Available online at http://scik.org
J. Math. Comput. Sci. 2024, 14:11
https://doi.org/10.28919/jmcs/8754
ISSN: 1927-5307

STUDY OF SUBDIFFUSION EQUATION WITH DIFFERENT FRACTIONAL DERIVATIVE AND THEIR ANALYSIS

JAGDISH SONAWANE[1,*], BHAUSAHEB SONTAKKE[2], KALYANRAO TAKALE[3]

[1]Department of Mathematics, GES R. H. Sapat College of Engineering, Management and Research, Nashik-422005 (M.S.), India

[2]Department of Mathematics, Pratishthan Mahavidyalaya, Paithan,Aurangabad, (M.S.), India

[3]Department of Mathematics, RNC Arts, JDB Commerce and NSC Science College, Nashik-Road, Nashik, India

Abstract. In this paper, we obtained a solution of the fractional subdiffusion equation with initial and boundary conditions. The fractional derivative is of type Caputo, Caputo-Fabrizio, and Atangana-Baleanu-Caputo sense. Furthermore, we developed the Crank-Nicolson finite difference method to obtain a numerical solution for the subdiffusion equation. We compare numerical solutions obtained by using various fractional derivatives and representing graphically by using Python software. Also, we discussed the stability and convergence of the scheme.

Keywords: Fractional subdiffusion equations; Atangana-Balenau-Caputo fractional derivatives; Caputo fractional derivatives; Caputo-Fabirizio fractional derivatives; Mittag-Leffler function; Python.

2020 AMS Subject Classification: 35R11, 80M20, 65M06.

1. Introduction

Recently, many researchers have found interest in fractional calculus. Many scientists and engineers have used fractional integrals and derivatives to simulate various phenomena in biology, finance, signal processing, material science, physics, etc. Classical definitions of fractional

*Corresponding author
E-mail address: jagdish.sonawane555@gmail.com
Received July 11, 2024

derivatives like Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, etc. have limitations in modeling non-local dynamics due to the singular kernel. To overcome this difficulty, researchers developed new definitions of the fractional derivative with a non-singular kernel, i.e., the Caputo-Fabrizio fractional derivative, the Atangana-Baleanu fractional derivative, etc.

In physics, fractional derivatives are used to model anomalous diffusion (i.e., subdiffusion or superdiffusion), where particles spread in a power law manner. Fractional derivatives describe anomalous diffusion processes more accurately than integer-order derivatives due to the advantage of the short-term memory principle. The subdiffusion equation is obtained by replacing the integer order derivative with the time-fractional derivative of order $(0 < \alpha < 1)$ in the classical diffusion equation.

In this paper, the following fractional subdiffusion equation is considered

$$\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha} = d\frac{\partial^2 \theta(x,t)}{\partial x^2}, \tag{1}$$

for $0 \le x \le L, 0 \le t \le T, 0 < \alpha < 1$.

$$\text{initial condition} \quad \theta(x,0) = g(x),\ 0 \le x \le L \tag{2}$$

$$\text{boundary conditions } \theta(0,t) = 0, \theta(L,t) = 0,\ 0 \le t \le T \tag{3}$$

where d is diffusion coefficient and $\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha}$ is fractional derivative of type Caputo, Caputo-Fabrizio, Atangana -Baleanu-Caputo sense.

Nowadays, a wide range of techniques are available in the literature for solving fractional diffusion equations. Analytical methods like the Laplace transform method, the Fourier transform method, the Mellin transform method, and so on do not work well due to the complexity and non-locality of fractional derivatives. Therefore, researchers need to establish numerical methods for solving such problems. Nowadays, many numerical methods exist for solving fractional differential equations, like the finite difference method, the finite element method, the spectral collocation method, etc. The solution of the fractional subdiffusion equation is obtained by Yuste's (2005) explicit finite difference method [12], Lewandowska's (2006), Mainardi's (2007) using an analytical method [2], and Liu (2009) using an implicit finite difference scheme [3].

Lukashcuuk (2011), Hristov (2011) using fractional time integral balance method [7], Mustapha by finite difference method [5], Wang (2014) by using compact finite difference method [11], Zeng (2015) using finite element method, Dehegan (2016) by using Legendre spectral element method [6], Yang (2019) using fractional linear multistep method, Qui (2020) by Galerkin method, and Dehestani (2022) using collocation method.

In this work, we developed the Crank-Nicolson finite difference method for the time-fractional subdiffusion equation, incorporating not only the Caputo fractional derivative but also the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives.

This paper is organized as follows:

In Section 2, we discussed various definitions of fractional derivatives. Section 3 is devoted to the discretization of Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives. The crank-Nicolson finite difference method for the time fractional subdiffusion equation is developed in Section 4. We discussed stability and convergence in Section 5. Graphical solutions and comparative analysis are discussed in Section 6.

2. Preliminaries

Definition 2.1. ***Riemann-Liouville Fractional integral:*** [8]

If $f(t) \in C[0,1]$ then

$$ {}_0J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{4}$$

is called the Riemann-Liouville fractional integral of order α. Where $\alpha \in (-\infty, \infty)$.

Definition 2.2. ***Riemann-Liouville Fractional Derivative:***[1]

If $f(t) \in C[0,1]$ then

$$ {}_0^RD_t^{\alpha} f(t) = D_0^1 I_t^{1-\alpha} f(t) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)ds, \tag{5}$$

is called the Riemann-Liouville fractional derivative of order α. Where $\alpha \in (0,1)$.

Definition 2.3. ***M.Caputo Fractional Derivative:***[8]

If $f(t) \in C[0,1]$ then

$$ {}_0^CD_t^{\alpha} f(t) = {}_0 I_t^{1-\alpha} D^1 f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{'}(s)ds, \tag{6}$$

is called the Caputo fractional derivative of order α. Where $\alpha \in (0,1)$.

Let $M(\alpha)$ is normalization function which satisfies $M(0) = M(1) = 1$

Definition 2.4. ***Caputo-Fabrizio Fractional Derivative:*** [5]

If $f(t) \in H^1(0,1)$ then

$$\frac{^{CF}\partial^\alpha f(t)}{\partial t^\alpha} = \frac{M(\alpha)}{(1-\alpha)} \int_0^t exp[\frac{-\alpha}{1-\alpha}(t-s)]f'(s)\,ds, \tag{7}$$

is called the Caputo-Fabrizio fractional derivative of order α. Where $\alpha \in (0,1)$.

Definition 2.5. ***Atangana-Baleanu-Caputo Fractional Derivative:***[4]

If $f(t) \in H^1(0,1)$ then

$$\frac{^{ABC}\partial^\alpha f(t)}{\partial t^\alpha} = \frac{M(\alpha)}{(1-\alpha)} \int_0^t E_\alpha[\frac{-\alpha}{1-\alpha}(t-s)^\alpha]f'(s)\,ds, \tag{8}$$

is called the Atangana-Baleanu-Caputo fractional derivative of order α. Where $\alpha \in (0,1)$ and $E_\alpha(z)$ is Mittag-Leffler function.

Definition 2.6. ***Mittag-Leffler function of one parameter*** [8]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha K + 1)} \tag{9}$$

Definition 2.7. ***Mittag-Leffler function of two parameter*** [8]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha K + \beta)} \tag{10}$$

3. Discretization of Fractional Derivatives

To develop discretization of different time fractional derivatives, we define $x_i = i\Delta x, i = 0,1,\ldots,M$ and $t_n = n\Delta t, n = 0,1,\ldots,N$; where $\Delta x = \frac{L}{M}$ and $\Delta t = \frac{T}{N}$.

Caputo Fractional Derivative: Discretization of the Caputo time fractional derivative of order α given by definition (6) at time level $t = t_{n+1}$,is as follow

$$\left(\frac{^C\partial^\alpha \theta}{\partial t^\alpha}\right)_n^i = \frac{\Delta t^{-\alpha}\, {}^C d_0}{\Gamma(2-\alpha)}\left[\theta_i^{n+1} - \theta_i^n\right] + \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{n} \left[\theta_i^{n-j+1} - \theta_i^{n-j}\right] {}^C d_j \tag{11}$$

$$\text{where} \quad {}^C d_j = [(j+1)^{1-\alpha} - j^{1-\alpha}] \text{and } {}^C d_0 = 1 \tag{12}$$

Caputo-Fabrizio Fractional Derivative: Discretization of the Caputo-Fabrizio time fractional derivative of order α given by definition (7) at time level $t = t_{n+1}$ with $M(\alpha) = 1$ and $\frac{-\alpha}{1-\alpha} = \alpha_1$ given by H. Yépez-Martínez and J.F. Gómez-Aguilar, is

$$\left(\frac{{}^{CF}\partial^{\alpha}\theta}{\partial t^{\alpha}}\right)_n^i = \frac{{}^{CF}d_0}{\alpha\Delta t}\left[\theta_i^{n+1} - \theta_i^n\right] + \frac{1}{\alpha\Delta t}\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]{}^{CF}d_j \tag{13}$$

$$\text{where} \qquad {}^{CF}d_j = \left[exp[\frac{-\alpha}{1-\alpha}j\Delta t] - \exp[\frac{-\alpha}{1-\alpha}(j+1)\Delta t]\right] \tag{14}$$

Atangana-Baleanu-Caputo Fractional Derivative: Discretization of the Atangana-Baleanu-Caputo(ABC) time fractional derivative of order α given by definition (9) at time level $t = t_{n+1}$ with $M(\alpha) = 1$, and $\alpha_1 = \frac{-\alpha}{1-\alpha}$, is

$$\left(\frac{{}^{ABC}\partial^{\alpha}\theta(x,t)}{\partial t^{\alpha}}\right)_i^n = \frac{{}^{ABC}d_0}{(1-\alpha)\Delta t}\left[\theta_i^{n+1} - \theta_i^n\right] + \frac{1}{(1-\alpha)\Delta t}\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]{}^{ABC}d_j \tag{15}$$

where

$${}^{ABC}d_j = \left[(j+1)E_{\alpha,2}\left[\alpha_1(j+1)^{\alpha}(\Delta t)^{\alpha}\right] - jE_{\alpha,2}\left[\alpha_1 j^{\alpha}(\Delta t)^{\alpha}\right]\right] \tag{16}$$

and

$$\int\left[E_{\alpha}[\alpha_1(t_{n+1}-s)^{\alpha}]\right]ds = -(t_{n+1}-s)\left[E_{\alpha,2}[\alpha_1(t_{n+1}-s)^{\alpha}]\right]$$

where $E_{\alpha,2}$ is Mittag-Leffler function of two parameter as given in definition (10).

Moreover, the discretization of time fractional derivative can be represented as

$$\left(\frac{\partial^{\alpha}\theta(x,t)}{\partial t^{\alpha}}\right)_i^n = \zeta d_0\left[\theta_i^{n+1} - \theta_i^n\right] + \zeta\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]d_j \tag{17}$$

Where

$$\zeta = \begin{cases} \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}, & \text{for Caputo time fractional derivative;} \\ \frac{1}{\alpha\Delta t}, & \text{for Caputo-Fabrizio time fractional derivative;} \\ \frac{1}{(1-\alpha)\Delta t} & \text{for ABC time fractional derivative} \end{cases}$$

And

$$d_j = \begin{cases} {}^{C}d_j, & \text{for Caputo time fractional derivative;} \\ {}^{CF}d_j, & \text{for Caputo-Fabrizio time fractional derivative;} \\ {}^{ABC}d_j & \text{for ABC time fractional derivative} \end{cases}$$

Space Derivative: Discretization of space derivative given in (1) by using Crank-Nicolson Method as follow:

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_n^i = \frac{1}{(2\Delta x)^2}\left[U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1} + U_{i-1}^n - 2U_i^n + U_{i+1}^n\right] + O((\Delta x)^2) \tag{18}$$

4. Main Results

In this section, we solved fractional subdiffusion equation (1)-(3) by using the Crank-Nicolson Finite difference method. we substitute equation (17) and (18) in equation (1), we get

$$\zeta d_0\left[\theta_i^{n+1} - \theta_i^n\right] + \zeta\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]d_j = \frac{d}{2(\Delta x)^2}\left[\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n\right]$$

$$\theta_i^{n+1} - \theta_i^n + \frac{1}{d_0}\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]d_j = \frac{d}{2(\Delta x)^2\zeta d_0}\left[\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n\right]$$

$$\theta_i^{n+1} - \theta_i^n = r\left[\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n\right] - h\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]d_j$$

where $r = \frac{d}{2(\Delta x)^2\zeta d_0}$ and $h = \frac{1}{d_0}$

(19)

$$-r\theta_{i-1}^{n+1} + (1+2r)\theta_i^{n+1} - r\theta_{i+1}^{n+1} = r\theta_{i-1}^n + (1-2r)\theta_i^n + r\theta_{i+1}^n - h\sum_{j=1}^{n}\left[\theta_i^{n-j+1} - \theta_i^{n-j}\right]d_j$$

Initial condition (2) is discretize as

$$\theta_i^0 = g(x_i),\ i = 1, 2, \ldots, M \tag{20}$$

And boundary conditions (3) are discretize as

$$\theta_0^n = 0, \text{and } \theta_M^n = 0,\ n = 0, 1, 2, \ldots, N \tag{21}$$

The complete discretization of subdiffusion equation (1)-(3) is given by

(22)

$$-r\theta_{i-1}^1 + (1+2r)\theta_i^1 - r\theta_{i+1}^1 = r\theta_{i-1}^0 + (1-2r)\theta_i^0 + r\theta_{i+1}^0 \text{ for } n = 0$$

(23)

$$-r\theta_{i-1}^{n+1} + (1+2r)\theta_i^{n+1} - r\theta_{i+1}^{n+1} = r\theta_{i-1}^n + \Lambda\theta_i^n + r\theta_{i+1}^n + h\sum_{j=1}^{n-1}(d_j - d_{j+1})\,\theta_i^{n-j} + h\,d_n\theta_i^0$$

for $n \geq 1$

$$\text{initial condition} \qquad \theta_i^0 = g(x_i),\ i = 1, 2, \dots, M \tag{24}$$

$$\text{boundary conditions } \theta_0^n = 0, \text{and } \theta_M^n = 0,\ n = 0, 1, 2, \dots, N \tag{25}$$

with $r = \frac{d}{2(\Delta x)^2 \zeta d_0}$, $\Lambda = 1 - 2r - h d_1$ and $h = \frac{1}{d_0}$, and $i = 1, 2, \dots, M$; $n = 0, 1, 2, \dots, N$. Note that value of ζ and d_j depends on choice of fractional derivative as discussed in previous section. Hence, matrix form of discretized Crank-Nicolson finite difference scheme (22)-(25) is as follow

$$A\theta^1 = B\theta^0, \qquad \text{for } n = 0, \tag{26}$$

$$A\theta^{n+1} = F\theta^n + h\sum_{j=1}^{n-1}\left[d_j - d_{j+1}\right]\theta^{n-j} + h d_n U^0 \qquad \text{for } n \geq 1, \tag{27}$$

where $\theta^n = \left(\theta_1^n, \theta_2^n, \dots, \theta_{M-1}^n\right)^T$ is the constant matrix, and

$$A = \begin{pmatrix} 1+2r & -r & & & \\ -r & 1+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 1+2r & -r \\ & & & -2r & 1+2r \end{pmatrix},$$

$$B = \begin{pmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{pmatrix}, \quad F = \begin{pmatrix} \Lambda & r & & & \\ r & \Lambda & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & \Lambda & r \\ & & & r & \Lambda \end{pmatrix},$$

with $r = \frac{d}{2(\Delta x)^2 \zeta d_0}$, $\Lambda = 1 - 2r - h d_1$ and $h = \frac{1}{C d_0}$.

5. Stability and Convergence

5.1. Stability.

Lemma 5.1. *The eigenvalues of $M \times M$ tri-diagonal matrix*

$$\begin{pmatrix} a_1 & a_2 & & & & & \\ a_3 & a_1 & a_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_3 & a_1 & a_2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_3 & a_1 & a_2 \\ & & & & & a_3 & a_1 \end{pmatrix}$$

are given as

$$\lambda_i = a_1 + 2\sqrt{a_2 a_3} cos(\frac{i\pi}{M+1})$$

where a_1, a_2 and a_3 are either real or complex numbers[9].

Lemma 5.2. *If $\lambda_i(A), i = 1, 2, \ldots, M-1$ represents eigen values of matrix A, then following conditions are hold*

(i) $\lambda_i(A) \geq 1$,

(ii) $\|A^{-1}\|_2 \leq 1$, *where $\|.\|_2$ is second norm of matrix.*

Proof. By the Gerschgorin's circle theorem [9] , each eigenvalue λ_i of a square matrix A lies in at least one of the following circle

$$|\lambda - a_{ij}| \leq \sum_{i=1, i\neq j}^{M} a_{ij}, \ j = 1, 2, 3, \ldots M. \tag{28}$$

Thus each Eigen value of square matrix A satisfy at least one of the following inequality

$$|\lambda| \leq \sum_{i=1}^{M} |a_{ij}| \tag{29}$$

$$|\lambda| \geq |a_{ij}| - \sum_{i=1, i\neq j}^{M} |a_{ij}| \tag{30}$$

Now, we use inequality (30) to prove the condition (i) for the matrix A.

$$|\lambda_1(A)| \geq |(1+2r)-r| = 1+r \geq 1$$

$$|\lambda_2(A)| \geq |(1+2r)-r-r| = 1$$

$$|\lambda_3(A)| \geq |(1+2r)-r-r| = 1$$

$$\vdots$$

$$|\lambda_{M-1}(A)| \geq |(1+2r)-r| = 1+r \geq 1$$

Thus $|\lambda_j| \geq 1,\ \ j = 1, 2, 3 \ldots, M$.

To prove condition (ii), we have

$$\|A\|_2 = \max_{1 \leq j \leq n} |\lambda_j(A)|$$

Therefore, from condition (i), we get $\|A\|_2 \geq 1$

Hence $\|A^{-1}\|_2 \leq 1$.

This completes the proof. □

Lemma 5.3. *The discretized fractional order Crank-Nicolson finite difference scheme with initial and boundary conditions given by* (22)-(25) *is solvable for each time step unconditionally.*

Proof. To prove the solvability of equations (22)-(25), it is enough to prove that matrix A is invertible [9, 10]. We observed that the first and last row of matrix A is diagonally dominant. For other rows, the diagonal element is $1+2r$ and the sum of the absolute values of the non-diagonal element in the same row is,

$$|(-r)| + |(-r)| = 2r$$

Hence, for each row, we have $1+2r > 2r$. Thus, matrix A is strictly diagonally dominant. Hence, matrix A is invertible. This shows that the solvability of the finite difference scheme. □

Lemma 5.4. *If $\lambda_i(B)$ and $\lambda_i(F), i = 1, 2, \ldots, M$ represents the eigenvalues of B and F respectively, then following conditions are hold*

(i) $|\lambda_i(B)| \leq 1, \quad |\lambda_i(F)| \leq 1, \quad i = 1, 2, \ldots, M$

(ii) $\|B\|_2 \leq 1, \quad \|F\|_2 \leq 1, \quad i = 1, 2, \ldots, M$

Theorem 5.1. *The numerical solution of the fractional order Crank-Nicolson finite difference scheme* (22)-(25) *for fractional order subdiffusion equation* (1)-(3)*, is unconditionally stable.*

Proof. To prove the numerical solution of finite difference scheme is unconditionally stable, we will prove that

$$\|\theta^n\|_2 \leq K\|\theta^0\|_2$$

where K is positive integer independent of x and t.
For $n = 1$

$$\theta^1 = A^{-1}B\theta^0\|\theta^1\|_2 \leq \|A^{-1}\|_2\,\|B\|_2\,\|\theta^0\|_2 \leq K\|\theta^0\|_2$$

Thus result is true for $n = 1$.
For $n \leq m$, let us assume that $\|\theta^m\|_2 \leq K\|\theta^0\|_2$
Now, for $n = m+1$

$$\theta^{m+1} = A^{-1}F\,\theta^m + hA^{-1}\sum_{j=1}^{m-1}\left[d_j - d_{j+1}\right]\theta^{m-j} + hA^{-1}d_m\theta^0$$

$$\begin{aligned}
\|\theta^{m+1}\|_2 &\leq \|\theta^m\|_2 + \sum_{j=1}^{m-1} h\left[d_j - d_{j+1}\right]\|\theta^{m-j}\|_2 + h\,d_m\,\|U^0\|_2 \\
&= \|\theta^m\|_2 + h\left[(d_1-d_2)\,\|\theta^{m-1}\|_2 + (d_2-d_3)\,\|\theta^{m-2}\|_2 + \ldots + (d_{m-1}-d_m)\,\|\theta^1\|_2\right] \\
&\quad + h\,d_m\,\|\theta^0\|_2 \\
&\leq K_1\,\|\theta^0\|_2 + h\left[(d_1-d_2) + (d_2-d_3) + \ldots + (d_{m-1}-d_m)\right]K_2\,\|\theta^0\|_2 + h\,d_m\,\|U^0\|_2 \\
&\leq \left[K_1 + h\,d_1 + (1-K_2)d_m\right]\|\theta^0\|_2 = K\|\theta^0\|_2
\end{aligned}$$

Hence, by induction, for all n, we have $\|\theta^n\|_2 \leq K\|\theta^0\|_2$.where K is a positive number independent of x and t. Note that, here we use notation d_j as common notation for ${}^{C}d_j$, ${}^{CF}d_j$, ${}^{ABC}d_j$. Therefore, this shows that the solution obtained by the scheme is unconditionally stable. This completes the proof. □

5.2. Convergence: In this section, we discuss the convergence of the scheme. Let Ω be the region $[0,L]\times[0,T]$.

We introduce the vector,

$$\overline{\theta}^n = \Big(\overline{\theta}(x_0,t_n), \overline{\theta}(x_1,t_n), \overline{\theta}(x_2,t_n), \ldots, \overline{\theta}(x_M,t_n)\Big)^T,$$

of size M+1, which represent the exact solution of the fractional order subdiffusion equation (1)-(3) at time level t_n. Let $\tau^n = \Big(\tau_1^n,\ \tau_2^n,\ \tau_3^n,\ \ldots,\ \tau_M^n\Big)^T$ be the vector of truncation error at time level t_n. Since $\overline{\theta}^n$ is the exact solution of the equation(1)-(3), we have

$$A\,\overline{\theta}^1 = B\overline{\theta}^0 + \tau^1, \qquad \text{for } n=0 \tag{31}$$

$$A\,\overline{\theta}^{n+1} = F\,\overline{\theta}^n + h\sum_{j=1}^{n-1}\Big[d_j - d_{j+1}\Big]\overline{\theta}^{n-j} + h\,d_n\overline{\theta}^0 + \tau^{n+1} \qquad \text{for } n\geq 1 \tag{32}$$

Lemma 5.5. *The coefficient $d_j,\ j=0,1,2,3,\ldots$ satisfy the following conditions* [10]

(i) $d_j > 0$

(ii) $d_j > d_{j+1}$ *and* $d_j \to 0$ *as* $j\to\infty$

Theorem 5.2. *The fractional order Crank-Nicolson finite difference scheme* (22)-(25) *for fractional order subdiffusion equation* (1)-(3)*, is unconditionally convergent.*

Proof. We set, $E^n = \overline{\theta}^n - \theta^n = \Big(e_1^n,\ e_2^n,\ e_3^n,\ \ldots,\ e_M^n\Big)^T$ be the error vector in the solution at time level t_n. Furthermore, we assume that

$$|e_l^n| = \max_{1\leq i\leq M}|e_i^n| = \|E^n\|_\infty$$

$$|\tau_l^n| = \max_{1\leq i\leq M}|\tau_i^n|, \text{ for } l=1,2,3,\ldots$$

using equation (22) we obtain

$$|e_l^1| = |-r\,e_{i-1}^1 + (1+2r)\,e_i^1 - r\,e_{i+1}^1| \leq r|e_{i-1}^0| + (1-2r)|e_i^0| + r|e_{i+1}^0| + |\tau_l^1|$$

$$\leq (r+1-2r+r)\max_{1\leq i\leq M}|e_i^0| + \max_{1\leq i\leq M}|\tau_i^1| \leq \|E^0\|_\infty + \max_{1\leq i\leq M}|\tau_i^1|$$

$$\therefore \|E^1\|_\infty \leq \|E^0\|_\infty + \max_{1\leq i\leq M}|\tau_i^1|$$

Now, from equation (23), we obtain

$$\begin{aligned}
|e_l^{m+1}| &= |-r\,e_{i-1}^{m+1} + (1+2r)\,e_i^{m+1} - r\,e_{i+1}^{m+1}| \\
&\le r\,|e_{i-1}^{m}| + (1-2r-h\,d_1)\,|e_i^{m}| + r\,|e_{i+1}^{m}| + h\sum_{j=1}^{m-1}\Big[d_j - d_{j+1}\Big]\,|e_i^{m-j}| + h\,d_m\,|e_i^{0}| + |\tau_l^{m+1}| \\
&= r\,|e_{i-1}^{m}| + (1-2r-h\,d_1)\,|e_i^{m}| + r\,|e_{i+1}^{m}| + h\,(d_1-d_2)\,|e_i^{m-1}| + h\,(d_2-d_3)\,|e_i^{m-2}| \\
&\qquad + \cdots + h\,(d_{m-1}-d_m)\,|e_i^{1}| + h\,d_m\,|e_i^{0}| + |\tau_l^{m+1}| \\
&\le \Big[r + 1 - 2r - h\,d_1 + r\Big]|e_l^{m}| + h\Big[(d_1-d_2) + (d_2-d_3) + \cdots + (d_{m-1}-d_m)\Big]|e_l^{m}| \\
&\qquad + h\,d_m\,|e_i^{l}| + \max_{1\le i\le M}|\tau_i^{m+1}| \\
&= \|E^m\|_\infty + \max_{1\le i\le M}|\tau_i^{m+1}|
\end{aligned}$$

This is true for every m, therefore we have

$$\|E^{m+1}\|_\infty \le \|E^m\|_\infty + \max_{1\le i\le M}|\tau_i^{m+1}|$$

Hence, by induction, we get

$$\|E^{n+1}\|_\infty \le \|E^n\|_\infty + \max_{1\le i\le M}|\tau_i^{n+1}|$$

As $\|E^0\|_\infty = 0$ implies $\|E^n\|_\infty = 0$

Therefore

$$\|E^n\|_\infty \le \max_{1\le i\le M}|\tau_i^{n+1}|.$$

Since

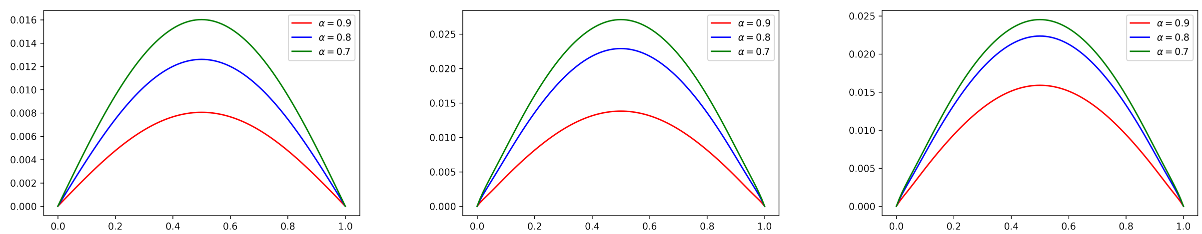
$$\max_{1\le i\le M}|\tau_i^{n+1}| \to 0 \text{ as } (\Delta x, \Delta t) \to (0,0)$$

implies that $\|E^n\|_\infty \to 0$ uniformaly on Ω an $(\Delta x, \Delta t) \to (0,0)$. Therefore, this shows that for any x and t, as $(\Delta x, \Delta t) \to (0,0)$, the vector θ^n converges to $\overline{\theta}^n$.

Hence, this completes the proof. □

6. NUMERICAL SIMULATIONS

In this section, we represent solution of fractional subdiffusion equations (1)-(3) graphically with initial condition $g(x) = x(1-x), L = 1, T = 1$ by using Python software. Also, we obtained absolute error between vales of $U(x,t)$ with Caputo, Caputo-Fabrizio, and Antagana-Balenau fractional derivatives. We observe that the graph of the solution of the subdiffusion



(A) Solution of subdiffusion equation with Caputo FD

(B) Solution of subdiffusion equation with Caputo-Fabrizio FD

(C) Solution of subdiffusion equation with Atangana Balenau FD

FIGURE 1. Comparison of temperature profiles along x direction at $t = 0.5$.

equation with Caputo-Fabrizio and Atangana-Baleneu fractional derivatives are similar while a solution of the subdiffusion equation with Caputo fractional derivative is slightly different, this difference is due to the singular and non-singular kernel in Caputo and Caputo-Fabrizio, Atangana-Baleneu fractional derivatives respectively.

x ($t = 0.5$)	$U(x,t)$ **by using** C	$U(x,t)$ **by using** CF	$\lvert C - CF \rvert$
0.1	0.00251619	0.00427791	0.00176172
0.2	0.00476929	0.00814101	0.00337172
0.3	0.00654142	0.01118644	0.00464502
0.4	0.00767103	0.01313604	0.00546501
0.5	0.00805875	0.01380633	0.00574758
0.6	0.00767103	0.01313604	0.00546501
0.7	0.00654142	0.01118644	0.00464502
0.8	0.00476929	0.00814101	0.00337172
0.9	0.00251619	0.00427791	0.00176172

TABLE 1. Absolute difference between values of $U(x,t)$ in the x direction at $t = 0.5$ and $\alpha = 0.9$ by using Caputo and CF FD

x ($t = 0.5$)	$U(x,t)$ **by using** C	$U(x,t)$ **by using** ABC	$\lvert C - ABC \rvert$
0.1	0.00251619	0.00495264	0.00243645
0.2	0.00476929	0.00940461	0.00463532
0.3	0.00654142	0.0128988	0.00635738
0.4	0.00767103	0.01512721	0.00745618
0.5	0.00805875	0.01589185	0.0078331
0.6	0.00767103	0.01512721	0.00745618
0.7	0.00654142	0.0128988	0.00635738
0.8	0.00476929	0.00940461	0.00463532
0.9	0.00251619	0.00495264	0.00243645

TABLE 2. Absolute difference between values of $U(x,t)$ in the x direction at $t = 0.5$ and $\alpha = 0.9$ by using Caputo and ABC FD

x ($t = 0.5$)	$U(x,t)$ **by using** CF	$U(x,t)$ **by using** ABC	$\lvert CF - ABC \rvert$
0.1	0.00427791	0.00495264	0.00067473
0.2	0.00814101	0.00940461	0.0012636
0.3	0.01118644	0.0128988	0.00171236
0.4	0.01313604	0.01512721	0.00199117
0.5	0.01380633	0.01589185	0.00208552
0.6	0.01313604	0.01512721	0.00199117
0.7	0.01118644	0.0128988	0.00171236
0.8	0.00814101	0.00940461	0.0012636
0.9	0.00427791	0.00495264	0.00067473

TABLE 3. Absolute difference between values of $U(x,t)$ in the x direction at $t = 0.5$ and $\alpha = 0.9$ by using CF and ABC FD

From tables, we observe that the absolute difference between values of $U(x,t)$ obtained by Caputo and Caputo-fabirizio fractional derivative, Caputo and Atangana-Baleneu fractional derivative for various values of α are approximately of order 10^{-2} while absolute difference between values of $U(x,t)$ obtained by Caputo-fabirizio and Atangana-Baleneu fractional derivative for various values of α are approximately of order 10^{-3}. This difference is due to kernel of fractional derivatives. Caputo fractional derivative having power kernel, Caputo-fabrizio having exponential kernel and Atangana-baleneu having Mittag-Leffler kernel. Note that Exponential is particular case of Mittag-Leffler.

7. CONCLUSION

We successfully developed Crank-Nicolson finite difference scheme for fractional subdiffusion equation by using Caputo, Caputo-Fabrizio, and Atangana-Baleneu fractional derivative. Further, we have shown that the scheme is unconditionally stable and convergent. We developed Python code successfully to obtain a numerical solution of a fractional subdiffusion equation and represent it graphically. We compared solutions of the fractional subdiffusion equation obtained by various fractional derivatives. We observe that solution obtained by Caputo-Fabrizio and Atangana-Baleneu fractional derivative are strongly agreed than solution obtained by Caputo fractional derivative. This is due to the difference between singular and non-singular kernels.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model, Therm. Sci. 20 (2016), 763–769.

[2] U. Kharde, K. Takale, S. Gaikwad, Crank-Nicolson method for time fractional drug concentration equation in central nervous system, Adv. Appl. Math. Sci. 22 (2022), 407–433.

[3] F. Liu, C. Yang, K. Burrage, Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term, J. Comput. Appl. Math. 231 (2009), 160–176.

[4] S.Yu. Lukashchuk, Estimation of parameters in fractional subdiffusion equations by the time integral characteristics method, Computers Math. Appl. 62 (2011), 834–844.

[5] J. Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 2 (2015), 87–92.

[6] F. Mainardi, A. Mura, G. Pagnini, et al. Sub-diffusion equations of fractional order and their fundamental solutions, in: K. Tas, J.A. Tenreiro Machado, D. Baleanu (Eds.), Mathematical Methods in Engineering, Springer Netherlands, Dordrecht, 2007: pp. 23–55.

[7] K. Mustapha, J. AlMutawa, A finite difference method for an anomalous sub-diffusion equation, theory and applications, Numer. Algor. 61 (2012), 525–543.

[8] I. Podlubny, Fractional differential equation, Academic Press, San Diego, 1999.

[9] G.D. Smith, Numerical solution of partial differential equations: Finite difference methods, third edition, Oxford Applied Mathematics and Computing Science Series, Oxford University Press, Oxford, New York, 1985.

[10] K.C. Takale, On some mathematical models in heat transfer, Ph.D. dissertation, Dept of Maths., Dr. B. A. M. Univ., Aurangabad, 2012.

[11] Y.M. Wang, A compact finite difference method for solving a class of time fractional convection-subdiffusion equations, BIT Numer. Math. 55 (2015), 1187–1217.

[12] S.B. Yuste, L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, SIAM J. Numer. Anal. 42 (2005), 1862–1874.