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COMPATIBILITY OF TYPE (P) AND FIXED POINT THEOREM IN FUZZY METRIC SPACES

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Abstract: Fuzzy metric space is first defined by Kramosil and Michalek in 1975. Many authors modified Fuzzy metric space and proved fixed point results in Fuzzy metric space. Singh B. and Chauhan were first introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem in 2000. Cho et al were introduced the concept of compatible mapping of type (P). In this paper, a fixed point theorem for six self-mappings is presented by using the concept of compatible maps of type (P) which is the generalized result.

Keywords: Common fixed points, fuzzy metric space, compatible maps, and weak compatible maps.

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1. Introduction

The concept of Fuzzy sets was initially investigated by Zadeh [12] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several

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researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [8] and modified by George and Veeramani [3]. Recently, Grabiec [4] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [11] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Pathak, Chang and Cho [10] introduced the concept of compatible mapping of type (P). Jain and Singh [5] proved a fixed point theorem for six self maps in a fuzzy metric space.

In this paper, a fixed point theorem for six self maps has been established using the concept of compatible maps of type (P), which generalizes the result of Cho [1]. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

In this section, we review some fundamentals of basic fuzzy metric spaces, which will be used the rest of this paper.

Definition 2.1. [9] A binary operation $* : [0,1]\times[0,1]\to[0,1]$ is called a *t-norm* if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for a, b, c, d $\in$ [0, 1].

Examples of t-norms are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 2.2. [1] The 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions: for all x, y, z $\in$ X and s, t > 0

(FM-1) M (x, y, 0) = 0,

(FM-2) M (x, y, t) =1 for all t > 0 if and only if x = y,

(FM-3) M (x, y, t) = M (y, x, t),

(FM-4) M (x, y, t) $*$ M (y, z, s) $\le$ M (x, z, t + s), (FM-5) M (x, y, .) : $[0,\infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty}$ M (x, y, t) =1.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [9] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and

$M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Then (X, M, $*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [9] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, $*$) is said to be a *Cauchy sequence* if and only if for each ε > 0, t > 0, there exists $n_0 \in N$ such that

M (x_n, x_m , t) > 1 - ε for all n, m $\geq n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each ε > 0, t > 0 there exists $n_0 \in$ N such that M (x_n, x, t) > 1 - ε for all n,m $\geq n_0$.

A Fuzzy metric space (X, M, $*$) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [11] Self mappings A and S of a Fuzzy metric space (X, M, $*$) are said to be *compatible* if and only if M (ASx_n, SAx_n, t) $\to$ 1 for all t > 0, whenever $\{x_n\}$ is a sequence in X such that Sx_n, $Ax_n \to$ p for some p in X as $n \to \infty$.

Definition 2.5. [10] Self maps A and S of a Fuzzy metric space (X, M, $*$) are said to be compatible of type (P) M (AAx_n, SSx_n, t) $\to$ 1 for all t >0, whenever $\{x_n\}$ is a sequence in X such that Sx_n, $Ax_n \to$ p for some p in X as $n \to \infty$.

Proposition 2.1. [5] In a fuzzy metric space (X, M, $*$) limit of a sequence is unique.

Lemma 2.1. [4] Let (X, M, $*$) be a fuzzy metric space. Then for all x, y $\in$ X, M (x, y,.)

is a non-decreasing function.

Lemma 2.2. [1] Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)$ $\forall t > 0$, then $x = y$.

Lemma 2.3. [5] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$ $\forall t > 0$ and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4. [7] The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Lemma 2.5. [3] Let $(X, M, *)$ be a fuzzy metric space if there exists $q \in (0,1)$ such that $M(x, y, qt) \geq M(x, y, t/q^n)$ for integer n. Taking limit as $n \to \infty$, $M(x, y, t) \geq 1$ and hence $x = y$.

3. Main results

Theorem 3.1. Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset ST(X)$, $Q(X) \subset AB(X)$;

(b) $AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$;

(c) Either AB or P is continuous;

(d) Pair (P, AB) is compatible and (Q, ST) is compatible map of type (P);

(e) There exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Px, Qy, qt) \geq M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) * M(Px, STy, t).$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof.

Let $x_0 \in X$. From (a) there exist $x_1, x_2 \in X$ such that $Px_0 = STx_1$ and $Qx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that:

$Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$ for n = 1, 2, 3...

Step 1. Put $x = x_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(Px_{2n}, Qx_{2n+1}, qt) \geq M(ABx_{2n}, STx_{2n+1}, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Px_{2n}, STx_{2n+1}, t).$$

$$= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t)$$

$$\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

From lemmas 2.1 and 2.2, we have

$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t)$.

Similarly, we have

$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t)$.

Thus, we have

$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t)$ for n = 1, 2,...

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n+1}, t/q)$$

$$\geq M(y_{n-2}, y_{n-1}, t/q^2)$$

$$\ldots \ldots \ldots\ldots$$

$$\geq M(y_1, y_2, t/q^n) \to 1 \text{ as } n \to \infty, \text{and}$$

hence $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$.

For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that

$M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$

For m, $n \in N$, we suppose $m \geq n$. Then we have

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) * \ldots * M(y_{m-1}, y_m, t/m-n)$$

$$\geq (1 - \varepsilon) * (1 - \varepsilon) * \ldots * (1 - \varepsilon) \text{ (m - n) times}$$

$$\geq (1 - \varepsilon) \text{ and hence } \{y_n\} \text{ is a Cauchy sequence in X.}$$

Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converge to the same point $z \in X$.

i.e. $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$ (1)

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$. (2)

Case I. Suppose AB is continuous.

Since AB is continuous, we have $(AB)^2x_{2n} \to ABz$ and $ABPx_{2n} \to ABz$.

As (P, AB) is compatible pair of type (P), we have

$M(PPx_{2n}, ABABx_{2n}, t) = 1$ for all $t > 0$ or $M(PPx_{2n}, ABz, t) = 1$.

Therefore $PPx_{2n} \to ABz$

Step 2. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(PABx_{2n}, Qx_{2n+1}, qt) \geq M(ABABx_{2n}, STx_{2n+1}, t) * M(PABx_{2n}, ABABx_{2n}, t)$$
$$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PABx_{2n}, STx_{2n+1}, t).$$

Taking $n \to \infty$, we get

$$M(ABz, z, qt) \geq M(ABz, z, t) * M(ABz, ABz, t) * M(z, z, t) * M(ABz, z, t)$$
$$\geq M(ABz, z, t) * M(ABz, z, t) \text{ i.e. } M(ABz, z, qt) \geq M(ABz, z, t).$$

Therefore, by using lemma 2.2, we get $ABz = z$. (3)

Step 3. Put $x = z$ and $y = x_{2n+1}$ in (e), we have

$$M(Pz, Qx_{2n+1}, qt) \geq M(ABz, STx_{2n+1}, t) * M(Pz, ABz, t)$$
$$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pz, STx_{2n+1}, t).$$

Taking $n \to \infty$ and using equation (1), we get

$$M(Pz, z, qt) \geq M(z, z, t) * M(Pz, z, t) * M(z, z, t) * M(Pz, z, t)$$
$$\geq M(Pz, z, t) * M(Pz, z, t) \text{ i.e. } M(Pz, z, qt) \geq M(Pz, z, t).$$

Therefore, by using lemma 2.2, we get $Pz = z$. Therefore, $ABz = Pz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (e), we get

$$M(PBz, Qx_{2n+1}, qt) \geq M(ABBz, STx_{2n+1}, t) * M(PBz, ABBz, t)$$
$$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PBz, STx_{2n+1}, t).$$

As $BP = PB$, $AB = BA$, so we have $P(Bz) = B(Pz) = Bz$ and $(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz$.

Taking $n \to \infty$ and using (1), we get

$$M(Bz, z, qt) \geq M(Bz, z, t) * M(Bz, Bz, t) * M(z, z, t) * M(Bz, z, t)$$

$$\geq M(Bz, z, t) * M(Bz, z, t) \text{ i.e. } M(Bz, z, qt) \geq M(Bz, z, t).$$

Therefore, by using lemma 2.2, we get

Bz = z and also we have ABz = z . Az = z.

Therefore, Az = Bz = Pz = z. (4)

Step 5. As $P(X) \subset ST(X)$, there exists $u \in X$ such that z = Pz = STu.

Putting $x = x_{2n}$ and y = u in (e), we get

$$M(Px_{2n}, Qu, qt) \geq M(ABx_{2n}, STu, t) * M(Px_{2n}, ABx_{2n}, t)$$
$$* \; M(Qu, STu, t) * M(Px_{2n}, STu, t).$$

Taking $n \to \infty$ and using (1) and (2), we get

$$M(z, Qu, qt) \geq M(z, z, t) * M(z, z, t) * M(Qu, z, t) * M(z, z, t)$$
$$\geq M(Qu, z, t) \text{ i.e. } M(z, Qu, qt) \geq M(z, Qu, t).$$

Therefore, by using lemma 2.2, we get Qu = z. Hence STu = z = Qu. Since (Q, ST) is compatible pair of type (P), therefore, by proposition 2.2, we have QSTu = STQu. Thus Qz = STz.

Step 6. Putting $x = x_{2n}$ and y = z in (e), we get

$$M(Px_{2n}, Qz, qt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t)$$
$$* \; M(Qz, STz, t) * M(Px_{2n}, STz, t).$$

Taking $n \to \infty$ and using (2) and step 5, we get

$$M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t)$$
$$\geq M(z, Qz, t) * M(z, Qz, t) \text{ i.e. } M(z, Qz, qt) = M(z, Qz, t).$$

Therefore, by using lemma 2.2, we get Qz = z.

Step 7. Putting $x = x_{2n}$ and y = Tz in (e), we get

$$M(Px_{2n}, QTz, qt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t)$$
$$* \; M(QTz, STTz, t) * M(Px_{2n}, STTz, t).$$

As QT = TQ and ST = TS, we have QTz = TQz = Tz and ST (Tz) = T (STz) = TQz = Tz.

Taking $n \to \infty$, we get

$M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t)$

$\geq M(z, Tz, t) * M(z, Tz, t)$ i.e. $M(z, Tz, qt) = M(z, Tz, t)$.

Therefore, by using lemma 2.2, we get Tz = z. Now STz = Tz = z implies Sz = z.

Hence Sz = Tz = Qz = z. (5)

Combining (4) and (5), we get Az = Bz = Pz = Qz = Tz = Sz = z. Hence, z is the common fixed point of A, B, S, T, P and Q.

Case II. Suppose P is continuous. As P is continuous, $P^2x_{2n} \rightarrow Pz$ and $P(AB)x_{2n} \rightarrow Pz$. As (P, AB) is compatible pair of type (P), we have

$M(PPx_{2n}, ABABx_{2n}, t) = 1$ for all t > 0 or $M(Pz, ABABx_{2n}, t) = 1$

Therefore $(AB)Px_{2n} \rightarrow Pz$.

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have

$M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t)$

$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t)$.

Taking $n \rightarrow \infty$, we get

$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t)$

$\geq M(Pz, z, t) * M(Pz, z, t)$ i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.

Therefore by using lemma 2.2, we have Pz = z. Further, using steps 5, 6, 7, we get Qz = STz = Sz = Tz = z.

Step 9. As $Q(X) \subset AB(X)$, there exists $w \in X$ such that z = Qz = ABw.

Put x = w and $y = x_{2n+1}$ in (e), we have

$M(Pw, Qx_{2n+1}, qt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t)$

$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pw, STx_{2n+1}, t)$.

Taking $n \rightarrow \infty$, we get

$M(Pw, z, qt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(Pw, z, t)$

$\geq M(Pw, z, t) * M(Pw, z, t)$ i.e. $M(Pw, z, qt) \geq M(Pw, z, t)$.

Therefore, by using lemma 2.2, we get Pw = z.Therefore, ABw = Pw = z. As (P, AB) is compatible pair of type (P), we have Pz = ABz. Also, from step 4, we get Bz = z.

Thus, Az = Bz = Pz = z and we see that z is the common fixed point of the six maps in this case also.

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q.

Then Au = Bu = Pu = Qu = Su = Tu = u. Put x = z and y = u in (e), we get

$M(Pz, Qu, qt) \geq M(ABz, STu, t) * M(Pz, ABz, t) * M(Qu, STu, .t) * M(Pz, STu, t).$

Taking $n \to \infty$, we get

$M(z, u, qt) \geq M(z, u, t) * M(z, z, t) * M(u, u, t) * M(z, u, t)$

$\geq M(z, u, t) * M(z, u, t)$ i.e. $M(z, u, qt) \geq M(z, u, t)$.

Therefore by using lemma 2.2, we get z = u. Therefore z is the unique common fixed point of self-maps A, B, S, T, P and Q.

Remark 3.1. If we take B = T = I, the identity map on X in Theorem 3.1, then condition (b) is satisfied trivially and we get

Corollary 3.1. Let $(X, M, *)$ be a complete fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;

(b) either A or P is continuous;

(c) (P, A) and (Q, S) is compatible maps of type (P);

(d) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$M(Px, Qy, qt) \geq M(Ax, Sy, t) * M(Px, Ax, t) * M(Qy, Sy, t) * M(Px, Sy, t).$

Then A, P, S and Q have a unique common fixed point in X

Remark 3.2. In view of remark 3.1, corollary 3.1, is a generalization of the result of Cho [1] in the sense that condition of compatibility of the pairs of self-maps has been restricted to compatibility of type (P) and only one map of the first pair is needed to be continuous.

4. Conclusion

In this paper, a fixed point theorem for six self-mappings is presented by using the concept of compatible maps of type (P) which is the generalized result.

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