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PIECEWISE DEFINED RECURSIVE SEQUENCES WITH APPLICATION IN MATRIX THEORY

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Abstract. In this paper we determine periodicity and boundedness of orbits of a piecewise defined difference equation. A corollary is that the real eigenvalues of certain arbitrarily large, sparse matrices may be computed exactly.

Keywords: piecewise defined sequence; eventually periodic; strictly increasing; sparse matrix; eigenvalue.

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1. Introduction

The geometric sequence $S_k = hS_{k-1}$ is a basic component of exponential growth models [2, 3, 5, 10]. In this note, we consider a limiting or harvesting condition on S_k and describe the resulting piecewise defined sequence. Precisely, we make the

Definition 1. *Let* h, v *and* w *be natural numbers and let* S_0 *be an integer. Define for* $k > 0$

$$S_k = \begin{cases} h\ S_{k-1} & \text{if } S_{k-1} \leq v \\ h\ S_{k-1} - w & \text{if } S_{k-1} > v \end{cases}$$

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If $S_r \le 0$ for some r, then S_k is geometric for $k \ge r$. Hence we restrict our attention to positive initial values.

We begin with a characterization of S_k, then discuss feasibility when hypotheses fail. A central condition in the treatment is $w = 2v + 1$. For example, suppose that $h = 2$, $v = 5$, $w = 11$ and $S_0 = 1$. Then S_k is periodic and all integers from 1 to $w - 1$ appear in the sequence:

$$1,\ 2,\ 4,\ 8,\ 5,\ 10,\ 9,\ 7,\ 3,\ 6,\ 1,\ 2,\ \ldots\ .$$

However, if $h = 2$, $v = 3$, $w = 7$ and $S_0 = 1$, then S_k is again periodic but not all such integers appear: 1, 2, 4, 1, 2, ... On the other hand, if $h = 3$, $v = 2$, $w = 5$ and $S_0 = 2$, then S_k is strictly increasing (to infinity): 2, 6, 13, 34, 97, 286, 853, 2554, 7657, ...

2. The Main Result

In general we have the following.

Theorem 2.1. *Let* h, v, w *and* S_0 *be positive integers.*

1. If $h = 1$, *then* S_k *reaches a point of equilibrium, i.e., there is a nonnegative integer* r *such that* $S_k = S_0 - rw$ *for all* $k \ge r$.

2. Let $h \ge 2$.

a. If $S_0 = w > v$, *then*

$$S_k = \frac{h^k(h-2)+1}{h-1}\, S_0 \quad \textit{for all } k.$$

b. If $S_0 > w \ge v$, *then* S_k *is strictly increasing.*

3. Let $h = 2$.

a. Suppose that $w = 2v + u$ *for some* $u = 0, 1$ *or* 2, *and* $1 \le S_0 \le w - 1$.

i. If $u = 1$, *then* S_k *is periodic and* $1 \le S_k \le w - 1$ *for all* k.

ii. If $u = 0$, *then* S_k *either reaches equilibrium at* w *or is eventually periodic such that* $1 \leq S_k \leq w - 1$ *for all* k.

iii. If $u = 2$, *then* S_k *either reaches equilibrium at* 0 *or is eventually periodic such that* $1 \leq S_k \leq w - 1$ *for all* k.

b. If $w > 2v + 2$, *then* $S_0 \geq w$ *(in which case, see 2) or* S_k *is either eventually nonpositive or eventually periodic.*

c. If $w \leq v$, *then* S_k *is strictly increasing.*

4. If $h \geq 3$ *and* $w \leq 2v + 1$, *then* S_k *is strictly increasing.*

Proofs. *1.* Assume $h = 1$. There is a nonnegative integer r such that S_0 satisfies

$$0 < v < v + w < \cdots < v + (r-1)w < S_0 \leq v + rw.$$

If $r = 0$, then $S_k = S_0$ for all k. And if $r > 0$, then

$$S_1 = S_0 - w > v, \quad \ldots, \quad S_{r-1} = S_0 - (r-1)w > v,$$

but

$$S_r = S_0 - rw \leq v$$

and it follows that $S_k = S_r$ for $k \geq r$.

2. Assume that $h \geq 2$.

a. If $S_0 = w > v$, then

$$S_1 = hS_0 - w = (h-1)w \geq w > v$$

$$S_2 = hS_1 - w = [h(h-1) - 1]w \geq w > v$$

$$S_3 = hS_2 - w = \{h[h(h-1) - 1] - 1\}w \geq w > v, \text{ etc.}$$

Therefore in general

$$S_k = (h^k - h^{k-1} - h^{k-2} - \cdots - 1)w = (h^k - \frac{h^k - 1}{h - 1})w = \frac{h^k(h-2) + 1}{h - 1}\, w.$$

b. Assume $S_0 > w \geq v$. Then

$$S_1 = hS_0 - w = S_0 + [(h-1)S_0 - w] > S_0 > w \geq v$$

and similarly by induction

$$S_k = hS_{k-1} - w = S_{k-1} + [(h-1)S_{k-1} - w] > S_{k-1} > w \geq v$$

for every $k \geq 2$. Thus $S_{k+1} > S_k > v$ for all k.

3a. Let $h = 2$, $1 \leq S_0 \leq w - 1$, and $w = 2v + u$ for some $u = 0, 1$ or 2. If $1 \leq S_0 \leq v$, then

$$2 \leq S_1 = 2S_0 \leq 2v = w - u. \tag{1}$$

On the other hand, if $v < S_0 \leq w - 1$, then $2v + 2 \leq 2S_0 \leq 2w - 2$ and

$$2 - u \leq S_1 = 2S_0 - w \leq w - 2 < w - 1. \tag{2}$$

i. Suppose that $u = 1$. By (1) and (2), it follows that $1 \leq S_1 \leq w - 1$, and by induction, $1 \leq S_k \leq w - 1$ for all k. Since S_k is a sequence of natural numbers, we have that some term S_r must repeat. We show that S_0 repeats: Suppose that $r > 0$ is the least integer such that $S_r = S_{r+s}$ for some $s > 0$. If S_r is even, then $S_r = 2S_{r-1}$, since the other possibility $2S_{r-1} - w$ is odd; and since $S_r = S_{r+s}$, it follows that $S_{r+s} = 2S_{r+s-1}$ so that $S_{r-1} = S_{r+s-1}$ in this case. If S_r is odd, then similarly $S_{r-1} = S_{r+s-1}$. Therefore $S_{r-1} = S_{r+s-1}$ in either case, which contradicts the minimality of r. Hence S_k is periodic when $u = 1$.

ii. Assume $u = 0$. If $S_N = w$ for some N, then since $h = 2$ and $w > v$, we have that $S_k = w$ for all $k \geq N$. Thus suppose that $S_k \neq w$ for all k. Hence by (1) and (2), $1 \leq S_1 \leq w - 1$, and an induction argument shows that $1 \leq S_k \leq w - 1$ for all k. As above, some term S_r must repeat so S_k is eventually periodic.

iii. Similar to (*ii*).

3b. Suppose that $h = 2$, $w > 2v + 2$, and $S_N \geq w$ for some N. Assume that $N > 0$. If $S_{N-1} \leq v$, then

$$S_N = 2S_{N-1} \leq 2v < w$$

which is impossible. Hence $S_{N-1} > v$ and $S_N = 2S_{N-1} - w \geq w$ so $S_{N-1} \geq w$. Continuing by induction, we have that $N = 0$ is the only possibility if $S_N \geq w$.

Therefore, if $S_0 \leq w - 1$, then $S_k \leq w - 1$ for all k. Hence either $S_N \leq 0$ for some N or $1 \leq S_k \leq w - 1$ for all k. It follows that either $S_k \leq 0$ for all $k \geq N$ or S_k is eventually periodic as in the proof of (*3a*).

3c. Assume $h = 2$ and $w \leq v$. Let l be the least integer such that $2^l S_0 > v$. Then $S_l > S_{l-1} > \cdots > S_0$, $S_l = 2^l S_0$, and

$$S_{l+1} = 2^{l+1} S_0 - w \geq 2^{l+1} S_0 - v > 2^l S_0 = S_l$$

since $2^l(2-1)S_0 > v$. Hence

$$S_{l+1} > S_l > v.$$

By induction

$$S_{k+1} > S_k > v \text{ for all } k \geq l$$

and thus S_k $(k \geq 0)$ is strictly increasing.

4. Assume $h \geq 3$ and $w \leq 2v + 1$. We show $S_{k+1} > S_k$ for all k. Let l denote the least integer such that

$$S_l = h^l S_0 > v.$$

Then $S_l > S_{l-1} > \cdots > S_0$, and

$$S_{l+1} = hS_l - w \geq 3S_l - w > S_l.$$

The last inequality follows since

$$S_l \geq v + 1 \text{ and } 2S_l \geq (2v + 1) + 1 \geq w + 1 > w.$$

Hence

$$S_{l+1} > S_l > v$$

and S_k is strictly increasing as in (*3c*). $\square$

Corollary 2.2. *Let* v *be a natural number,* $h = 2$, $w = 2v + 1$, *and let* R *be the relation defined on the set* $\{1, 2, \ldots, n = 2v\}$ *by:* xRy *if and only if there exist terms* S_0 *and* S_k *such that* $x = S_0$ *and* $y = S_k$ *for some* $k > 0$. *Then* R *is an equivalence relation on* $\{1, 2, \ldots, n\}$.

Proof. By (*3ai*) of Theorem 2.1, for any given S_0 in $\{1, 2, \ldots, n\}$ there exists a unique integer $p = p(S_0) > 0$ such that the sequence S_k is

$$S_0,\ S_1,\ \ldots,\ S_{p-1},\ S_p = S_0,\ S_{p+1} = S_1,\ \ldots \tag{3}$$

and $S_0, S_1, \ldots, S_{p-1}$ are distinct.

(i) *Reflexive*: Let $x = S_0$. By (3), $x = S_p$ so xRx.

(ii) *Symmetric*: Suppose xRy. Thus $x = S_0$ and $y = S_k$ where by (3) we may assume $1 \leq k < p$. Redefine $y = S_0^{'}$ (another starting value). Then by (3), $x = S_{p-k}^{'}$ where $p - k > 0$ so yRx.

(iii) *Transitive*: Assume xRy and yRz. As above, by the definition of the sequence S_k,

$$x = S_0, \quad y = S_k = S_0^{'}, \quad z = S_l^{'} = S_{k+l}$$

for some positive integers k and l. Therefore $x = S_0$ and $z = S_{k+l}$ where $k + l > 0$, thus xRz. □

The following is a fundamental result from Algebra [8, 9]:

Let R *be an equivalence relation on a set* S. *For any* s *in* S, *the equivalence class of* s *under* R, *denoted* $[s]$, *is the subset of* S *consisting of all elements* t *of* S *such that* tRs. *Then every element of* S *is in exactly one equivalence class under* R. *That is, the equivalence classes partition* S *into a family of mutually disjoint nonempty subsets.*

The equivalence classes of Corollary 2.2 are, moreover, *ordered* sets

$$[S_0] = \{S_0,\ S_1,\ \ldots,\ S_{p-1}\}$$

and we have for example

$$\{1,\, 2,\, \ldots,\, 14\} = [1] \cup [3] \cup [5] \cup [7],$$

$$\{1,\, 2,\, \ldots,\, 16\} = [1] \cup [3],$$

$$\{1,\, 2,\, \ldots,\, 18\} = [1],$$

$$\{1,\, 2,\, \ldots,\, 20\} = [1] \cup [3] \cup [5] \cup [7] \cup [9].$$

An interesting problem in this algebraic context is to determine all even values n such that $\{1,\, 2,\, \ldots,\, n\} = [1]$.

Example 1. The following table illustrates possible situations in $(3a)$ and $(3b)$ of Theorem 2.1 where $h = 2$:

S_0	v	w	sequence
1	3	8	$1, 2, 4, 0, 0, 0, \ldots$
3	10	20	$3, 6, 12, 4, 8, 16, 12, 4, \ldots$
5	10	20	$5, 10, 20, 20, 20, \ldots$
8	2	7	$8, 9, 11, 15, 23, 39, 71, 135, \ldots$
1	10	24	$1, 2, 4, 8, 16, 8, 16, 8, \ldots$
1	9	22	$1, 2, 4, 8, 16, 10, -2, -4, \ldots$

3. Pathology

Theorem 2.1 describes piecewise defined recursive sequences when either $w \le v$, $w = 2v$ or $w = 2v + 1$ for any h and S_0. The specific cases not covered are

1. $h = 2$ and $v < w < 2v$
2. $h \ge 3$ and $w > 2v + 1$.

In each case, $w > v$, so by the reasoning in the proofs of (2) and (3) of Theorem 2.1, one of the following situations must hold for some N:

i. $S_N = w$ (and $S_{N+k} = \frac{h^k(h-2)+1}{h-1} w$ for every k)

ii. $S_N > w$ (and S_k is strictly increasing for $k \ge N$)

iii. $1 \le S_k \le w - 1$ for all $k \ge N$ (and S_k is eventually periodic)

iv. $S_N \le 0$ (and $S_{N+k} = h^k S_N$ for every k)

We illustrate these possibilities as follows.

Example 2. $(h = 2$ and $v < w < 2v)$ In this case, we show that $S_k \geq 2$ for all $k \geq 1$ so (iv) is not feasible: Let l be the least natural number such that $S_l = h^l S_0 > v$. Then

$$S_{l+1} = h^{l+1} S_0 - w > h^{l+1} S_0 - 2v = 2(h^l S_0 - v) \geq 2.$$

Let l' be the least natural number such that $S_{l+l'} = h^{l'-1}(h^{l+1} S_0 - w) > v$. Then

$$S_{l+l'+1} = h S_{l+l'} - w > 2(S_{l+l'} - v) \geq 2.$$

Continuing similarly by induction, the result follows.

The other situations are possible: with $S_0 = 1$, we have

v	w	S_k type
9	16	i
7	13	ii
11	20	iii

For case 2, any of (i) - (iv) are feasible:

Example 3. $(h = 3$ and $w > 2v + 1)$ Choosing $S_0 = 1$ again, we calculate the table

v	w	S_k type
11	27	i
7	19	ii
16	72	iii
7	18	iii
7	30	iv
2	9	iv

It is easy to generate periodic S_k with arbitrary initial values from known examples. If S_k is periodic and α is a positive integer, the piecewise defined sequence S_k' with $h' = h,\ v' = \alpha v$ and $w' = \alpha w$ is periodic and $S_0' = \alpha S_0$. For $h = 2$, periodic sequences S_k with $S_0 = 1$ are given by the theorem *(3a)* where $w = 2v + u$ $(u = 0, 1, 2)$. If

$\alpha > 2$ and $u > 0$, then S'_k is periodic and satisfies (*3b*). We can similarly modify the following results.

Example 4. For $h \geq 2$, there are general choices of v and w such that $w > 2v+1$, $S_0 = 1$, and S_k is either periodic, strictly increasing or reaches an equilibrium point:

a. For any positive integers r and s, let $t = (r+1)s$ and define

$$v = \frac{(h-1)(h^{r+t} - h^r)}{h^{r+1} - 1} \quad \text{and} \quad w = hv + h - 1.$$

We begin by showing $h^{t-1} < v < h^t$. The right inequality is equivalent to

$$h^r + h^t < h^{r+1} + h^{r+t}$$

which is trivial. The left inequality is equivalent to

$$h^{t-1}(h^{r+1} - 1) \leq h^r(h^t - 1)(h-1)$$

which clearly holds when $s = 1$ since $t = (r+1)s$ and $h \geq 2$. Thus assume that $s \geq 2$. Then the inequality becomes

$$2\, h^{r+t} + h^{r+1} \leq h^{r+t+1} + h^r + h^{t-1}$$

where $2\, h^{r+t} \leq h\, h^{r+t}$ since $h \geq 2$, and $h^{r+1} \leq h^r + h^{t-1}$ (or $h \leq 1 + h^{t-(r+1)}$) since $t = (r+1)s \geq 2(r+1)$ and $r \geq 0$. Thus $h^{t-1} < v < h^t$ and straightforward calculations show that

$$w = \frac{(h-1)(h^{r+t+1} - 1)}{h^{r+1} - 1} \quad \text{and} \quad h^{t+1} - w > v.$$

More generally, for $1 \leq k < r$,

$$h^{t+k} - \frac{h^k - 1}{h - 1} w = h^{t+k} - \frac{(h^k - 1)(h^{r+t+1} - 1)}{h^{r+1} - 1} > v$$

if and only if

$$h^{r+t} + h^{r+1} + h^k > h^{k+t} + h^r + 1,$$

which holds since $h \geq 2$.

It follows that since $S_0 = 1$,

$$S_{t+k} = h^{t+k} - \frac{h^k - 1}{h - 1} w \qquad (k = 1, \ \ldots, \ r).$$

Since $S_{t+r} = v + 1$, we have that $S_{t+r+1} = h(v+1) - (hv + h - 1) = 1$ and S_k is periodic of period $t + r + 1 = (r+1)(s+1)$.

For example, if $h = 3$, $r = 1$ and $s = 3$, then $v = 546$, $w = 1640$ and the sequence is

$$1, \ 3, \ 9, \ 27, \ 81, \ 243, \ 729, \ 547, \ 1, \ \ldots$$

b. Let t be a positive integer, $v = h^t$ and $w = h^{t+2} - h^{t+1} + h - 1$ (which satisfy (*3ai*) of Theorem 2.1 if $h = 2$). The sequence S_k is then computed as follows:

$$1, \ h, \ \ldots, \ h^{t+1}, \ h^{t+1} - h + 1, \ h^{t+1} - h^2 + 1, \ \ldots, \ h^{t+1} - h^t + 1, \ 1, \ \ldots \ .$$

For example, if $h = 4, v = 64$ and $w = 771$, then the sequence is

$$1, \ 4, \ 16, \ 64, \ 256, \ 253, \ 241, \ 193, \ 1, \ \ldots \ .$$

c. Let t and v be any positive integers that satisfy

$$h^{t-1} < v \leq h^t - h^{t-1} - 1,$$

and let $w = hv + h - 1$ as above. Then $S_k = h^k$ $(0 \leq k \leq t)$, $S_t > v$, and

$$S_{t+1} = hS_t - w = h^{t+1} - hv - h + 1 \geq h^t + 1 > S_t > v.$$

By induction,

$$S_{t+k+1} > S_{t+k} > v$$

for all k and thus S_{t+k}, and therefore S_k, are strictly increasing.

d. Let t and v be positive integers such that

$$h^{t-1} \leq v < h^t,$$

and let $w = h^{t+1} - h^t$. Then $S_k = h^k$ for $k \leq t$ and $S_{t+1} = h^{t+1} - w = h^t$. Thus $S_k = h^t$ for all $k \geq t$.

4. Application in Matrix Theory

The following sparse matrices arose in [1] while considering certain vector spaces of magic squares.

Definition 2. *The C-matrix* $A = (a_{ij})$ *is the square matrix of order* n *such that its* nonzero *elements are defined as follows where either* $n = 2k$ or $n = 2k + 1$:

$$a_{ij} = \begin{cases} 1 \text{ for } j = 2i \text{ when } 1 \leq i \leq k \\ 1 \text{ for } j = 2i - (n+1) \text{ when } n - k < i \leq n \\ -2 \text{ for } j = i \end{cases}$$

The ones appear as the moves of the knight on a chessboard. Odd ordered C-matrices are distinguished from even ones by a middle row without ones.

According to Gerschgorin's Disk Theorem (see [7]), the eigenvalues of C-matrices lie in the unit circle with centre $(-2, 0)$ in the complex plane. We show that the real bounds -1 and -3 of the circle will indeed be eigenvalues in many cases. Moreover we note that 0 is not contained in the Gerschgorin disk so C-matrices are invertible. (This also follows since they are strictly diagonally dominant.)

For any C-matrix of odd order we note that -2 is an eigenvalue of A since the matrix $A + 2I$ has the zero row as its middle row. We conjecture that -2 is the *only* eigenvalue when the order of A is $n = 2^l - 1$. This is illustrated in the following

Example 5. Let A be the C-matrix of order 15. Suppose by way of contradiction that $\beta \neq -2$ is a (real or complex) eigenvalue of A and let $\alpha = \beta + 2$. Since $\alpha \neq 0$, by the definition of C-matrix, if $x = (x_1,\ x_2,\ ...,\ x_{15})^t$ is a nonzero vector in the kernel of $A-\beta I$, then $x_8=0$ and $x_i=x_{i+8}$ $(1\leq i\leq 7)$. Moreover, if $x_{2i}=0$ for some $i=1,2,\ldots,7$, then $x_i=0$. It follows in order that

$$0=x_4=x_2=x_1=x_{4+8}=x_{2+8}=x_{1+8}=x_6=x_3=x_5=x_{6+8}=x_{3+8}=x_{5+8}=x_7=x_{7+8}.$$

Thus, x is the zero vector, a contradiction. Therefore, -2 is the only eigenvalue of A.

Another possible eigenvalue of odd ordered C-matrices is -1:

Proposition 4.1. *Let A be a C-matrix of order $n=4l+1$. Then -1 is an eigenvalue of A.*

Proof. Let $n=4l+1$. For each row of the matrix $A+I$ except the middle row we have one entry 1, one entry -1, and the other entries 0. The ones occur in even numbered cells. Column $2l+1$ is the middle column so it contains no ones. If we sum the columns of $A+I$ except the middle column, then we obtain the zero vector. Hence, the determinant of $A+I$ is zero. □

We now turn to the eigenvalues of C-matrices of even order. The following is similar to the above result.

Proposition 4.2. *Let A be a C-matrix of order $n=6l+2$. Then -3 is an eigenvalue of A.*

Proof. Let $n=6l+2$. We show in this case that rows $4l+2$ and $2l+1$ of $A+3I$ are identical. By the definition of C-matrix, the main diagonal of $A+3I$ consists of ones, and row $2l+1$ has one in the entries $(2l+1,\,2l+1)$ and $(2l+1,\,2(2l+1))$ since $2l+1<3l+1$. On the other hand, row $4l+2$ has one in the entries $(4l+2,\,4l+2)$ and $(4l+2,\,2(4l+2)-(n+1)))=(4l+2,2l+1)$. □

We can extend the idea behind the above proof for other C-matrices of even order. We consider matrices where the sum of several rows of $A+3I$ is identical to the sum of another set of rows. For example, let A be the C-matrix of order 4. We obtain

$$A+3I=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The sum of the first and fourth rows is the same as the sum of the second and third rows. Thus -3 is an eigenvalue of A. In general, we have the following procedure.

Theorem 4.3. *Let A be a C-matrix of even order $n=2k$. Then -1 is an eigenvalue of A.*

Conversely, if λ is a real eigenvalue of A, then $\lambda=-1$ or $\lambda=-3$.

Define the sequence Q_l of positive integers as follows: let $Q_0=1$ and for $l \geq 0$ let $[Q_l, Q_{l+1}]$ denote the positions in row Q_l of the ones in the matrix $A+3I$. Then the sequence Q_l is periodic. If the period of Q_l is even, then -3 is an eigenvalue of A.

Proof. Let A be a C-matrix of order $n=2k$. Then each column of $A+I$ has exactly one entry 1, one entry -1 and the remaining entries 0. Hence, the sum of all rows of $A+I$ is the zero row so $\mid A+I \mid = 0$.

Let λ be a real eigenvalue of A. We argue indirectly. Assume that $\mid \lambda+2 \mid < 1$ by Gerschgorin's theorem. We rearrange the columns C_i of $A-\lambda I$ in the order

$$C_2,\ C_4,\ ...,\ C_n,\ C_1,\ C_3,\ ...,\ C_{n-1}.$$

The resulting matrix is strictly diagonally dominant, and is therefore invertible with nonzero determinant, a contradiction.

Let Q_l be given as above. Note that rows with ones in positions $[Q_l, Q_{l+1}]$ and $[Q_{l+1}, Q_{l+2}]$ have a one in the same position Q_{l+1}. Thus, for $l \geq 1$, row Q_l also shares a one with row Q_{l-1}. By the definition of C-matrix of order $2k$, with $Q_0 = 1$, for $l \geq 1$,

$$Q_l = \begin{cases} 2\,Q_{l-1} & \text{if } Q_{l-1} \leq k \\ 2\,Q_{l-1} - (2k+1) & \text{if } Q_{l-1} > k \end{cases}$$

By (*3ai*) of Theorem 2.1, Q_l is periodic. Hence if the period of Q_l is p, then $Q_p = 1$ and row 1 with ones in $[Q_0, Q_1]$ and row Q_{p-1} with ones in $[Q_{p-1}, Q_p]$ have position $Q_p = Q_0$ in common. Since $Q_0, Q_1, \ldots, Q_{p-1}$ are distinct, we have that if p is even, then the sum of the rows $Q_0, Q_2, \ldots, Q_{p-2}$ coincides with the sum of the rows $Q_1, Q_3, \ldots, Q_{p-1}$ and hence the determinant of $A + 3I$ is zero. □

Example 6. We can readily list the sequences Q_l. Two of the first eleven even ordered C-matrices have sequences Q_l with odd periods:

n	Q_l
2	$1, 2, 1, \ldots$
4	$1, 2, 4, 3, 1, \ldots$
6	$1, 2, 4, 1, \ldots$
8	$1, 2, 4, 8, 7, 5, 1, \ldots$
10	$1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1, \ldots$
12	$1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, \ldots$
14	$1, 2, 4, 8, 1, \ldots$
16	$1, 2, 4, 8, 16, 15, 13, 9, 1, \ldots$
18	$1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1, \ldots$
20	$1, 2, 4, 8, 16, 11, 1, \ldots$
22	$1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 1, \ldots$

For example, the period of Q_l is even for $n = 16$ and by Theorem 4.3, the sum of the rows 1, 4, 16 and 13 of $A + 3I$ is identical to the sum of the rows 2, 8, 15 and 9.

We can compute the period of Q_l in some general cases:

Corollary 4.4. *Let $l \geq 2$ be an even integer and let A be a C-matrix of order $n = 2^l - 2$. Then -3 is an eigenvalue of A.*

Proof. We have $Q_0 = 1, Q_1 = 2, Q_2 = 4, \ldots, Q_{l-1} = 2^{l-1}$ since $2^{l-2} \leq \frac{n}{2}$. But $2^{l-1} = \frac{n}{2} + 1 > \frac{n}{2}$ so $Q_l = 2(\frac{n}{2}+1) - (n+1) = 1$. □

Corollary 4.5. *Let A be a C-matrix of order $n = 2^l$* where $l \geq 2$. *Then -3 is an eigenvalue of A.*

Proof. Assume that A is a C-matrix of order $n = 2^l$ where $l \geq 2$. Then $Q_0 = 1$, $Q_1 = 2, Q_2 = 4, \ldots, Q_l = 2^l$. We prove by induction that

$$Q_{l+i} = Q_l - 2^i + 1 > 2^{l-1} = \frac{n}{2}$$

for $i = 0, 1, \ldots, l-1$. The initialization $i = 0$ is clear. Assume the statement holds for some $i < l - 1$. Then

$$Q_{l+i+1} = 2(Q_l - 2^i + 1) - (n+1) = Q_l - 2^{i+1} + 1 > 2^{l-1}$$

since $i + 1 \leq l - 1$ and $2^{l-1} + 1 > 2^{i+1}$.

In particular,

$$Q_{2l-1} = Q_{l+(l-1)} = Q_l - 2^{l-1} + 1 = \frac{n}{2} + 1 > \frac{n}{2}$$

so $Q_{2l} = 2(\frac{n}{2}+1) - (n+1) = 1$. Therefore the period of Q_l is even. □

If the period of Q_l is odd, then we can not deduce any information about the value -3. For example, the period of Q_l is three for the C-matrix A of order six and its eigenvalues are

$$-\frac{5}{2} \pm \frac{\sqrt{3}}{2}i, \ -\frac{5}{2} \pm \frac{\sqrt{3}}{2}i, \ -1, \ -1.$$

On the other hand, -3 is an eigenvalue of the C-matrix of order 366 although Q_l has period 183.

We computed the eigenvalues of the C-matrices of even orders up to order 4780 and found the following orders for which -3 is not an eigenvalue:

6, 22, 30, 46, 48, 70, 72, 78, 88, 102, 126, 150, 160, 166, 190, 198, 216, 222, 232, 238, 262, 270, 310, 328, 336, 342, 358, 430, 438, 496, 510, 552, 600, 622, 630, 712, 720, 880, 888, 910, 918, 936, 960, 1056, 1102, 1288, 1392, 1432, 1456, 1518, 1560, 1678, 1800, 1896, 2046, 2088, 2142, 2200, 2262, 2350, 2358, 2592, 2686, 2758, 2920, 3016, 3190, 3390, 3478, 3472, 3576, 3936, 4056, 4176, 4206, 4512, 4576, 4680.

References

[1] A. Al-Zahawi, S. Al – Ashhab, Linear systems resulting from pandiagonal magic squares, Almanara journal, Al-Albayt University, Vol. 13, No. 6, pp 179-199, 2007.

[2] P. Blanchard, R. L. Devaney, and G. R. Hall, Differential Equations, Brooks/Cole, Pacific Grove, California, 2002.

[3] C. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, 1976.

[4] G. J. Etgen and W. L. Morris, An Introduction to Ordinary Differential Equations with Difference Equations, Numerical Methods and Applications, Harper & Row, New York, 1977.

[5] E. G. Hutchinson, An Introduction to Population Ecology, Yale University Press, New Haven, Connecticut, 1978.

[6] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, New York, 1995.

[7] E. Kresyzig, Advanced Engineering Mathematics, 7th ed., John Wiley & sons, 1993.

[8] S. MacLane and G. Birkhoff, Algebra, Macmillan, New York, 1968.

[9] D. Saracino, Abstract Algebra: A First Course, Waveland Press, Prospect Heights, Illinois, 1992.

[10] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer Verlag, New York, 1990.

[11] T. Yi and Z. Zhou, Periodic solutions of difference equations, J. Math. Anal. Appl. 286 (2003), pp. 220-229.

[12] L. Zhang, G. Zhang, and H. Liu, Periodicity and attractivity for a rational recursive sequence, J. Appl. Math. Comput. 19 (2005), pp. 191-201.