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SKEW SEMI-INVARIANT SUBMANIFOLDS OF A METALLIC RIEMANNIAN MANIFOLD

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Abstract. In this paper, we define and study skew semi-invariant submanifolds ($(SSI)sub_M$s) of metallic Riemannian manifold. We calculate the sectional curvature and mean curvature of $(SSI)sub_M$s and obtain some interesting results. We also construct an example of $(SSI)sub_M$.

Keywords: metallic structure; Riemannian manifold; skew semi-invariant submanifolds; mixed totally geodesics; mean curvature; sectional curvature.

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1. Introduction

A. Bejancu generalized the geometry of invariant and anti-invariant submanifolds in 1984 to define $(SI)sub_M$s of locally product Riemannian manifolds [11]. Holomorphic (invariant) and totally real (anti-invariant) submanifolds are generalized to form $(SI)sub_M$s. If a $(SI)sub_M$ is neither a totally real submanifold nor a holomorphic submanifold, it is said to be proper. Under the impact of the nearly contact structure, the tangent space of holomorphic submanifolds is invariant. On the other hand, the tangent space is anti-invariant in totally real submanifolds, that is, it is mapped into the normal space. Another generalization of invariant and anti-invariant

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submanifolds were defined by Chen [14] as slant submanifolds. The idea that semi-slant submanifolds can obtain slant submanifolds and CR-submanifolds as specific cases, was first introduced by Papaghine in [21]. Carriazo [13] introduced bi-slant submanifolds as a generalization of semi-slant submanifolds. One of the cases of such submanifolds is that of anti-slant submanifolds. Since the term anti-slant implies that the submanifold lacks a slant factor, Sahin [23] named these submanifolds as hemi-slant submanifolds.

Skew CR-submanifolds of a Kahlerian manifold were first defined by Ronsse in [22]. Such submanifolds are generalizations of bi-slant submanifolds. Consequently, invariant, anti-invariant, CR, slant, semi-slant and hemi-slant submanifolds are particular cases of skew CR-submanifolds. We observe that CR-submanifolds in Kahlerian manifolds correspond to $(SI)sub_M$s in local product Riemannian manifolds [11]. Therefore, skew CR-submanifolds in Kahlerian manifolds correspond to $(SSI)sub_M$s in locally product Riemannian manifolds. Ximin and Shao firstly studied these type of submanifolds in [28].

The generalization of the golden mean is known as the metallic means family or metallic proportions [1]. The positive solution for the equation $x^2 - \xi x - \lambda = 0$, where ξ and λ are positive integers, is member of the metallic means family. According to the values of ξ and λ, the members of the metallic means family get the name of a metal, such as the golden mean, the silver mean, the bronze mean, the copper mean, and many others [6], [7]. Hretcanu and Crasmareanu have examined metallic structure on Riemannian manifolds [19],[20]. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold have been studied by Blaga and Hretcanu in [12].

The geometry of $(SI)sub_M$s has been studied on several manifolds by many geometers such as nearly Kenmotsu manifolds [2], [3], [4], golden Riemannian manifolds [15], [17], [18], Lorentzian para Sasakian manifolds [5], Sasakian manifolds [10], [24] and nearly Sasakian manifolds [16]. $(SSI)sub_M$s have been studied by several geometers in [8], [25], [26], [28]. Ahmad & Qayyoom studied warped product $(SSI)sub_M$s on golden Riemannian manifold [9].

We define and investigate $(SSI)sub_M$s of a metallic Riemannian manifold in this research, which is inspired by the studies mentioned above. The following describes how the paper is structured: We introduce the concept of metallic structure and metallic Riemannian manifold in

Section 2. Section 3 introduces the term ”skew semi-invariant submanifold”, explores some of its characteristics, and constructs an example of $(SSI)sub_M$ of metallic Riemannian manifold.

Abbreviation: We have used following abbreviations in this paper-

(1) $(SI)sub_M$: Semi-invariant submanifold

(2) $(SSI)sub_M$: Skew semi-invariant submanifold

(3) SC : Sectional curvature

2. Definition and Preliminaries

Consider an $n-$dimensional manifold $(\overline{Q}, q)$ with a $(1,1)$ tensor field ψ such that

$$\psi^2 = \xi\psi + \lambda I, \tag{2.1}$$

where ξ and λ are positive integers and I is the identity transformation on $\Gamma(T\overline{Q})$. Then, the structure ψ is called a metallic structure. If ψ is a self-adjoint operator with respect to metric tensor q which is a symmetric $(0,2)$ tensor field on $\overline{Q}$, i.e.

$$q(\psi F, G) = q(F, \psi G) \tag{2.2}$$

for all F, G vector fields on $\Gamma(T\overline{Q})$, then the metric q is said to be ψ-compatible and $(\overline{Q}, q, \psi)$ is called metallic Riemannian manifold. From (2.1) and (2.2) we have

$$q(\psi F, \psi G) = \xi q(F, \psi G) + \lambda q(F, G)$$

for any $F, G \in \Gamma(T\overline{Q})$.

Proposition 2.1. [1], [12] *The following characteristics pertain to the metallic structure on the manifold $\overline{Q}$:*

(i) $\psi^n = F_n\psi + \lambda F_{n-1}I$,

where $\{F_n\}_{n\in N}$ is the generalized secondary Fibonacci sequence which is defined by $F_{n+1} = \xi F_n + \lambda F_{n-1}, n \geq 1$ with $F_0 = 0, F_1 = 1$ and ξ, λ real numbers.

(ii) ψ is an isomorphism on the tangent space $T_x\overline{Q}$ for every $x \in \overline{Q}$. As a result, ψ is invertible and its inverse whichis represented as $\overline{\psi} = \psi^{-1} = (1/\lambda)\psi - (\xi/\lambda)I$, is not a metallic structure. Itis still a polynomial, more specifically a quadratic one:

$$\lambda\overline{\psi}^2 + \xi\overline{\psi} - 1 = 0$$

or

$$\overline{\psi}^2 = -(\xi/\lambda)\overline{\psi} + (1/\lambda).$$

(iii) *The characteristic values of* ψ *are the metallic numbers* σ *and* $\xi - \sigma$*, where*

$$\sigma_{\xi,\lambda} = \frac{\xi + \sqrt{\xi^2 + 4\lambda}}{2}.$$

(iv) *A metallic structure* ψ *is called integrable if its Nijenhuis tensor field* $N_\psi(F,G) = [\psi F, \psi G] - \psi[\psi F, G] - \psi[F, \psi G] + \psi^2[F,G]$ *vanishes.*

Proposition 2.2. [20] (i) *There are two complementary distributions* D *and* $D^\perp$ *on a metallic manifold* $(\overline{Q}, \psi)$ *that correspond to the projection operators*

$$t = \frac{\sigma_{\xi,\lambda}}{2\sigma_{\xi,\lambda} - \xi}.I - \frac{1}{2\sigma_{\xi,\lambda} - \xi}.\psi$$

and

$$f = \frac{\sigma_{\xi,\lambda} - \xi}{2\sigma_{\xi,\lambda} - \xi}.I + \frac{1}{2\sigma_{\xi,\lambda} - \xi}.\psi.$$

(ii) *The operators* t *and* f *verify*

$$t + f = I,$$

$$t^2 = t, f^2 = f,$$

$$\psi t = t\psi = (\xi - \sigma_{\xi,\lambda})t,$$

$$\psi f = f\psi = \sigma_{\xi,\lambda} f.$$

Therefore, complementary distributions D *and* $D^\perp$ *for these projections are defined by* t *and* f*, respectively. So, we have* $\psi = (\xi - \sigma_{\xi,\lambda})t + \sigma_{\xi,\lambda} f$.
(iii) *The complementary distributions* D *and* $D^\perp$ *are orthogonal with respect to the* ψ*-compatible metric* q*, i.e.* $q \circ (t \times f) = 0$.

3. Skew Semi-Invariant Submanifolds

Let $(\overline{Q}, q, \psi)$ be a metallic Riemannian manifold. If the tangent bundle TQ of a submanifold Q of $\overline{Q}$ has the following decomposition,

$$TQ = D^0 \oplus D^1 \oplus D^\alpha$$

such as, the distribution on Q,
(i) D^0 is invariant, that is $\psi(D^0_x) = D^0_x \ \forall\, x \in Q$,
(ii) D^1 is anti-invariant, that is $\psi(D^1_x) \subset T_xQ^\perp \ \forall\, x \in Q$,
(iii) D^α is neither invariant nor anti-invariant, that is $tF_x \neq 0$ and $fF_x \neq 0$ for any $x \in Q$ and $F_x \in D_x$.
Then, Q is said to be $(SSI)sub_M$ of $\overline{Q}$ with induced metric q.

Remark 3.1. [26] *A $(SSI)sub_M$ Q of a metallic Riemannian manifold $\overline{Q}$ is said to be*
(i) *semi-invariant if $D^\alpha = 0$,*
(ii) *semi slant if $D^1 = 0$,*
(iii) *hemi-slant if $D^0 = 0$,*
(iv) *slant if $D^0 = 0$ and $D^1 = 0$,*
(v) *invariant if $D^1 = 0$ and $D^\alpha = 0$,*
(vi) *anti-invariant if $D^0 = 0$ and $D^\alpha = 0$.*

We denote the Riemannian connection in $\overline{Q}$ by $\overline{\nabla}$, induced Riemannian connection in Q by ∇ and induced normal Riemannian connection in $T^\perp Q$ by $\nabla^\perp$. We have Gauss and Weingarten equations respectively

$$\overline{\nabla}_F G = \nabla_F G + \bar{b}(F,G), \tag{3.1}$$

$$\overline{\nabla}_F \eta = -A_\eta F + \nabla^\perp_F \eta \tag{3.2}$$

for $F, G \in TQ$ and $\eta \in T^\perp Q$. For the second fundamental form $\bar{b}$ related to A, we have

$$q(\bar{b}(F,G), \eta) = q(A_\eta F, G).$$

Now, for tangent vector $F \in TQ$ and normal vector $\eta \in T^\perp Q$ we write,

$$\psi F = tF + fF \tag{3.3}$$

and

$$\psi\eta = b\eta + c\eta, \tag{3.4}$$

where $tF, b\eta \in TQ$ and $fF, c\eta \in T^{\perp}Q$.

We have,

$$q(\psi F, G) = q(tF, G)$$

$$q(\psi F, G) = q(F, tG).$$

So, t and t^2 are symmetric operators on the tangent space TQ. Let $\phi(x)$ is the characteristic value of t^2 at $x \in Q$. Since, t^2 is a composition of an isometry and a projection, hence $\phi(x) \in [0,1]$. Now, we set $D_x^{\phi} = ker(t^2 - \phi(x)I)$ for each $x \in Q$, where I is the identity transformation on T_xQ. Obviously,

$$D_x^0 = ker(f) \quad and \quad D_x^1 = ker(t).$$

D_x^1 is the maximal ψ anti-invariant subspace of T_xQ and D_x^0 is the maximal ψ invariant subspace of T_xQ.

T_xQ has the following decomposition as the direct sum of characteristic spaces that are mutually orthogonal:

$$T_xQ = D_x^{\phi_1} \oplus D_x^{\phi_2} \oplus .... \oplus D_x^{\phi_k},$$

where $\phi_1(x), \phi_2(x), ..., \phi_k(x)$ are all characteristic values of t^2 at x.

Now, we know that

$$\psi(\overline{\nabla}_F G) = \overline{\nabla}_F \psi G - G\overline{\nabla}_F \psi$$

$$\psi(\overline{\nabla}_F G) = \overline{\nabla}_F \psi G.$$

From (3.1) and (3.3), we get

$$\psi(\nabla_F G + \bar{b}(F,G)) = \overline{\nabla}_F(tG + fG).$$

From (3.1) and (3.2), we obtain

$$\psi(\nabla_F G) + \psi(\bar{b}(F,G)) = \nabla_F tG + \bar{b}(F,tG) - A_\eta F + \nabla_F^{\perp} fG$$

(3.3) and (3.4) give

$$t(\nabla_F G)+f(\nabla_F G)+b\bar{b}(F,G)+c\bar{b}(F,G)$$

$$=\nabla_F tG+\bar{b}(F,tG)-A_{fG}F+\nabla^{\perp}_F fG \tag{3.5}$$

for all $F,G\in TQ$. Comparing tangential and normal components in (3.5), we have

$$t(\nabla_F G)=\nabla_F tG-b\bar{b}(F,G)-A_{fG}F \tag{3.6}$$

and

$$f(\nabla_F G)=\bar{b}(F,tG)+\nabla^{\perp}_F fG-c\bar{b}(F,G). \tag{3.7}$$

Now,

$$t[F,G]=t\nabla_F G-t\nabla_G F$$

$$=\nabla_F tG-b\bar{b}(F,G)-A_{fG}F$$

$$-\nabla_G tF+b\bar{b}(F,G)+A_{fF}G$$

$$t[F,G]=\nabla_F tG-\nabla_G tF+A_{fF}G-A_{fG}F. \tag{3.8}$$

Similarly,

$$f[F,G]=\bar{b}(F,tG)-\bar{b}(G,tF)+\nabla^{\perp}_F fG-\nabla^{\perp}_G fF. \tag{3.9}$$

Lemma 3.2. [28] *Consider a metallic Riemannian manifold $(\overline{Q},j)$ and its $(SSI)sub_M$ Q, then the distribution*
(i) D^0 is integrable if and only if $A_{\psi F}G=A_{\psi G}F$ $\forall$ $F,G\in D^0$,
(ii) D^1 is integrable if and only if $\bar{b}(F,\psi G)=\bar{b}(\psi F,G)$ $\forall$ $F,G\in D^1$.

Now, we define the covariant derivative of t and f for all $F,G\in TQ$ as follows:

$$(\nabla_F t)G=\nabla_F tG-t\nabla_F G,$$

$$(\nabla_F f)G=\nabla^{\perp}_F fG-f\nabla_F G.$$

Using (3.8) and (3.6), we get

$$(\nabla_F t)G=b\bar{b}(F,G)+A_{fG}F$$

and using (3.9) and (3.7), we get

$$(\nabla_F f)G = c\bar{b}(F,G) - \bar{b}(F,tG). \tag{3.10}$$

Remark 3.3. *Let D_1 and D_2 are any two distributions on a submanifold Q of a metallic Riemannian manifold $(\overline{Q},q)$, then D_1 is said to be parallel w.r.t. D_2 if $\nabla_F G = 0$ for all $F \in D_1$ and $G \in D_2$.*

Definition 3.4. [20] *A distribution D_0 on a submanifold Q of a metallic Riemannian manifold $(\overline{Q},q)$ is said to be totally geodesic if for all $F,G \in D_0$ we have $\bar{b}(F,G) = 0$. For any two distributions D_1 and D_2 on Q, we say that Q is $D_1 - D_2$ mixed totally geodesic if $\bar{b}(F,G) = 0$ for all $F \in D_1$ and $G \in D_2$.*

Proposition 3.5. *Let D^{ϕ} be any distribution on a $(SSI)sub_M$ Q of a metallic Riemannian manifold $(\overline{Q},q)$. If $A_\eta tF = tA_\eta F \ \ \forall\, F \in D^{\phi}$ and $\eta \in T^{\perp}Q$, where $\phi \neq \theta$, then Q is $D^{\phi} - D^{\theta}$ mixed totally geodesic.*

Proof. : As we have assumed earlier,

$$t^2 A_\eta F - \phi A_\eta F = 0.$$

This concludes that $A_\eta F \in D^{\phi}$. So $\forall\, G \in D^{\theta}, \eta \in T^{\perp}Q, \phi \neq \theta$ we have

$$q(A_\eta F, G) = 0$$

$$q(\bar{b}(F,G), \eta) = 0,$$

which implies $\bar{b}(F,G) = 0$ for all $F \in D^{\phi}$ and $G \in D^{\theta}$. Therefore, Q is $D^{\phi} - D^{\theta}$ mixed totally geodesic.

From (3.3), for all $F_x \in T_xQ$ we have

$$\psi F_x = tF_x + fF_x$$

$$\psi^2 F_x = \psi(\psi F_x) - \psi(tF_x + fF_x)$$

$$(\xi\psi + \lambda I)F_x = \psi(\psi F_x) - \psi(tF_x + fF_x)$$

$$\xi(tF_x + fF_x) + \lambda F_x = t^2 F_x + ftF_x + bfF_x + cfF_x.$$

Comparing normal parts, we get

$$\xi fF_x = ftF_x + cfF_x$$

$$cfF_x = \xi fF_x - ftF_x. \tag{3.11}$$

Similarly, from (3.4) we have

$$\psi\eta = b\eta + c\eta$$

$$\psi^2\eta = \psi(\psi\eta) = \psi(b\eta + c\eta)$$

$$(\xi\psi + \lambda I)\eta = \psi(b\eta) + \psi(c\eta)$$

$$\xi(b\eta + c\eta) + \lambda\eta = tb\eta + fb\eta + bc\eta + c^2\eta.$$

Comparing normal parts, we get

$$\xi c\eta + \lambda\eta = fb\eta + c^2\eta,$$

$$fb\eta = \lambda\eta + \xi c\eta - c^2\eta \tag{3.12}$$

for all $F_x \in T_xQ, \eta \in T_x^{\perp}Q$.

Further, for $F_x \in D_x^{\phi_i}, \phi \in \{\phi_1, \phi_2, \ldots.., \phi_k\}$, we have from (3.11)

$$c^2 fF_x = (\xi + \phi_x) fF_x.$$

Also, if $F_x \in D_x^0$, then it is obvious that $t^2 fF_x = 0$. Therefore, if F_x is an characteristic vector of t^2 w.r.t. the characteristic value $\phi(x)$, then fF_x is an characteristic vector of c^2 having the characteristic value $(\xi + \phi(x))$.

From (3.12) if $c\eta = 0$, we get

$$fb\eta = \lambda\eta - c^2\eta$$

$$fb\eta = (\lambda - c^2)\eta.$$

$(\xi + \phi(x))$ is an characteristic value of c^2, if and only if $(\lambda - \xi - \phi(x))$ is an characteristic value of fb. The characteristic spaces of fb and c^2 are orthogonal as these are symmetric operators on the normal bundle $T^{\perp}Q$. The dimension of the characteristic space of fb that corresponds to the characteristic value $(\lambda - \xi - \phi(x))$ is same as that of D^{ϕ}. As a result, we conclude

Lemma 3.6. *Consider a metallic Riemannian manifold $(\overline{Q}, q, \psi)$. A submanifold Q of $\overline{Q}$ is a $(SSI)sub_M$ if and only if the characteristic values of fb are constant and the characteristic spaces of fb have constant dimension.*

Theorem 3.7. *Consider a metallic Riemannian manifold $(\overline{Q}, j, \psi)$, where $\overline{\nabla}\psi = 0$. If $\nabla t = 0$, then a submanifold Q of $\overline{Q}$ is a $(SSI)sub_M$. Additionally, all of the $t-$invariant distributions D^0, D^1 and D_i^{ϕ}, $1 \leq i \leq k$ are parallel.*

Proof. : Consider $G_x \in D^{\phi_i}$ for any $x \in Q$ and $F, G \in TQ$. Suppose G be a parallel translation of T_xQ along the integral curve of F. Since $(\nabla_F t)G = 0$, we get

$$\nabla_F((t^2 - \phi(x))G) = t^2\nabla_F G - \phi(x)\nabla_F G = 0.$$

This shows that the characteristic values of t^2 are constant. The dimension of each D^{ϕ} is constant and Q is a $(SSI)sub_M$ as the parallel translation of T_xQ along any curve is an isometry that preserves each D^{ϕ}.

Now, if $G \in D^{\phi}$, we have

$$t^2G = \phi G,$$

where ϕ is constant.

$$t^2\nabla_F G = \phi \nabla_F G,$$

which shows that D^{ϕ} is parallel.

Theorem 3.8. *Consider a submanifold Q of metallic Riemannian manifold $\overline{Q}$. If $\nabla f = 0$, then Q is $(SSI)sub_M$.*

Proof. : Consider $F_x \in D_x^{\phi}, \phi \neq 1$ for any point $x \in Q$. Let $\eta_x = fF_x$ then by (3.12) η_x is an characteristic vector of fb with characteristic value $(\lambda - \xi - \phi_x)$.

Now, consider $G \in TQ$ and the translation of η_x in T_xQ along an integral curve of G is η, we have

$$\nabla_G^{\perp}(fb\eta + (\xi - \lambda + \phi_x)\eta) = \nabla_G^{\perp} fb\eta + (\xi - \lambda + \phi_x)\nabla_G^{\perp}\eta$$

From Lemma 3.6, we get

$$\nabla^{\perp}_{G}(fb\eta+(\xi-\lambda+\phi_x)\eta) = fb\nabla^{\perp}_{G}\eta+(\xi-\lambda+\phi_x)\nabla^{\perp}_{G}\eta$$

$$=(fb+(\xi-\lambda+\phi_x))\nabla^{\perp}_{G}\eta.$$

Since, $(fb\eta+(\xi-\lambda+\phi_x)\eta)=0$ at x, hence $(fb\eta+(\xi-\lambda+\phi_x)\eta)=0$ on Q. fb are invariable and their characteristic spaces have constant dimension. Then, according to Lemma 3.6, Q is a $(SSI)sub_M$.

Proposition 3.9. *Consider a $(SSI)sub_M$ Q of metallic Riemannian manifold $\overline{Q}$. If $\nabla f=0$, then for all $\phi\neq\theta$, Q is a $D^{\phi}-D^{\theta}$ mixed totally geodesic. Further, if $F\in D^{\phi}$, then either $\bar{b}(F,G)=0$ or $\bar{b}(F,G)$ is a characteristic vector of c^2 having characteristic value ϕ.*

Proof. : From (3.10) and $\nabla f=0$, for each $F,G\in TQ$, we have

$$c\bar{b}(F,G)=\bar{b}(F,tG).$$

If $G\in D^{\theta}$, then

$$c^2\bar{b}(F,G)=c\bar{b}(F,tG)=\bar{b}(F,t^2G)=\bar{b}(F,\phi G)$$

$$c^2\bar{b}(F,G)=\phi\bar{b}(F,G)$$

$$(c^2-\phi)\bar{b}(F,G)=0$$

$$\bar{b}(F,G)=0.$$

Therefore, Q is $D^{\phi}-D^{\theta}$ mixed totally geodesic. Obviously, the further result follows.

4. Sectional Curvature and Mean Curvature

We denote the curvature tensor of $\overline{Q}$ and Q by $\overline{R}$ and R respectively. The Gauss equation becomes,

$$q(R(F,G)U,V)=q(\overline{R}(F,G)U,V)+q(\bar{b}(F,V),\bar{b}(G,U))-q(\bar{b}(F,U),\bar{b}(G,V)) \tag{4.1}$$

for $F,G,U,V\in TQ$.

For two orthogonal unit vectors $F,G\in T\overline{Q}$, the SC of a plane section of $\overline{Q}$ is given as

$$K_{\overline{Q}}(F\wedge G)=q(\overline{R}(F,G)G,F). \tag{4.2}$$

For two orthogonal unit vectors $F, G \in TQ$, the SC of a plane section of Q is given by

$$K_Q(F \wedge G) = q(R(F,G)G,F). \tag{4.3}$$

$$K_Q(F \wedge G) = \frac{c}{4}\{1+q(F,\psi tG)^2\} + q(\bar{b}(F,F),\bar{b}(G,G)) - q(\bar{b}(F,G),\bar{b}(G,F)), \tag{4.4}$$

where c is constant holomorphic curvature of plane section of Q.

From (4.1), (4.2), (4.3) and (4.4) we get

$$K_Q(F \wedge G) - K_{\overline{Q}}(F \wedge G) = q(R(F,G)G,F) - q(\overline{R}(F,G)G,F)$$

$$= q(\bar{b}(F,F),\bar{b}(G,G)) - q(\bar{b}(F,G),\bar{b}(G,F))$$

$$= q(\bar{b}(F,F),\bar{b}G,G)) - |\,\bar{b}(F,G)\,|^2$$

$$K_Q(F \wedge G) = K_{\overline{Q}}(F \wedge G) + q(\bar{b}(F,F),\bar{b}(G,G)) - |\,\bar{b}(F,G)\,|^2. \tag{4.5}$$

Remark 4.1. [27] *The holomorphic SC H of Q determined by a unit vector $F \in D$ is the SC determined by $\{F, \psi F\}$.*

Hence, from (4.4) we have

$$H(F) = \frac{c}{4}\{1+q(F,\psi^2 tF)^2\} + q(\bar{b}(F,F),\bar{b}(\psi F,\psi F)) - q(\bar{b}(F,\psi F),\bar{b}(\psi F,F))$$

$$H(F) = \frac{c}{4}\{1+q(F,(\xi\psi+\lambda I)tF)^2\} + q(\bar{b}(F,F),\bar{b}(\psi F,\psi F)) - q(\bar{b}(F,\psi F),\bar{b}(\psi F,F))$$

$$H(F) = \frac{c}{4}\{1+q(F,\lambda tF)^2\} + q(\bar{b}(F,F),\bar{b}(\psi F,\psi F)) - q(\bar{b}(F,\psi F),\bar{b}(\psi F,F)).$$

Proposition 4.2. *Let $(\overline{Q},q,\psi)$ be a metallic Riemannian manifold and Q be its $(SSI)sub_M$, where $\overline{\nabla}_F\psi = 0$. If $\nabla f = 0$, then*

$$K_Q(F \wedge G) = K_{\overline{Q}}(F \wedge G)$$

for unit vectors $F \in D^{\phi}$, $G \in D^{\theta}$, $\phi \neq \theta$.

Proof. : From Proposition 3.5 we have $\hbar(F,G)=0$ for $F\in D^{\phi}, G\in D^{\theta}$.

By (4.5) we have,

$$K_Q(F\wedge G)=K_{\overline{Q}}(F\wedge G)+j(\hbar(F,F),\hbar(G,G))-\mid \hbar(F,G)\mid^2.$$

Now, $\hbar(F,F)\in D^{\phi}$ and $\hbar(G,G)\in D^{\theta}$ as $F\in D^{\phi}$ and $G\in D^{\theta}$, which gives $j(\hbar(F,F),\hbar(G,G))=0$ as D^{ϕ} and D^{θ} are mutually orthogonal. Thus,

$$K_Q(F\wedge G)=K_{\overline{Q}}(F\wedge G).$$

Lemma 4.3. *Let $(\overline{Q},j,\psi)$ be metallic Riemannian manifold and $\overline{\nabla}_F\psi=0$. If Q is a $(SSI)sub_M$ of $\overline{Q}$, then for all $F,G\in D^{\phi}$, following are equivalent:*

(i) $(\nabla_F f)G-(\nabla_G f)F=0$,

(ii) $\bar{b}(tF,G)=\bar{b}(F,tG)$,

(iii) $f[F,G]=\nabla^{\perp}_F fG-\nabla_G fF$,

(iv) $A_\eta tG-tA_\eta G$ is perpendicular to D^{ϕ} for $\eta\in T^{\perp}Q$.

Proof. : From (3.10), we get

$$(\nabla_F fG)=c\bar{b}(F,G)-\bar{b}(F,tG), \tag{4.6}$$

$$(\nabla_G fF)=c\bar{b}(G,F)-\bar{b}(G,tF). \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$(\nabla_F fG)-(\nabla_G fF)=\bar{b}(G,tF)-\bar{b}(F,tG).$$

Comparing the tangential and normal parts, we get

$$\nabla_F fG-\nabla_G fF=0,$$

$$\bar{b}(G,tF)-\bar{b}(F,tG)=0$$

for all $F,G\in D^{\phi}$. From equality (3.9), i.e.

$$f[F,G]=\bar{b}(F,tG)-\bar{b}(G,tF)+\nabla^{\perp}_F fG-\nabla^{\perp}_G fF$$

$$f[F,G]=\nabla^{\perp}_F fG-\nabla^{\perp}_G fF.$$

Now,

$$q(A_\eta tG - tA_\eta G, G) = 0.$$

This shows that $(A_\eta tG - tA_\eta G)$ is perpendicular to D^ϕ.

We assume a local orthonormal basis $E^1, E^2, ...E^{n(\phi)}$ for each t invariant D^ϕ. D^ϕ mean curvature vector is determined by $H^\phi = \sum_{i=1}^{n(\phi)} \bar{b}(E^i, E^i)$, then mean curvature vector is defined by

$$H = \frac{1}{n}(H^0 + H^1 + H^\phi), n = dimQ.$$

Let $\overline{Q}$ be metallic Riemannian manifold with $\nabla\psi = 0$ and Q be its $(SSI)sub_M$. If $H^\phi = 0$, Q is called D^ϕ-minimal and if $H = 0$, Q is called minimal.

For unit vector $F \in D^\phi, \phi \neq 0$ and $c \approx 0$, we establish the ϕ-SC of $\overline{Q}$ and Q respectively, by

$$\overline{H}_\phi(F) = K_{\overline{Q}}(F \wedge G) \quad and \quad H_\phi(F) = K_Q(F \wedge G),$$

where $G = \frac{tF}{\sqrt{\phi}}$. From (4.5), we obtain

$$H_\phi(F) = \overline{H}_\phi(F) + \frac{1}{\phi} q(\bar{b}(F,F), \bar{b}(tF,tF)) - \frac{1}{\phi} | \bar{b}(F,tF) |^2.$$

Proposition 4.4. *Consider a $(SSI)sub_M$ Q of metallic Riemannian manifold $\overline{Q}$ alongwith $\overline{\nabla}\psi =$ 0. For non zero ϕ, if t is ϕ-commutative, then*

$$H_\phi(F) = \overline{H}_\phi(F) + | \bar{b}(F,F) |^2 - \frac{1}{\phi} | \bar{b}(F,tF) |^2.$$

Proof. : From equality (4.5) and $G = \frac{tF}{\sqrt{\phi}}$, we get

$$H_\phi(F) = \overline{H}_\phi(F) + q(\bar{b}(F,F), \bar{b}(\frac{tF}{\sqrt{\phi}}, \frac{tF}{\sqrt{\phi}}) - | \bar{b}(F, \frac{tF}{\sqrt{\phi}}) |^2$$

$$H_\phi(F) = \overline{H}_\phi(F) + \frac{1}{\phi} q(\bar{b}(F,F), \bar{b}(F,t^2F)) - \frac{1}{\phi} | \bar{b}(F,tF) |^2.$$

Since, $t^2F = \phi F, F \in D^\phi$, we have

$$H_\phi(F) = \overline{H}_\phi(F) + \frac{1}{\phi} q(\bar{b}(F,F), \bar{b}(F,\phi F) - \frac{1}{\phi} | \bar{b}(F,tF) |^2$$

$$H_\phi(F) = \overline{H}_\phi(F) + | \bar{b}(F,F) |^2 - \frac{1}{\phi} | \bar{b}(F,tF) |^2.$$

5. Example

Let the metallic Riemannian manifold $(R^{10}=R^5\times R^5, j, \psi)$ with $\overline{\nabla}\psi=0$. We can define ψ as

$$\psi(\frac{\partial}{\partial\mu_i},\frac{\partial}{\partial\nu_i})=(\sigma\frac{\partial}{\partial\mu_1},\sigma\frac{\partial}{\partial\mu_2},\overline{\sigma}\frac{\partial}{\partial\mu_3},\overline{\sigma}\frac{\partial}{\partial\mu_4},\overline{\sigma}\frac{\partial}{\partial\mu_5},\sigma\frac{\partial}{\partial\nu_1},\sigma\frac{\partial}{\partial\nu_2},\sigma\frac{\partial}{\partial\nu_3},\overline{\sigma}\frac{\partial}{\partial\nu_4},\overline{\sigma}\frac{\partial}{\partial\nu_5}),$$

where $0<i,j\le 5$.

$$\psi^2(\frac{\partial}{\partial\mu_i},\frac{\partial}{\partial\nu_i})$$
$$=(\sigma^2\frac{\partial}{\partial\mu_1},\sigma^2\frac{\partial}{\partial\mu_2},\overline{\sigma}^2\frac{\partial}{\partial\mu_3},\overline{\sigma}^2\frac{\partial}{\partial\mu_4},\overline{\sigma}^2\frac{\partial}{\partial\mu_5},\sigma^2\frac{\partial}{\partial\nu_1},\sigma^2\frac{\partial}{\partial\nu_2},\sigma^2\frac{\partial}{\partial\nu_3},\overline{\sigma}^2\frac{\partial}{\partial\nu_4},\overline{\sigma}^2\frac{\partial}{\partial\nu_5})$$

$$\psi^2(\frac{\partial}{\partial\mu_i},\frac{\partial}{\partial\nu_i})=((\xi\sigma+\lambda)\frac{\partial}{\partial\mu_1},(\xi\sigma+\lambda)\frac{\partial}{\partial\mu_2},(\xi\overline{\sigma}+\lambda)\frac{\partial}{\partial\mu_3},(\xi\overline{\sigma}+\lambda)\frac{\partial}{\partial\mu_4},$$
$$\xi(\overline{\sigma}+\lambda)\frac{\partial}{\partial\mu_5},(\xi\sigma+\lambda)\frac{\partial}{\partial\nu_1},(\xi\sigma+\lambda)\frac{\partial}{\partial\nu_2},(\xi\sigma+\lambda)\frac{\partial}{\partial\nu_3},(\xi\overline{\sigma}+\lambda)\frac{\partial}{\partial\nu_4},(\xi\overline{\sigma}+\lambda)\frac{\partial}{\partial\nu_5})$$

$$\psi^2(\frac{\partial}{\partial\mu_i},\frac{\partial}{\partial\nu_i})=\xi\psi(\frac{\partial}{\partial\mu_1},\frac{\partial}{\partial\mu_2},\frac{\partial}{\partial\mu_3},\frac{\partial}{\partial\mu_4},\frac{\partial}{\partial\mu_5},\frac{\partial}{\partial\nu_1},\frac{\partial}{\partial\nu_2},\frac{\partial}{\partial\nu_3},\frac{\partial}{\partial\nu_4},\frac{\partial}{\partial\nu_5})$$
$$+\lambda(\frac{\partial}{\partial\mu_1},\frac{\partial}{\partial\mu_2},\frac{\partial}{\partial\mu_3},\frac{\partial}{\partial\mu_4},\frac{\partial}{\partial\mu_5},\frac{\partial}{\partial\nu_1},\frac{\partial}{\partial\nu_2},\frac{\partial}{\partial\nu_3},\frac{\partial}{\partial\nu_4},\frac{\partial}{\partial\nu_5})$$
$$\psi^2=\xi\psi+\lambda I.$$

Consider a submanifold Q of $\overline{Q}=(R^{10},q,\psi)$ given by

$$f(\mu,\nu,w,x,y)=(\mu+\nu,\mu-\nu,\mu cosx,\nu sinx,w,-w,\mu,2\nu,\mu cosy,\mu siny).$$

Now, we can get $TQ=span\{\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5\}$ where,

$$\zeta_1=\frac{\partial}{\partial\mu_1}+\frac{\partial}{\partial\mu_2}+cosx\frac{\partial}{\partial\mu_3}+sinx\frac{\partial}{\partial\mu_4}+\frac{\partial}{\partial\mu_5}+cosy\frac{\partial}{\partial\nu_4}+siny\frac{\partial}{\partial\nu_5},$$
$$\zeta_2=\frac{\partial}{\partial\mu_1}-\frac{\partial}{\partial\mu_2}+\frac{\partial}{\partial\nu_3},$$
$$\zeta_3=i\frac{\partial}{\partial\nu_1}+\frac{\partial}{\partial\nu_2},$$
$$\zeta_4=-\mu sinx\frac{\partial}{\partial\mu_3}+\mu cosx\frac{\partial}{\partial\mu_4},$$
$$\zeta_5=-\mu siny\frac{\partial}{\partial\nu_4}+\mu cosy\frac{\partial}{\partial\nu_5},$$

where $\sigma = \frac{\xi+\sqrt{\xi^2+4\lambda}}{2}$ is the metallic ratio and $\overline{\sigma} = \xi - \sigma$.

We have $\psi(\zeta_2) = \sigma\zeta_2, \psi(\zeta_4) = \overline{\sigma}\zeta_4, \psi(\zeta_5) = \overline{\sigma}\zeta_5$ and

$$\psi(\zeta_1) = \sigma\frac{\partial}{\partial\mu_1} + \sigma\frac{\partial}{\partial\mu_2} + \overline{\sigma}cosx\frac{\partial}{\partial\mu_3} + \overline{\sigma}sinx\frac{\partial}{\partial\mu_4} + \overline{\sigma}\frac{\partial}{\partial\mu_5} + \overline{\sigma}cosy\frac{\partial}{\partial\nu_4} + \overline{\sigma}siny\frac{\partial}{\partial\nu_5},$$

$$\psi(\zeta_3) = i\sigma\frac{\partial}{\partial\nu_1} + \sigma\frac{\partial}{\partial\nu_2}.$$

Further, we have

$\|\psi(\zeta_1)\|^2 = -\xi\sigma + 5\lambda + 3\xi^2, \|\psi(\zeta_2)\|^2 = 3(\xi\sigma + \lambda), \|\psi(\zeta_3)\|^2 = 0, \|\psi(\zeta_4)\|^2 = \mu^2(-\xi\sigma + \lambda + \xi^2) \quad and \quad \|\psi(\zeta_5)\|^2 = \mu^2(-\xi\sigma + \lambda + \xi^2)$.

Now, we can obtain $< \psi(\zeta_1), \zeta_1 >= 3\xi - \sigma$ and $< \psi(\zeta_i), \zeta_k) >= 0 \; \forall \; i \neq k$ and $0 < i,k \leq 5$.

$$cos\phi = \frac{< \psi(\zeta_1), \zeta_1 >}{\|\zeta_1\|.\|\psi(\zeta_1)\|}$$
$$cos\phi = \frac{3\xi - \sigma}{\sqrt{5(-\xi\sigma + 5\lambda + 3\xi^2)}}.$$

We define $D^\phi = span\{\zeta_1\}$ is a slant distribution with slant angle $\phi = arccos[\frac{3\xi-\sigma}{\sqrt{5(-\xi\sigma+5\lambda+3\xi^2)}}]$ and $D^1 = span\{\zeta_3\}$ is an anti-invariant distribution. Since, $\psi(\zeta_3) \perp TQ$ and $D^0 = span\{\zeta_2, \zeta_4, \zeta_5\}$, so $\psi(D^0) \subset D^0$. Therefore, by above calculation, it can be concluded that Q is a proper $(SSI)sub_M$ of $\overline{Q}$. Let D^ϕ be slant and D^0 be invariant distribution on Q. If $\bar{b}(F,G) = 0$, where $F \in D^\phi$ and $G \in D^0$, then Q is (D^ϕ, D^0)-mixed totally geodesic.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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