Available online at http://scik.org
J. Math. Comput. Sci. 2024, 14:19
https://doi.org/10.28919/jmcs/8882
ISSN: 1927-5307

STOCHASTIC MODELLING OF M/M/2 PRODUCTION INVENTORY SYSTEMS WITH VACATION SCHEDULING FOR SERVERS AND PRODUCTION UNIT

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Abstract. The article analyzes an M/M/2 multiple vacation production inventory system, where a production facility consists of two heterogeneous servers with multiple vacations. The inventory is restocked according to (s,S) policy. The production facility takes a vacation once the inventory level reaches S, and after completing the vacation, if the inventory level exceeds s, the production facility goes on another random vacation with the same distribution rate. The matrix geometric method is used to estimate the steady state distribution of the obtained Markov chain. Various performance measures and cost analysis are done along with numerical results.

Keywords: multiple vacations; production vacation; matrix geometric method; cost analysis.

2020 AMS Subject Classification: 60K25, 90B05, 91B70.

1. Introduction

A production vacation refers to a planned break or downtime in the manufacturing process during which production activities are temporarily paused. This helps to prevent overproduction by ensuring that production matches actual demand and allowing time to assess and adjust inventory levels. Additionally, by aligning production vacations with periods of low demand, companies can avoid overproduction and better match their output to market needs. The policy of most manufacturing companies is to make items based on customer demand, stock the items

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Received September 2, 2024

at that level, and then begin production based on need. If servers are allowed to take advantage of vacation time, this vacation time can be used to do some other work, which will improve the profitability of an organization.

Levy and Yechiali [7] were the first to study the queueing systems with one or more vacations. Yue and Qin [13] analyzed a production inventory system with single server and production vacations. Further understanding of vacation models can be found in studies by Teghem [12] and Doshi [4], as well as Takagi's [11] monograph. Baek and Moon [1] analyzed a production facility with c servers and lost sales that use a regenerative process to analyze the model. Beena and Jose [2] investigated a production inventory model and compared the cost function incurred when using heterogeneous and homogeneous servers. Jeganathan [5] compared two-server Markovian inventory systems with homogeneous servers and heterogeneous servers with server interruptions. Beena and Jose [3] analyzed the production inventory system with multiple servers and production vacation. The impact of production vacation parameter on cost function is analyzed in the model. Li et al. [8] analyzed a two-state production system (in-control and out-of-control state) with inventory deterioration, rework process, and back-ordering. Jose and Beena [6] discussed a retrial inventory system with production and two servers in which one server is static in the system and another takes multiple vacations. Vishwanath Maurya [9] demonstrated a mathematical model for analyzing a Markovian queueing system with two heterogeneous servers and a working vacation.

The next section, section 2 describes the system model. Section 3 comprises the stability condition and steady state probability vector. Performance measure and cost analysis are given in sections 4 and 5. Concluding remarks is given in section 6.

2. System Model

A (s,S) production inventory model that produces a single type of product is considered, where demand from customers is identified one by one according to a Poisson process with rate λ. The production unit takes an exponentially distributed random vacation with rate θ when the inventory level reaches the maximum S. The production rate is η. When the production facility returns from vacation, if the level of inventory declines to s, then it is straight away switched ON and it stops production when the inventory level reaches S. On the other hand if the level of stock is greater than s, then it takes another vacation of same length. This goes on until the level of stock reaches s. Servers avail vacation if there are no customers in the system or inventory level is zero or both. The servers yield exponential service to customers with rate μ_1 and μ_2. The duration of the vacation of server 1 and 2 are assumed to be exponential with parameters θ_1 and θ_2. When the server returns from vacation if the system is still empty or inventory level is zero or both it goes for another vacation. The servers continue in the same manner until they find the system nonempty with a positive inventory level. Vacation periods taken by the production system and servers can be used to assess inventory levels, manage surpluses, or address shortages.

Notations used in this model are

$N(t)$: Number of customers in the system at time t.

$I(t)$: Inventory level at time t.

$$C(t) : \begin{cases} 0, \text{ if both the servers are on vacation} \\ 1, \text{ if server 1 is busy and the server 2 is on vacation} \\ 2, \text{ if server 1 is on vacation and the server 2 is busy} \\ 3, \text{ if both servers are busy} \end{cases}$$

$$P(t) : \begin{cases} 0, \text{ if the production process gets switched OFF} \\ 1, \text{ if the production process gets switched ON} \end{cases}$$

$\mathbf{e}$: $(1,1,1,...,1)'$, column vector of appropriate dimension containing all ones.

It is clear that $\{X(t) = (N(t), C(t), P(t), I(t))t \geq 0\}$ is a continuous time Markov chain (CTMC) with state space $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3$ where,

$$\Omega_0 = (i,0,0,k)|0 \leq k \leq S \cup (i,0,1,k)|0 \leq k \leq S-1; i \geq 0$$

$$\Omega_1 = (i,1,0,k)|1 \leq k \leq S \cup (i,1,1,k)|1 \leq k \leq S-1; i \geq 1$$

$$\Omega_2 = (i,2,0,k)|1 \leq k \leq S \cup (i,2,1,k)|1 \leq k \leq S-1; i \geq 1$$

$$\Omega_3 = (i,3,0,k)|2 \leq k \leq S \cup (i,3,1,k)|2 \leq k \leq S-1; i \geq 2$$

The transitions in the Markov chain is as follows.

a) Transitions due to arrival of customers

$$(i,k,t,j) \xrightarrow{\lambda} (i+1,k,t,j); \begin{cases} \text{for } k=0; t=0, i\geq 0, 1\leq j\leq S \\ \text{for } k=0; t=1, i\geq 0, 1\leq j\leq S-1 \\ \text{for } k=1,2; t=0, i\geq 1, 1\leq j\leq S \\ \text{for } k=1,2; t=1, i\geq 1, 1\leq j\leq S-1 \\ \text{for } k=3, t=0, i\geq 2, 2\leq j\leq S \\ \text{for } k=3, t=1, i\geq 2, 2\leq j\leq S-1 \end{cases}$$

b) Transitions due to the completion of service

$$(1,1,t,j) \xrightarrow{\mu_1} (0,0,t,j-1); \begin{cases} \text{for } t=0, 1\leq j\leq S \\ \text{for } t=1, 1\leq j\leq S-1 \end{cases}$$

$$(1,2,t,j) \xrightarrow{\mu_2} (0,0,t,j-1); \begin{cases} \text{for } t=0, 1\leq j\leq S \\ \text{for } t=1, 1\leq j\leq S-1 \end{cases}$$

$$(i,1,t,1) \xrightarrow{\mu_1} (i-1,0,t,0);\ t=0,1, i\geq 2$$

$$(i,1,t,j) \xrightarrow{\mu_1} (i-1,1,t,j-1); \begin{cases} \text{for } t=0, i\geq 2, 2\leq j\leq S \\ \text{for } t=1, i\geq 2, 2\leq j\leq S-1 \end{cases}$$

$$(i,2,t,1) \xrightarrow{\mu_2} (i-1,0,t,0);\ t=0,1, i\geq 2$$

$$(i,2,t,j) \xrightarrow{\mu_2} (i-1,2,t,j-1); \begin{cases} \text{for } t=0, i\geq 2, 2\leq j\leq S \\ \text{for } t=1, i\geq 2, 2\leq j\leq S-1 \end{cases}$$

$$(i,3,0,2) \xrightarrow{\mu_2} (i-1,1,0,1);\ i\geq 2$$

$$(i,3,0,2) \xrightarrow{\mu_1} (i-1,2,0,1);\ i\geq 2$$

$$(i,3,t,j) \xrightarrow{\mu_1+\mu_2} (i-1,3,t,j-1); \begin{cases} \text{for } t=0, i\geq 2, 3\leq j\leq S \\ \text{for } t=1, i\geq 2, 3\leq j\leq S-1 \end{cases}$$

c) Transitions due to the completion of production of an item

$$(i,k,1,j) \xrightarrow{\eta} (i,k,1,j+1); \begin{cases} \text{for } k=0, i \geq 0, 0 \leq j \leq S-2 \\ \text{for } k=1,2, i \geq 1, 1 \leq j \leq S-2 \\ \text{for } k=3, i \geq 2, 2 \leq j \leq S-2 \end{cases}$$

$$(i,k,1,S-1) \xrightarrow{\eta} (i,k,0,S); \begin{cases} \text{for } k=0, i \geq 0 \\ \text{for } k=1,2, i \geq 1 \\ \text{for } k=3, i \geq 2 \end{cases}$$

d) Transitions due to the completion of vacation of the production facility

$$(i,k,0,j) \xrightarrow{\theta} (i,k,0,j); \begin{cases} \text{for } k=0, i \geq 0, 0 \leq j \leq s \\ \text{for } k=1,2, i \geq 1, 1 \leq j \leq s \\ \text{for } k=3, i \geq 2, 2 \leq j \leq s \end{cases}$$

e) Transitions due to the completion of vacation of servers

$$(i,0,t,j) \xrightarrow{\theta_1} (i,1,t,j); \begin{cases} \text{for } t=0, i \geq 2, 2 \leq j \leq S \\ \text{for } t=1, i \geq 2, 2 \leq j \leq S-1 \end{cases}$$

$$(i,0,t,j) \xrightarrow{\theta_2} (i,2,t,j); \begin{cases} \text{for } t=0, i \geq 2, 2 \leq j \leq S \\ \text{for } t=1, i \geq 2, 2 \leq j \leq S-1 \end{cases}$$

$$(i,1,t,j) \xrightarrow{\theta_2} (i,3,t,j); \begin{cases} \text{for } t=0, i \geq 2, 2 \leq j \leq S \\ \text{for } t=1, i \geq 2, 2 \leq j \leq S-1 \end{cases}$$

$$(i,2,t,j) \xrightarrow{\theta_1} (i,3,t,j); \begin{cases} \text{for } t=0, i \geq 2, 2 \leq j \leq S \\ \text{for } t=1, i \geq 2, 2 \leq j \leq S-1 \end{cases}$$

The infinitesimal generator of the process is given by

$$Q = \begin{bmatrix} C_{00} & C_{01} & 0 & 0 & 0 & \dots \\ C_{10} & C_{11} & C_{12} & 0 & 0 & \dots \\ 0 & C_{21} & C_1 & C_0 & 0 & \dots \\ 0 & 0 & C_2 & C_1 & C_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where C_0, C_1, C_2 are square matrices of order $8S-4$

3. Stability Condition and Computation of Steady State Probability Vector

The stability of the system can be derived by defining a matrix $\tilde{C} = C_0 + C_1 + C_2$. $\tilde{C}$ is irreducible so there exist a row stationary probability vector Π of order $8S-4$ satisfying $\Pi\tilde{C} = 0$ and $\Pi e = 1$. The matrix $\tilde{C}$ can be written as

$$\tilde{C} = \begin{bmatrix} D_{00} & D_{01} & D_{02} & 0 \\ D_{10} & D_{11} & 0 & D_{13} \\ D_{20} & 0 & D_{22} & D_{23} \\ 0 & D_{31} & D_{32} & D_{33} \end{bmatrix}$$

From the renowned result of the standard drift condition of Neuts [10] $\Pi C_0 e < \Pi C_2 e$ is necessary and sufficient condition for the stability of the QBD process. Under the stability condition of the system, there exist a steady state probability vector X=$(X_0, X_1, \dots)$ satisfying $XQ = 0, Xe = 1$.

$$X_0 = (y_{0,0,0,0}, \dots, y_{0,0,0,S}, y_{0,0,1,0} \cdots, y_{0,0,1,S-1})$$

$$X_1 = (y_{1,0,0,0}, \dots, y_{1,0,0,S}, y_{1,0,1,0} \cdots, y_{1,0,1,S-1}, y_{1,1,0,1}, \dots, y_{1,1,0,S}, y_{1,1,1,1} \cdots, y_{1,1,1,S-1},$$

$$y_{1,2,0,1}, \dots, y_{1,2,0,S}, y_{1,2,1,1}, \dots, y_{1,2,1,S-1})$$

$$X_i = (y_{i,0,0,0}, \dots, y_{i,0,0,S}, y_{i,0,1,0} \cdots, y_{i,0,1,S-1}, y_{i,1,0,1}, \dots, y_{i,1,0,S}, y_{i,1,1,1} \cdots, y_{i,1,1,S-1},$$

$$y_{i,2,0,1}, \dots, y_{i,2,0,S}, y_{i,2,1,1}, \dots, y_{i,2,1,S-1},$$

$$y_{i,3,0,2}, \dots, y_{i,3,0,S}, y_{i,3,1,2}, \dots, y_{i,3,1,S-1}), (i \geq 2)$$

. The sub vectors of X can be derived by solving

$$X_0C_{00}+X_1C_{10}=0 \tag{1}$$

$$X_0C_{01}+X_1C_{11}+X_2C_{21}=0 \tag{2}$$

$$X_1C_{12}+X_2[C_1+RC_2]=0 \tag{3}$$

$$X_i=X_{i-1}*R,\ i=3,4,5\ldots \tag{4}$$

The normalizing equation is

$$X_0e+X_1e+X_2(I-R)^{-1}e=1 \tag{5}$$

where R is the minimal non-negative solution of the matrix quadratic equation $R^2C_2+RC_1+C_0=0$, and is computed from $R=-C_0(C_1)^{-1}-R^2C_2(C_1)^{-1}$. R is approximated by the successive substitution method developed by Neuts [10] namely $R_0=0, R_{n+1}=-C_0(C_1)^{-1}-R_n^2C_2(C_1)^{-1}$, $n=0,1,2,\ldots$. The sub vectors X_0, X_1, and X_2 and $X_i, i\geq 3$ can be calculated using equations 1,2,3,4 and 5.

4. Performance Measures

(i) Expected number of buyers in the system:

$$N_{EC}=X_1e+X_2[(I-R)^{-1}+(I-R)^{-2}]e$$

(ii) Expected inventory level:

$$N_{EI}=\sum_{i=0}^{\infty}\sum_{k=1}^{S}ky_{i,0,0,k}+\sum_{i=0}^{\infty}\sum_{k=1}^{S-1}ky_{i,0,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S}ky_{i,1,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}ky_{i,1,1,k}$$
$$+\sum_{i=1}^{\infty}\sum_{k=1}^{S}ky_{i,2,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}ky_{i,2,1,k}+\sum_{i=2}^{\infty}\sum_{k=1}^{S}ky_{i,3,0,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S-1}ky_{i,3,1,k}$$

(iii) Expected inventory level when both the servers are active:

$$N_{EIACT}=\sum_{i=2}^{\infty}\sum_{k=2}^{S}ky_{i,3,0,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S-1}ky_{i,3,1,k}$$

(iv) Expected inventory level when both the servers are in vacation:

$$N_{EIVAC}=\sum_{i=0}^{\infty}\sum_{k=1}^{S}ky_{i,0,0,k}+\sum_{i=0}^{\infty}\sum_{k=1}^{S-1}ky_{i,0,1,k}$$

(v) Mean on-hand inventory level when production unit is ON:

$$N_{EIPON}=\sum_{i=0}^{\infty}\sum_{k=1}^{S-1}ky_{i,0,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}ky_{i,1,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}ky_{i,2,1,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S-1}ky_{i,3,1,k}$$

(vi) Average on-hand inventory level when production unit is OFF:

$$N_{EIPOFF}=\sum_{i=0}^{\infty}\sum_{k=1}^{S-1}ky_{i,0,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S}ky_{i,1,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S}ky_{i,2,0,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S}ky_{i,3,0,k}$$

(vii) Expected number of departures after completing service:

$$N_{EDS}=\mu_1\left[\sum_{i=1}^{\infty}\sum_{k=1}^{S}y_{i,1,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}y_{i,1,1,k}\right]+\mu_2\left[\sum_{i=1}^{\infty}\sum_{k=1}^{S}y_{i,2,0,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}y_{i,2,1,k}\right]$$
$$+(\mu_1+\mu_2)\left[\sum_{i=2}^{\infty}\sum_{k=1}^{S}y_{i,3,0,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S-1}y_{i,3,1,k}\right]$$

(viii) Mean Production rate:

$$N_{EPR}=\eta\left[\sum_{i=0}^{\infty}\sum_{k=0}^{S-1}y_{i,0,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}y_{i,1,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{S-1}y_{i,2,1,k}+\sum_{i=2}^{\infty}\sum_{k=2}^{S-1}y_{i,3,1,k}\right]$$

5. Cost Analysis and Numerical Results

For the construction of cost function, the following costs can be defined as follows:

C : Fixed cost/unit/unit time

c_1 : The production unit running cost per unit per unit time

c_2 : The holding cost of inventory per unit per unit time

c_3 : The holding cost of customers per unit per unit time

c_4 : The cost due to service per unit per unit time

The expected total cost (T_{COST}) of the system per unit per unit time is given by

$$T_{COST}=(C+(S-s)c_1)N_{SWR}+c_2N_{EI}+c_3N_{EC}+c_4N_{EDS}$$

The impact of arrival rate on various system performance measures and expected total cost is detailed in Table 1.

TABLE 1. Variations in the arrival rate λ

λ	N_{EI}	N_{EC}	N_{EDS}	N_{EPR}	T_{COST}
0.6000	3.9161	0.5741	0.1165	0.5933	114.4509
0.7000	3.7675	0.5793	0.0974	0.5851	107.6813
0.8000	3.6196	0.5851	0.0835	0.5796	102.7802
0.9000	3.4763	0.5944	0.0749	0.5781	**100.1402**
1.0000	3.3413	0.6117	0.0723	0.5823	100.4174
1.1000	3.2182	0.6445	0.0762	0.5940	104.5807
1.2000	3.1101	0.7044	0.0872	0.6149	114.0046
1.3000	3.0195	0.8103	0.1055	0.6463	130.6794

$$\mu_1 = 1.5, \mu_2 = 1.2, \eta = 1.5, \theta_1 = 4, \theta_2 = 3, \theta = 5, S = 10, s = 4$$

$$c_1 = 350, c_2 = 4, c_3 = 100, c_4 = 1$$

From table 1, the optimum value of T_{COST} is 100.1402 and is obtained when λ= 0.9.

The fluctuations in the system measures and expected cost due to the changes in the value of the production vacation parameter are illustrated in Table 2. The lowest expected cost is 20.6872 and is obtained at $\theta = 7.5$.

TABLE 2. Variations in production vacation parameter θ

θ	N_{EI}	N_{EC}	N_{EDS}	N_{EPR}	T_{COST}
4.5000	3.4757	0.5947	0.0749	0.5787	20.6907
5.5000	3.4768	0.5941	0.0749	0.5777	20.6881
6.5000	3.4778	0.5937	0.0749	0.5770	20.6873
7.5000	3.4786	0.5934	0.0749	0.5765	**20.6872**
8.5000	3.4792	0.5932	0.0749	0.5761	20.6875
9.5000	3.4798	0.5931	0.0749	0.5759	20.6879
10.5000	3.4803	0.5929	0.0749	0.5756	20.6883

$$\mu_1 = 1.5, \mu_2 = 1.2, \eta = 1.5, \theta_1 = 4, \theta_2 = 3, \lambda = 0.9, S = 10, s = 4$$

$$c_1 = 3.5, c_2 = 4, c_3 = 10, c_4 = 1$$

6. Concluding Remarks

A detailed study of a production inventory system in which a production facility and two servers take multiple vacations is considered. A production vacation is a strategic tool that can lead to long-term gains in cost savings, efficiency, and overall business performance. Based on the model's system performance metrics, a cost function was constructed and numerically analyzed. For future research, extending the model using more than two servers and any arbitrary distribution for lead time is possible.

Conflict of Interests

The author declares that there is no conflict of interests.

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