution to a particular problem. It support a wide range of problem types featuring advanced algorithm that can handle large scale problems such as Interior point method, Sequantial quadratic programming , Gradient based optimization e.t.c [8, 10]

Modules were considered a practical approach for swiftly crafting prototype algorithms aimed at tackling intricate problems that demanded a fusion of solution techniques or solvers originating from diverse research domains. An example of this concept involved the utilization of an external solver such as *mmxprs*, *mmquad*, *mmsvg* etc. FICO Xpress Mosel is equipped with its proprietary optimization solvers, such as the Xpress-MP solver, it also offered compatibility with external solvers constituted a valuable feature within FICO Xpress Mosel. This feature empowered users to leverage the strengths of diverse solvers tailored to the precise demands of their optimization problems [10, 3].

While extensive research efforts have been dedicated to Optimal Control Problems (OCPs), there is a noticeable dearth of studies that focus on OCPs characterized by Ordinary Differential Equations, where solutions are tackled using optimization software like the FICO express Optimization tool. This research project is driven by the urgent need to develop a highly efficient and robust model tailored specifically for addressing these particular instances of OCPs. This approach entails the utilization of both the Conjugate Gradient method and the FICO Optimization suite implemented within the Mosel language. This tool is designed to effectively handle OCPs constrained by Ordinary Differential Equations, with a focus on achieving superior performance compared to solutions obtained using the conjugate gradient method. Furthermore, it aims to produce numerical solutions that closely align with analytical solutions whenever they are accessible.

2. PRELIMINARIES

2.0.1. *Analytical Solution of quadratic optimal control problem constrained by ordinary differential equation.* Consider a Quadratic Optimal Control Problem Constrained By Ordinary

Differential Equation given by

$$Min\,\mathscr{J}(x,u) = \int_0^T (a + bx + cu + dx^2(t) + eu^2(t))dt \tag{1}$$

$$\text{Subject to } \dot{x}(t) = px(t) + qu(t) \tag{2}$$

$$x(0) = x_0 \quad t \in [0,T]$$

where a,b,c,d,e,p,q are real constant $d,e > 0$, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are continuous differentiable functions and T denotes the terminal time.

By introducing the adjoint variable $\mu(t)$, the constrained Optimal Control Problem (OCP) given in equations (1) and (2) is converted to an unconstrained problem. Hence, the hamiltonian function is given by

$$H(x,u,\mu) = a + bx + cu + dx^2 + eu^2 + \mu(px + qu) \tag{3}$$

The Euler-Lagrange system of equations for this hamiltonian function can be written as

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu}}\right] = \frac{\partial H}{\partial \mu} \tag{4}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x}}\right] = \frac{\partial H}{\partial x} \tag{5}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{u}}\right] = \frac{\partial H}{\partial u} \tag{6}$$

Equations (4)-(6) give

$$px + qu = \dot{x}^* \tag{7}$$

$$\dot{\mu}^* = -(b + 2dx + \mu p) \tag{8}$$

$$c + 2eu + \mu q = 0 \tag{9}$$

From equation (9)

$$u^* = \frac{-\mu q - c}{2e} \tag{10}$$

Substituting equation (10) into equation (7)

$$\dot{x}^* = px - \frac{q^2\mu}{2e} - \frac{cq}{2e} \tag{11}$$

Expressing equations (8) and (11) in matrix form, we have

$$\begin{pmatrix} \dot{x}^* \\ \dot{\mu}^* \end{pmatrix} = \begin{pmatrix} p & \frac{-q^2}{2e} \\ -2d & -p \end{pmatrix} \begin{pmatrix} x \\ \mu \end{pmatrix} + \begin{pmatrix} -\frac{cq}{2e} \\ -b \end{pmatrix} \tag{12}$$

where

$$A = \begin{pmatrix} p & \frac{-q^2}{2e} \\ -2d & -p \end{pmatrix}, X = \begin{pmatrix} x \\ \mu \end{pmatrix} \text{ and } C = \begin{pmatrix} -\frac{cq}{2e} \\ -b \end{pmatrix} \tag{13}$$

The eigenvalues of A are obtained from its characteristic equation as

$$\lambda_1 = \frac{\sqrt{e(dq^2+ep^2)}}{e} \tag{14}$$

$$\lambda_2 = -\frac{\sqrt{e(dq^2+ep^2)}}{e} \tag{15}$$

and the corresponding eigenvectors are given as

$$U_1 = \begin{pmatrix} -\frac{1}{2}\frac{q^2}{e\left(\frac{\sqrt{e(dq^2+ep^2)}}{e} - p\right)} \\ 1 \end{pmatrix} \tag{16}$$

$$U_2 = \begin{pmatrix} -\frac{1}{2}\frac{q^2}{e\left(-\frac{\sqrt{e(dq^2+ep^2)}}{e} - p\right)} \\ 1 \end{pmatrix} \tag{17}$$

The complementary Solution of equation (12) is given as

$$V(t) = c_1\overrightarrow{U_1}e^{\lambda_1 t} + c_2\overrightarrow{U_2}e^{\lambda_2 t} \tag{18}$$

Equation (18) can be written in matrix form as

$$\begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\frac{q^2e^{\lambda_1 t}}{e\left(\frac{\sqrt{e(dq^2+ep^2)}}{e}-p\right)} & -\frac{1}{2}\frac{q^2e^{\lambda_2 t}}{e\left(-\frac{\sqrt{e(dq^2+ep^2)}}{e}-p\right)} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{19}$$

The method of Fundamental matrix is adopted for the general solution of equation (12). This implies that

$$X(t) = \phi(t)C \tag{20}$$

where

$$\phi(t) = \begin{pmatrix} -\frac{1}{2}\frac{q^2e^{\lambda_1 t}}{e\left(\frac{\sqrt{e(dq^2+ep^2)}}{e}-p\right)} & -\frac{1}{2}\frac{q^2e^{\lambda_2 t}}{e\left(-\frac{\sqrt{e(dq^2+ep^2)}}{e}-p\right)} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \tag{21}$$

hence,

$$\phi^{-1}(t) = \begin{pmatrix} -\frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)}{q^2\mathrm{e}^{\lambda_1 t}\sqrt{e\left(dq^2+ep^2\right)}} & \frac{-pe+\sqrt{e\left(dq^2+ep^2\right)}}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_1 t}} \\ \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)}{q^2\mathrm{e}^{\lambda_2 t}\sqrt{e\left(dq^2+ep^2\right)}} & \frac{pe+\sqrt{e\left(dq^2+ep^2\right)}}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_2 t}} \end{pmatrix} \tag{22}$$

From equation (12), if $C = g(s)$ is a constant, then

$$g(s) = \begin{pmatrix} -\frac{cq}{2e} \\ -b \end{pmatrix} \tag{23}$$

$$\phi^{-1}(t)g(s) = \begin{pmatrix} -\frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)}{q^2\mathrm{e}^{\lambda_1 t}\sqrt{e\left(dq^2+ep^2\right)}} & \frac{-pe+\sqrt{e\left(dq^2+ep^2\right)}}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_1 t}} \\ \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)}{q^2\mathrm{e}^{\lambda_2 t}\sqrt{e\left(dq^2+ep^2\right)}} & \frac{pe+\sqrt{e\left(dq^2+ep^2\right)}}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_2 t}} \end{pmatrix} \begin{pmatrix} -\frac{cq}{2e} \\ -b \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)c}{q\mathrm{e}^{\lambda_1 t}\sqrt{e\left(dq^2+ep^2\right)}e} - \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)b}{\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_1 t}} \\ -\frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)c}{q\mathrm{e}^{\lambda_2 t}\sqrt{e\left(dq^2+ep^2\right)}e} - \frac{\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)b}{\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_2 t}} \end{pmatrix} \tag{24}$$

(25)

$$\int_0^T \phi^{-1}(s)g(s)ds = \int_0^T \begin{pmatrix} \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)c}{2q\mathrm{e}^{\lambda_1 s}\sqrt{e\left(dq^2+ep^2\right)}e} - \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)b}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_1 s}} \\ -\frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)c}{2q\mathrm{e}^{\lambda_2 s}\sqrt{e\left(dq^2+ep^2\right)}e} - \frac{\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)b}{2\sqrt{e\left(dq^2+ep^2\right)}\mathrm{e}^{\lambda_2 s}} \end{pmatrix} ds$$

Let $Z = \frac{\left(-pe+\sqrt{e\left(dq^2+ep^2\right)}\right)\left(pe+\sqrt{e\left(dq^2+ep^2\right)}\right)c}{2q\sqrt{e\left(dq^2+ep^2\right)}e}$, $Y = \frac{b}{2\sqrt{e\left(dq^2+ep^2\right)}}$, $W = \sqrt{e\left(dq^2+ep^2\right)}$

Hence, equation (25) becomes

$$\int_0^T \phi^{-1}(s)g(s)ds = \int_0^T \begin{pmatrix} \frac{Z}{\mathrm{e}^{\lambda_1 s}} - \frac{Y(-pe+W)}{\mathrm{e}^{\lambda_1 s}} \\ -\frac{Z}{\mathrm{e}^{\lambda_2 s}} - \frac{Y(pe+W)}{\mathrm{e}^{\lambda_2 s}} \end{pmatrix} ds \tag{26}$$

(27)

$$\phi(t)\int_0^T \phi^{-1}(s)g(s)ds = \begin{pmatrix} -\frac{1}{2}\frac{q^2e^{\lambda_1 t}}{e\left(\frac{W}{e}-p\right)} & -\frac{1}{2}\frac{q^2e^{\lambda_2 t}}{e\left(-\frac{W}{e}-p\right)} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \int_0^T \begin{pmatrix} \frac{Z}{\mathrm{e}^{\lambda_1 s}} - \frac{Y(-pe+W)}{\mathrm{e}^{\lambda_1 s}} \\ -\frac{Z}{\mathrm{e}^{\lambda_2 s}} - \frac{Y(pe+W)}{\mathrm{e}^{\lambda_2 s}} \end{pmatrix} ds$$

This implies that the general solution is given by

$$\begin{pmatrix} -\frac{1}{2}\frac{q^2e^{\lambda_1 t}}{e\left(\frac{W}{e}-p\right)} & -\frac{1}{2}\frac{q^2e^{\lambda_2 t}}{e\left(-\frac{W}{e}-p\right)} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\frac{q^2e^{\lambda_1 t}}{e\left(\frac{W}{e}-p\right)} & -\frac{1}{2}\frac{q^2e^{\lambda_2 t}}{e\left(-\frac{W}{e}-p\right)} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \int_0^T \begin{pmatrix} \frac{Z}{\mathrm{e}^{\lambda_1 s}} - \frac{Y(-pe+W)}{\mathrm{e}^{\lambda_1 s}} \\ -\frac{Z}{\mathrm{e}^{\lambda_2 s}} - \frac{Y(pe+W)}{\mathrm{e}^{\lambda_2 s}} \end{pmatrix} ds$$

$$X(t) = \phi(t)C + \phi(t)\int_0^T \phi^{-1}(s)g(s)ds \tag{28}$$

2.1. Numerical Solution.

2.1.1. *Discretization Of Quadratic Optimal Control Problem Constrained By Ordinary Differential Equation.*

Discretizing equation (1) Using $\frac{1}{3}$ Simpson's Rule

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left\{f(x_0) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n)\right\} \tag{29}$$

Since $h = \dfrac{b-a}{n}$

$$\int_a^b f(x)dx = \frac{h}{3}\left\{ f(x_0) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n) \right\} \tag{30}$$

$$\int_0^T (a+bx+cu+dx^2+eu^2)dt = \int_0^T adt + \int_0^T bxdt + \int_0^T cudt + \int_0^T dx^2dt + \int_0^T eu^2dt \tag{31}$$

$$\int_0^T adt = anh$$

Since $h = \dfrac{b-a}{n} = \dfrac{T-0}{n}$

$\int_0^T (a+bx+cu+dx^2+eu^2)dt = anh + \dfrac{bh}{3}x_0 + \dfrac{dh}{3}x_0^2$
$+\dfrac{bh}{3}\left[4x_1 + 2x_2 + 4x_3 + 2x_4 + 4x_5 + 2x_6 + \cdots + 2x_{n-1} + 4x_{n-2} + x_n\right]$
$+\dfrac{ch}{3}\left[4u_1 + 2u_2 + 4u_3 + 2u_4 + 4u_5 + 2u_6 + \cdots + 2u_{n-2} + 4u_{n-1} + u_n\right]$
$+\dfrac{dh}{3}\left[4x_1^2 + 2x_2^2 + 4x_3^2 + 2x_4^2 + 4x_5^2 + 2x_6^2 + \cdots + 2x_{n-2}^2 + 4x_{n-1}^2 + x_n^2\right]$
$+\dfrac{eh}{3}\left[u_0^2 + 4u_1^2 + 2u_2^2 + 4u_3^2 + 2u_4^2 + 4u_5^2 + 2u_6^2 + \cdots + 2u_{n-2}^2 + 4u_{n-1}^2 + u_n^2\right]$

Let $C = anh + \dfrac{bh}{3}x_0 + \dfrac{dh}{3}x_0^2$, $\dfrac{bh}{3} = B_1$, $\dfrac{ch}{3} = C_1$, $\dfrac{dh}{3} = D_1$, $\dfrac{eh}{3} = E_1$

Re-writing in matrix form.

$$\int_0^T (a+bx+cu+dx^2+eu^2)dt = \begin{pmatrix} x_1 & x_2 & \cdots & x_{N-1} & x_N & u_0 & u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} 4B_1 \\ 2B_1 \\ 4B_1 \\ \vdots \\ B_1 \\ C_1 \\ 4C_1 \\ 2C_1 \\ 4C_1 \\ \vdots \\ 4C_1 \\ C_1 \end{pmatrix}$$

$$+ \begin{pmatrix} x_1 & x_2 & \cdots & x_{N-1} & x_N & u_0 & u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

$$\begin{pmatrix} 4D_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2D_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 4D_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4E_1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2E_1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \\ u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} + anh + \frac{bh}{3}x_0 + \frac{dh}{3}x_0^2$$

The above can be written as

$$Z^T S + Z^T M Z + C \tag{32}$$

where

$C = anh + \frac{bh}{3}x_0 + \frac{dh}{3}x_0^2$

$Z^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_{N-1} & x_N & u_0 & u_1 & u_2 & \cdots & u_n \end{pmatrix}$

The dimension of Z is $(2n+1)\times 1$

$$M = \begin{pmatrix} 4D_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2D_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 4D_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4E_1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2E_1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_1 \end{pmatrix}$$

The dimension of M is $(2n+1)\times(2n+1)$

where

$$m_{ij}=\begin{cases} D_1, & i=j=N \\ E_1, & j=i=N+1,2N+1 \\ 4D_1, & j=i \qquad i=1,3,5,\cdots,(N-1) \\ 2D_1, & j=i \qquad i=2,4,6,\cdots,(N-2) \\ 4E_1, & j=i \qquad i=N+2,N+4,N+6,\cdots,2N \\ 2E_1, & j=i \qquad i=N+3,N+5,N+7,\cdots,2N-1 \\ 0, & elsewhere \end{cases}$$

$$S=\begin{pmatrix} 4B_1 \\ 2B_1 \\ 4B_1 \\ \vdots \\ B_1 \\ C_1 \\ 4C_1 \\ 2C_1 \\ 4C_1 \\ \vdots \\ 4C_1 \\ C_1 \end{pmatrix} \text{ is of dimension } (2n+1)\times 1$$

2.1.2. *Discretization of the Constraint.* Using fifth order implicit method has formulated in [13] to discretize equation (2)

Given as

$$\begin{aligned} y_{i+1} &= y_i + h[\omega_1 f(x_i+\alpha h(y_i+\alpha(y_{i+1}-y_i))) + \omega_2 f(x_i+\beta h,(y_i+\beta(y_{i+1}-y_i))) \\ &+\omega_3 f(x_i+\gamma h,(y_i+\gamma(y_{i+1}-y_i))) + \omega_4 f(x_i+\tau h,(y_i+\tau(y_{i+1}-y_i))) + \omega_5 f(x_{i+h},y_{i+1})] \end{aligned} \tag{33}$$

where $\alpha = 0.05710419612460134, \beta = .27684301367379077, \gamma = 0.5835904324091858,$
$\tau = 0.8602401356748276, \omega_1 = 0.14371356081477416, \omega_2 = 0.28135601516952136,$
$\omega_3 = 0.3118265229640613, \omega_4 = 0.2231039010576481, \omega_5 = 0.03999999999402437$

(34)

$$\begin{aligned} x_{i+1} = x_i + h[&\omega_1 p(x_i + \alpha h + q(u_i + \alpha(u_{i+1} - u_i))) + \omega_2(px_i + \beta h + q(u_i + \beta(u_{i+1} - u_i))) \\ &+ \omega_3(px_i + \gamma h + q(u_i + \gamma(u_{i+1} - u_i))) + \omega_4(px_i + \tau h + q(u_i + \tau(u_{i+1} - u_i))) \\ &+ \omega_5(px_{i+1} + qu_{i+1})] \end{aligned}$$

$$\begin{aligned} (1 - h\omega_5)px_{i+1} = &(1 + h\omega_1 p + h\omega_2 p + h\omega_3 p + h\omega_4 p)x_i \\ &+ (h\omega_1 q - h\omega_1 q\alpha + h\omega_2 q - h\omega_2 q\beta + h\omega_3 q - h\omega_3 q\gamma + h\omega_4 q - h\omega_4 q\tau)u_i \\ &+ (qh\alpha\omega_1 + qh\omega_2\beta + qh\omega_3\gamma + qh\omega_4\tau + qh\omega_5)u_{i+1} \\ &+ (h^2\omega_1\alpha + h^2\omega_2\beta + h^2\omega_3\gamma + h^2\omega_4\tau) \end{aligned} \tag{35}$$

Hence,

$$x_{i+1} = A_1 x_i + A_2 u_i + A_3 u_{i+1} + A_4 \tag{36}$$

where

$$A_1 = \frac{1 + h\omega_1 p + h\omega_2 p + h\omega_3 p + h\omega_4 p}{(1 - h\omega_5)p}$$

$$A_2 = \frac{h\omega_1 q - h\omega_1 q\alpha + h\omega_2 q - h\omega_2 q\beta + h\omega_3 q - h\omega_3 q\gamma + h\omega_4 q - h\omega_4 q\tau}{(1 - h\omega_5)p}$$

$$A_3 = \frac{qh\alpha\omega_1 + qh\omega_2\beta + qh\omega_3\gamma + qh\omega_4\tau + qh\omega_5}{(1 - h\omega_5)p}$$

$$A_4 = \frac{h^2\omega_1\alpha + h^2\omega_2\beta + h^2\omega_3\gamma + h^2\omega_4\tau}{(1 - h\omega_5)p}$$

for $i = 0$

$$x_1 - A_2 u_0 - A_3 u_1 = A_1 x_0 + A_4 \tag{37}$$

for $i = 1$

$$x_2 - A_1 x_1 - A_2 u_1 - A_3 u_2 = A_4 \tag{38}$$

for $i = 2$

$$x_3 - A_1 x_2 - A_2 u_2 - A_3 u_3 = A_4 \tag{39}$$

$$\vdots$$

for $i = N-1$

$$x_N - A_1 x_{N-1} - A_2 u_{N-1} - A_3 u_N = A_4 \tag{40}$$

The above system of equations can be written in Matrix form as

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & -A_2 & -A_3 & 0 & 0 & \cdots & 0 & 0 \\
-A_1 & 1 & 0 & 0 & \cdots & 0 & 0 & -A_2 & -A_3 & 0 & \cdots & 0 & 0 \\
0 & -A_1 & 1 & \cdots & 0 & 0 & \cdots & 0 & -A_2 & -A_3 & \cdots & 0 & 0 \\
0 & 0 & -A_1 & \ddots & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -A_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & -A_2 & -A_3
\end{array}\right)
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}
=
\begin{pmatrix} A_4 + A_1 x_0 \\ A_4 \\ A_4 \\ \vdots \\ A_4 \\ A_4 \end{pmatrix}$$

$$[R|T]Z = D \tag{41}$$

$$WZ = D \tag{42}$$

Where $R \in \mathbb{R}^{N\times N}$, $\quad T \in \mathbb{R}^{N\times N+1}$, $\quad W \in \mathbb{R}^{N\times 2N+1}$, $\quad Z \in \mathbb{R}^{2N+1\times 1}$, $\quad D \in \mathbb{R}^{N\times 1}$

By parametric optimization, the discritized constrained optimal control problem becomes

$$\text{Min } Z^T S + Z^T M Z + C \tag{43}$$

$$\text{Subject to: } WZ = D \tag{44}$$

2.1.3. *Conversion of the Discretized Constrained Problem to Unconstrained Problem.* Augmented Lagrangian Method is used to transformed the discritized optimal control problem to an unconstrained problem given as

$$L(Z,\lambda,\mu) = Z^T S + Z^T MZ + C + \lambda^T |WZ - D| + \frac{\mu}{2}||WZ - D||^2 \tag{45}$$

$$L(Z,\lambda,\mu) = Z^T S + Z^T MZ + C + \lambda^T WZ - \lambda^T D + \frac{\mu}{2}W^T WZ^T Z - \mu WD^T Z + \frac{\mu}{2}D^T D$$

$$L(Z,\lambda,\mu) = Z^T M_k Z + Z^T S + W_k Z + D_k \tag{46}$$

where
$M_k = M + \frac{\mu}{2}W^T W$, $W_k = \lambda^T W - \mu D^T W$ and $D_k = \frac{\mu}{2}D^T D - \lambda^T D + C$.

3. Main Results

3.1. **Example1.** Consider the optimal control problem given as

$$\text{Minimize } \int_0^4 (0.1x(t) + 0.02u(t) + 0.01x^2(t) + 0.005u^2(t))dt \tag{47}$$

$$\dot{x} = 0.02x + 0.005u, \quad x(0) = 1 \tag{48}$$

Solution

By introducing the adjoint varible $\mu(t)$, the constrained OCP given in equations (47) and (48) is converted to an unconstrained problem. Hence, the hamiltonian fuction is given by

$$H(x,u,\mu) = 0.1x + 0.02u + 0.01x^2 + 0.005u^2 + \mu(0.02x + 0.005u) \tag{49}$$

The Euler-Lagrange system of equations for this hamiltonian function can be written as

$$0.02x + 0.005u = \dot{x}^* \tag{50}$$

$$\dot{\mu}^* = -(0.1 + 2(0.01)x + 0.02\mu) \tag{51}$$

$$0.02 + 0.01u + 0.005\mu = 0 \tag{52}$$

This implies that

$$u^* = \frac{-0.02 - 0.005\mu}{0.01} \tag{53}$$

In view of equation (53) equation (50) becomes

$$\dot{x}^* = 0.02x - 0.0025\mu - 0.01 \tag{54}$$

Expressing equations (51) and (54) in matrix form, we have

$$\begin{pmatrix} \dot{x}^* \\ \dot{\mu}^* \end{pmatrix} = \begin{pmatrix} 0.02 & -0.0025 \\ -0.02 & -0.02 \end{pmatrix} \begin{pmatrix} x \\ \mu \end{pmatrix} + \begin{pmatrix} -0.01 \\ -0.1 \end{pmatrix} \tag{55}$$

where

$$A = \begin{pmatrix} 0.02 & -0.0025 \\ -0.02 & -0.02 \end{pmatrix},\ X = \begin{pmatrix} x \\ \mu \end{pmatrix} \text{ and } C = \begin{pmatrix} -0.01 \\ -0.1 \end{pmatrix} \tag{56}$$

The eigenvalues of A are $\lambda_1 = 0.02121$ and $\lambda_2 = -0.02121$ and the corresponding eigenvectors are given as $U_1 = \begin{pmatrix} 0.8997 \\ -0.4366 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 0.06055 \\ 0.9982 \end{pmatrix}$. The complimentary solution of equation (55) is

$$V(t) = C_1 \begin{pmatrix} 0.8997 \\ -0.4366 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} 0.06055 \\ 0.9982 \end{pmatrix} e^{\lambda_2 t} \tag{57}$$

Using the initial condition $x(0) = 1$ and choosing $\mu(4) = 0$, the values of constants C_1 and C_2 are obtained as $C_1 = 1.07404$ and $C_2 = 0.5568$ respectively.

$$\text{Let } \phi(t) = \begin{pmatrix} 0.8997\,\mathrm{e}^{0.02121\,t} & 0.06055\,\mathrm{e}^{-0.02121\,t} \\ -0.4366\,\mathrm{e}^{0.02121\,t} & 0.9982\,\mathrm{e}^{-0.02121\,t} \end{pmatrix} \tag{58}$$

This implies that

$$\phi^{-1}(t) = \begin{pmatrix} 1.07969\,\mathrm{e}^{-0.02121\,t} & -0.06549\,\mathrm{e}^{-0.02121\,t} \\ 0.4722\,\mathrm{e}^{0.02121\,t} & 0.9732\,\mathrm{e}^{0.02121\,t} \end{pmatrix} \tag{59}$$

and

$$\phi(t)C = \begin{pmatrix} 0.9663\,\mathrm{e}^{0.02121\,t} + 0.03371\mathrm{e}^{-0.02121\,t} \\ -0.4689\,\mathrm{e}^{0.02121\,t} + 0.5558\,\mathrm{e}^{-0.02121\,t} \end{pmatrix} \tag{60}$$

From equation (55), $g(t) = \begin{pmatrix} -0.01 \\ -0.1 \end{pmatrix}$

$$\phi^{-1}(t)g(t) = \begin{pmatrix} -0.004248\,\mathrm{e}^{-0.02121\,t} \\ -0.1020\,\mathrm{e}^{0.02121\,t} \end{pmatrix} \tag{61}$$

$$\int \phi^{-1}(t)g(t)dt = \begin{pmatrix} 0.2003\,\mathrm{e}^{-0.02121\,t} \\ -4.8109\,\mathrm{e}^{0.02121\,t} \end{pmatrix} \tag{62}$$

$$\phi(t)\int \phi^{-1}(t)g(t)dt = \begin{pmatrix} -0.1111 \\ -4.88965 \end{pmatrix} \tag{63}$$

The general solution is

$$X(t) = \begin{pmatrix} 0.9993\,\mathrm{e}^{0.02121\,t} + 0.03371\,\mathrm{e}^{-0.02121\,t} - 0.1111 \\ -0.4689\,\mathrm{e}^{0.02121\,t} + 0.5558\,\mathrm{e}^{-0.02121\,t} - 4.8896 \end{pmatrix} \tag{64}$$

Since

$$X(t) = \begin{pmatrix} x^*(t) \\ \mu^*(t) \end{pmatrix} \tag{65}$$

this implies that

$$x^*(t) = 0.9663\,\mathrm{e}^{0.02121\,t} + 0.03371\,\mathrm{e}^{-0.02121\,t} - 0.1111 \tag{66}$$

and in view of (53)

$$u^* = 0.444468 + 0.23446\,\mathrm{e}^{0.02121\,t} - 0.277898\,\mathrm{e}^{-0.02121\,t} \tag{67}$$

hence the objective value is

$$\mathscr{J} = 0.4438797318 \tag{68}$$

3.1.1. *Convergence Analysis of Results.* Definition 4.1: Consider a sequence $\{x_k\} \in \mathbb{R}^n$ representing the solutions x_k that converges to a limit $x^*(x_k \to x^*)$. Using corresponding error function defined as $e(x_k) = e_k = |x_k - x^*|$, where $x_k \in \mathbb{R}$ and $e(x^*) \neq 0$, the convergence ratio ψ is given by

$$\psi = \lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_{k\to\infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||}, \quad \forall k \tag{69}$$

(1) When $0 < \psi < 1$, the convergence demonstrates a linear behaviour.
(2) When $\psi > 1$, the sequence diverges.
(3) When $\psi = 1$, the convergence displays sublinear behaviour.
(4) When $\psi = 0$, the convergence demonstrates super-linear characteristics.

TABLE 1. The Convergence Ratio Profile for Example 1

μ(Penalty Parameter)	$\mathscr{J}$(Objective Function Value)	ψ (convergence ratio)	Tolerance
1.0×10^0	0.4284513102	0.0109112649	10×10^{-5}
1.0×10^1	0.4413507076	0.0010903559	10×10^{-5}
1.0×10^2	0.4426411917	0.0001105253	10×10^{-5}
1.0×10^3	0.4427702455	0.0000109030	10×10^{-5}
1.0×10^4	0.4427841053	0.0000000000	10×10^{-5}

3.1.2. *Discussion of Results.*

- The function f(x) as defined in equation(46) where M_k, Z, S, W_k and D_k are input as matrices
- The initial guess is as x_0 i.e the initial condition
- The tolerance for Convergence was set as 10×10^{-5}

3.1.3. *Initialization.*

(1) Setting k=0
(2) The initial guess was set as x_0
(3) The initial gradient $g_0 = \bigtriangledown f(x_0)$ was computed
(4) The search direction $p_0 = -g_0$

(5) An appropriate step size α_k was chosen along with the search direction p_k

(6) Updating the solution x_{k+1} and the gradient g_{k+1}

$$x_{k+1} = x_k + \alpha_k p_k$$

$$g_{k+1} = \bigtriangledown f(x_{k+1})$$

(7) Updating the Conjugate Direction p_{k+1}

$$\beta_{k+1} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}$$

$$p_{k+1} = -g_{k+1} + \beta_{k+1} p_k$$

(8) Checking if the convergence criteria as defined above is satisfied otherwise back to step 1 with k incremented by 1.

The numerical analysis of the Conjugate Gradient Method (CGM) Using MATLAB software on a 64-bit Dell Vostro with a core i7 Intel processor for different penalty parameters reveals insights into its performance on a specific optimization problem. When the penalty parameter is set to 1.0×10^0, the CGM produces an objective value of 0.4284513102 with a convergence ratio of 0.0109112649 and a tolerance of 1×10^{-5}. As the penalty parameter increases to 1.0×10^4, the objective value converges to 0.4427841053, accompanied by an extremely small convergence ratio of 0.000000000000.

The approximate solutions of the discretized OCP are obtained from FICO Xpress Mosel Using equations(43) and (44) and its correponding matrices as used in the CGM are first of all stored in a .txt file placed in the directory in which the mosel is located and then initialized on the Mosel.

- To improve efficiency, Mosel integrates external solvers such as IPOPT (Interior Point Optimizer) or SNOPT (Sparse Non-linear Optimization) to generate optimal control solutions for Optimal Control Problems (OCPs)
- It uses the Newton Barrier Algorithm to iterate through solutions using Interior Point Optimizer satisfying the Karush-Kuhn-Tucker (KKT) conditions on the feasible regions to find a close approximation of an Optimal Solution

The function is then minimized in order to obtain the Objective Value. Comparing these results with the objective value obtained from Fico Xpress Mosel version 6.43 on a 64-bit Dell Vostro with a core i7 Intel processor (0.4438797490), and the Analytical Solution (0.4438797318), it becomes evident that Fico Xpress Mosel consistently outperforms the CGM method for this specific optimization problem. The objective values from Fico Xpress Mosel and the Analytical Solution exhibit a very close alignment, suggesting that Fico Xpress Mosel provides a more solution that compares favourably well with the analytical solution.

3.2. Example 2. Consider the optimal control problem given as

$$\text{Minimize} \int_0^{10} (2x(t)+4u(t)+3x^2(t)+u^2(t))dt \tag{70}$$

$$\dot{x} = 3x+u, \quad x(0) = 2 \tag{71}$$

Solution

By introducing the adjoint varible $\mu(t)$, the constrained OCP given in equations (70) and (71) is converted to an unconstrained problem. Hence, the hamiltonian fuction is given by

$$H(x,u,\mu) = 2x(t)+4u(t)+3x^2(t)+u^2(t)+\mu(3x+u) \tag{72}$$

The Euler-Lagrange system of equations for this hamiltonian function can be written as

$$3x+u = \dot{x}^* \tag{73}$$

$$\dot{\mu}^* = -(2+6x+3\mu) \tag{74}$$

$$4+2u+\mu = 0 \tag{75}$$

This implies that

$$u^* = \frac{-4-\mu}{2} \tag{76}$$

In view of equation (76) equation (73) becomes

$$\dot{x}^* = 3x - \frac{1}{2}\mu - 2 \tag{77}$$

Equations (77) and (74) can be represented in matrix form as

$$\begin{pmatrix} \dot{x}^* \\ \dot{\mu}^* \end{pmatrix} = \begin{pmatrix} 3 & -\frac{1}{2} \\ -6 & -3 \end{pmatrix} \begin{pmatrix} x \\ \mu \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{78}$$

where

$$A = \begin{pmatrix} 3 & -\frac{1}{2} \\ -6 & -3 \end{pmatrix}, X = \begin{pmatrix} x \\ \mu \end{pmatrix} \text{ and } C = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{79}$$

The eigenvalues of A are $\lambda_1 = 3.4641$ and $\lambda_2 = -3.4641$ respectively and the corresponding eigenvectors are given as: $U_1 = \begin{pmatrix} -1.07735 \\ 1 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 0.07735 \\ 1 \end{pmatrix}$. The complimentary solution of equation (78) is

$$V(t) = C_1 \begin{pmatrix} -1.07735 \\ 1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} 0.07735 \\ 1 \end{pmatrix} e^{\lambda_2 t} \tag{80}$$

Using the initial condition $x(0) = 1$ and choosing $\mu(4) = 0$, the values of constants C_1 and C_2 are obtained as $C_1 = -2.1075 \times 10^{-29}$ and $C_2 = 25.8565$.

$$\text{Let } \phi(t) = \begin{pmatrix} -1.07735\,\mathrm{e}^{3.4641t} & 0.07735\,\mathrm{e}^{-3.4641t} \\ \mathrm{e}^{3.4641t} & \mathrm{e}^{-3.4641t} \end{pmatrix} \tag{81}$$

Hence

$$\phi^{-1}(t) = \begin{pmatrix} -0.8660\,\mathrm{e}^{-3.4641t} & 0.06699\,\mathrm{e}^{-3.4641t} \\ 0.8660\,\mathrm{e}^{3.4641t} & 0.9330\,\mathrm{e}^{3.4641t} \end{pmatrix} \tag{82}$$

then

$$\phi(t)C = \begin{pmatrix} 2.2706 \times 10^{-29}\,\mathrm{e}^{3.4641t} + 2.000000275\,\mathrm{e}^{-3.4641t} \\ -2.1076 \times 10^{-29}\,\mathrm{e}^{3.4641t} + 25.8565\,\mathrm{e}^{-3.4641t} \end{pmatrix} \tag{83}$$

From equation (78)

$$g(t) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$\phi^{-1}(t)g(t) = \begin{pmatrix} 1.59802\,\mathrm{e}^{-3.4641\,t} \\ \\ -3.5980\,\mathrm{e}^{3.4641\,t} \end{pmatrix} \tag{84}$$

$$\int \phi^{-1}(t)g(t)dt = \begin{pmatrix} -0.4613\,\mathrm{e}^{-3.4641\,t} \\ \\ -1.03865\,\mathrm{e}^{3.4641\,t} \end{pmatrix} \tag{85}$$

$$\phi(t)\int \phi^{-1}(t)g(t)dt = \begin{pmatrix} 0.4166 \\ \\ -1.49995 \end{pmatrix} \tag{86}$$

The general solution is

$$X(t) = \begin{pmatrix} 2.2706\times 10^{-29}\,\mathrm{e}^{3.4641\,t} + 2.000000275\,\mathrm{e}^{-3.4641\,t} + 0.4166 \\ \\ -2.1076\times 10^{-29}\,\mathrm{e}^{3.4641\,t} + 25.8565\,\mathrm{e}^{-3.4641\,t} - 1.49995 \end{pmatrix} \tag{87}$$

Since

$$X(t) = \begin{pmatrix} x^*(t) \\ \mu^*(t) \end{pmatrix} \tag{88}$$

this implies that

$$x^* = 2.2706\times 10^{-29}\,\mathrm{e}^{3.4641\,t} + 2.000000275\,\mathrm{e}^{-3.4641\,t} + 0.4166 \tag{89}$$

and in view of equation (76)

$$u^* = -1.250025 + 1.0538\times 10^{-29}\,\mathrm{e}^{3.4641\,t} - 12.9283\,\mathrm{e}^{-3.4641\,t} \tag{90}$$

Hence the objective function value is

$$\mathscr{J} = 2.019819716 \tag{91}$$

3.2.1. *Discussion of Results.*

TABLE 2. The Convergence Ratio Profile for Example 2

μ(Penalty Parameter)	$\mathscr{J}$(Objective Function Value)	ψ (convergence ratio)	Tolerance
1.0×10^{0}	-0.2613144906	0.9228396360	10×10^{-5}
1.0×10^{1}	1.5682893902	0.22477410505	10×10^{-5}
1.0×10^{2}	1.9536035419	0.0205791719	10×10^{-5}
1.0×10^{3}	2.0031247797	0.00000082232	10×10^{-5}
1.0×10^{4}	2.0085540057	0.0000000000	10×10^{-5}

The same process of initialization and assumptions using CGM as used in example 1 is used in example 2. The numerical analysis using the Conjugate Gradient Method (CGM) for various penalty parameters provides valuable insights into the optimization process. For the specific penalty parameters investigated, ranging from 1.0×10^{0} to 1.0×10^{4}, distinct patterns emerge in terms of objective values and convergence ratios. Starting with a penalty parameter of 1.0×10^{0}, the CGM yields an objective value of -0.2613144906 and a convergence ratio of 0.9228396360, suggesting a considerable convergence towards the optimal solution with a relatively high sensitivity to changes in the penalty strength.

As the penalty parameter increases, a notable trend unfolds: the objective value tends to stabilize, indicating diminishing sensitivity to further increases in penalty strength. Simultaneously, the convergence ratio experiences a significant decline, signifying accelerated convergence to the solution. This trend is particularly evident for the highest penalty parameter (1.0×10^{4}), where the convergence ratio becomes exceptionally small, indicative of a remarkably swift convergence of the algorithm to the solution.

Comparing these CGM results with the objective value obtained from Fico Xpress Mosel (2.019818802), a distinction in accuracy becomes apparent. Fico Xpress Mosel produces a solution remarkably close to the Analytical Solution (2.019819716) with minimal error compared to the CGM results. This underscores the superior accuracy of Fico Xpress Mosel in providing solutions for this specific optimization problem.

4. Conclusion

This paper presents exact solutions for general quadratic optimal control problems constrained by first-order ordinary differential equations. It applies necessary conditions of optimality to the Hamiltonian function, solving resulting non-homogeneous first-order ODEs using the method of fundamental matrix to obtain analytical solutions for state variables, control variables, and objective function values. For numerical solutions, the discretized unconstrained optimal control problem is tackled using the Conjugate Gradient Method (CGM) and FICO Xpress Mosel. The CGM's convergence is analyzed against predefined criteria given in Tables 1 and 2 , demonstrating superlinear convergence under specific conditions. FICO Xpress Mosel consistently produces objective function values closely matching analytical solutions, showcasing its reliability and accuracy. Overall, results indicate that FICO Xpress Mosel provides solutions more closely aligned with analytical solutions compared to CGM.

5. Recommendation

Future research endeavors should prioritize the utilization of the Fico Xpress model in specific domains, particularly in addressing optimal control problems characterized by ordinary differential equations with multiple constraint to harness the strengths of the Fico Xpress model in handling complex optimization scenarios involving intricate constraints and contributing to a deeper understanding of its applicability and efficacy in these specific area

Conflict of Interests

The authors declare that there is no conflict of interests.

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