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SOME RESULTS ON DISJUNCTIVE DOMINATION IN GRAPHS

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Abstract. A set $D \subseteq V$ is a disjunctive dominating set of a graph $G = (V, E)$, if for any x not in D, x is either a neighbor of a vertex in D or there are at least two vertices in D at a distance two from x. Minimum cardinality of such a subset D of V is called the disjunctive domination number of G. In this paper we obtain the values of the disjunctive domination number of splitting graph of path and cycle graphs.

Keywords: domination in graphs; Disjunctive domination; splitting graph of a graph.
2020 AMS Subject Classification: 05C69.

1. Introduction

In graph theory, understanding how vertices interact and influence one another is essential for various applications ranging from network design to algorithm development. The concept of domination in graphs provides valuable insights into the ways in which the subsets of vertices in a graph can control or influence other vertices in it. Over the years, researchers have studied various varieties of domination parameters, each with distinct characteristics and applications. In 1998, Haynes, Hedetniemi, and Slater published a comprehensive survey of the theory of domination in graphs through two seminal books: *Fundamentals of Domination in Graphs* and

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Domination in Graphs: Advanced Topics. These works provided an extensive foundation for understanding various domination concepts.

One of the many varients of domination is disjunctive domination [2], where a set of vertices D in a graph G ensures that every vertex is either in D, adjacent to a vertex in D, or has at least 2 vertices in D at a distance 2 from it in G. The disjunctive domination number of G, denoted by $\gamma_2^d(G)$, is the minimum cardinality of a disjunctive dominating set in G. This type of domination offers a more flexible framework compared to traditional domination. Since its introduction in 2014, several works were done on this topic by many researchers. Some of the important works can be seen in [3, 4, 6].

Graph operators are transformations or modifications applied to a graph that generate new graphs from an original one. In this article, we focus on a specific graph operator known as the Splitting Graph, which was introduced by Sampathkumar and Walikar [9]. This operator constructs a new graph, $S'(G)$ by adding a new vertex x' corresponding to each vertex x of the original graph G, and joining x' to all vertices of G adjacent to x. We call the vertex x' in S'(G) the twin vertex of x in G. Different domination parameters of splitting graph of a graph are studied in [1]. In this paper we find the disjunctive domination number of splitting graph of path and cycle graphs.

For basic graph theoretic terminologies, we refer to the book [5] and for domination related concepts, we refer to the book [7].

2. Main Results

We use the following notations in the next lemma and theorem. Let P_n be the path graph on n vertices. Let $v_1, v_2, ..., v_n$ be the vertices on P_n and $v'_1, v'_2, ..., v'_n$ be the twin vertices of $v_1, v_2, ..., v_n$ respectively in the splitting graph $S'(P_n)$ of P_n.

Let

$$\begin{aligned} V_i &= \{v_i, v'_i\} \quad \text{for } i = 1, 2, ..., n \\ &= \emptyset \quad \text{if } i < 1 \text{ or } i > n \end{aligned}$$

and

$$\begin{aligned} W_i &= V_1 \cup V_2 \cup ... \cup V_i \quad \text{for } i = 1, 2, ..., n \\ &= W_n \quad \text{for } i > n. \end{aligned}$$

Lemma 2.1. If D is any disjunctive dominating set of $S(P_n)$, then

(i) $|D \cap W_3| \geq 2$ or $D \cap W_2 \neq \emptyset$ and $|D \cap W_4| \geq 2$.

(ii) If $|D \cap W_4| = 2$, then $|D \cap V_4| \leq 1$.

(iii) $|D \cap (W_n \setminus W_{n-3})| \geq 2$ or $D \cap (W_n \setminus W_{n-2}) \neq \emptyset$ and $|D \cap (W_n \setminus W_{n-4})| \geq 2$.

(iv) If $|D \cap (W_n \setminus W_{n-4})| = 2$, then $|D \cap V_{n-3}| \leq 1$

Proof. (i) and (ii) are easily followed from the observation that the vertices in V_i has no contribution towards the disjunctive domination of vertices in V_{i-3} or V_{i+3}. Because of symmetry of the graph, results (iii) and (iv) follow from results (i) and (ii). □

Lemma 2.2. If $n \geq 6$ and D is any disjunctive dominating set of $S'(P_n)$, then $|D \cap W_8| \geq 3$ and if $|D \cap W_8| = 3$, then $|D \cap (W_8 \setminus W_4)| \leq 1$.

Proof. From Lemma 2.1 (i) and (ii), we see that if $|D \cap W_4| = 2$, then it will contribute at most half towards the disjunctive domination of V_6. Hence D must contain at least one more vertex from its first or second neighborhood. Thus $|D \cap W_8| \geq 3$. If $|D \cap W_8| = 3$, it also follows from Lemma 2.1(i) that $|D \cap (W_8 \setminus W_4)| \leq 1$. □

Lemma 2.3. If $n \geq 8$ and D is any disjunctive dominating set of $S'(P_n)$, then $|D \cap W_{10}| \geq 4$ and if $|D \cap W_{10}| = 4$, then $|D \cap (W_{10} \setminus W_8)| \leq 1$.

Proof. From Lemma 2.2, we see that if $|D \cap W_8| = 3$, then it will contribute at most half towards the disjunctive domination of at least one vertex in V_7 or V_8. Hence D must contain at least one more vertex from its first or second neighborhood. Thus $|D \cap W_{10}| \geq 4$. If $|D \cap W_{10}| = 4$, it also shows that $|D \cap (W_{10} \setminus W_8)| \leq 1$. □

Theorem 2.4. For $n \geq 2$

$$\gamma_2^d(S'(P_n)) = \begin{cases} 2\lceil \frac{n-1}{6} \rceil + 1 & \text{if } n \equiv 0, 1 \ (mod\ 6) \\ 2\lceil \frac{n}{6} \rceil & \text{if } n \equiv 2, 3, 4, 5 \ (mod\ 6) \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on P_n. Let v_i', V_i, W_i, $i = 1, 2, \ldots, n$ be as defined above.

Case (i) $2 \leq n \leq 5$**.** Singletons in W_2 will not dominate $S'(P_2)$. On the other hand, there are two element subsets of W_2 that are disjunctive dominating sets of $S'(P_2)$. Hence $\gamma_2^d(S'(P_2)) = 2$. If $n = 3,4,5$, the set $V_3 = \{v_3, v_3'\}$ form a disjunctive dominating set of $S'(P_n)$. Also since there is no universal vertex, $\gamma_2^d(S'(P_n) \geq 2$. Thus $\gamma_2^d(S'(P_n) = 2$ if $2 \leq n \leq 5$. γ_2^d sets of $S'(P_5)$ are depicted in Figure 1.

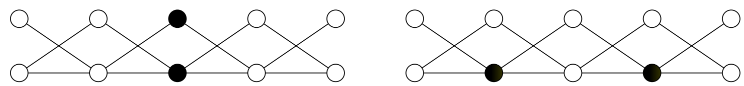

FIGURE 1. γ_2^d sets of $S'(P_5)$

Case (ii) $n = 6,7$**.** $\{v_2, v_4, v_6\}$ is a disjunctive dominating set of $S'(P_n)$. Hence $\gamma_2^d(S'(P_n)) \leq 3$. Now from Lemma 2.1 (i), we see that $|D \cap W_4| \geq 2$. If $|D \cap W_4| = 2$, then by Lemma 2.1(ii), D will contribute at most half towards the disjunctive domination of V_6. Hence D must contain at least one more vertex. Thus $|D| \geq 3$. Therefore $\gamma_2^d(S'(P_n)) = 3$ if $n = 6,7$. γ_2^d-set of $S'(P_7)$ is depicted in Figure 2.

From cases (i) and (ii) we get

$$\gamma_2^d(S'(P_n)) = \begin{cases} 2 & \text{if } 2 \leq n \leq 5 \\ 3 & \text{if } n = 6,7 \end{cases}$$

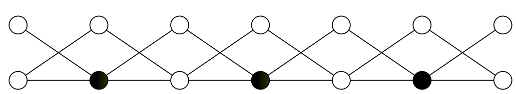

FIGURE 2. γ_2^d set of $S'(P_7)$

Case (iii) $8 \leq n \leq 11$**.** From Lemma 2.1(i), we see that $|D \cap W_4| \geq 2$. Similarly from Lemma 2.1(iii) we get $|D \cap (W_8 \setminus W_4)| \geq 2$. As $W_4 \cap (W_8 \setminus W_4) = \emptyset$, it follows that $|D| \geq 4$. On the other hand, $D = \{v_2, v_4, v_6, v_8\}$ is a disjunctive dominating set of $S'(P_8)$. Hence $\gamma_2^d(S'(P_8)) = 4$.

Now $D = \{v_2, v_4, v_8, v_{10}\}$ is a disjunctive set of $S'(P_{11})$. Hence $\gamma_2^d(S'(P_n) \leq 4$ if $n = 9,10,11$. As $\gamma_2^d(S'(P_8)) = 4$, we conclude that $\gamma_2^d(S'(P_n)) \geq 4$ for $n = 9,10,11$. Thus $\gamma_2^d(S'(P_n)) = 4$ for $n = 8,9,10,11$.

Case (iv) $n \geq 12$**.**

Let $n \equiv 0,1$ $(mod\ 6)$. If $n = 6k$, partition the vertices in $S'(P_n)$ into three sets W_8, $X = W_n \setminus W_{n-4}$ and $Y = V(S'(P_n)) \setminus (W_8 \cup X)$.

Then by Lemma 2.2, we get $|D \cap W_8| \geq 3$. By Lemma 2.1 (iii) and (iv) we see that $|D \cap X| \geq 2$. Now consider any subset $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$ of Y for any six consecutive indices $i, i+1, i+2, i+3, i+4, i+5$. For the disjunctive domination of $V_{i+2} \cup V_{i+3}$ it is clear that D must contain at least two vertices from $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$. As this is true for any set of six consecutive sets V_i in Y and since $|Y| = 6(k-2)$ we get $|D \cap Y| \geq 2(k-2)$. Thus $|D| \geq 3 + 2(k-2) + 2 = 2k+1 = 2\lceil \frac{n-1}{6} \rceil + 1$.

On the other hand $D = \{v_{6i+2}, v_{6i+4} : i = 0,1,2,...,k-1\} \cup \{v_{6k}\}$ is a disjunctive dominating set of $S'(P_{6k})$. As the number of vertices in this set is $2k+1$, we get $\gamma_2^d(S'(P_n)) \leq 2\lceil \frac{n-1}{6} \rceil + 1$. Thus if $n = 6k$, $\gamma_2^d(S'(P_n)) = 2\lceil \frac{n-1}{6} \rceil + 1$.

If $n = 6k+1$, then $\gamma_2^d(S'(P_n)) \geq 2k+1$ as $\gamma_2^d(S'(P_{6k})) = 2k+1$. $D = \{v_{6i+2}, v_{6i+4} : i = 0,1,2,...,k-1\} \cup \{v_{6k}\}$ is also a disjunctive dominating set of $S'(P_n)$ for $n = 6k+1$. Hence $\gamma_2^d(S'(P_n)) = 2k+1 = 2\lceil \frac{n-1}{6} \rceil + 1$ for $n = 6k+1$.

Let $n \equiv 2,3,4,5$ $(mod\ 6)$. Let $n = 6k+2$. Partition vertices in $S'(P_n)$ into three sets W_4, $X = W_n \setminus W_{n-4}$ and $Y = V(S'(P_n)) \setminus (W_4 \cup X)$. As in the above case we see that $|D \cap W_4| \geq 2$, $|D \cap X| \geq 2$ and $|D \cap Y| \geq 2(k-1)$. Hence $|D| \geq 2 + 2(k-1) + 2 = 2k+2 = 2\lceil \frac{n}{6} \rceil$.

Thus $\gamma_2^d(S'(P_n)) \geq 2\lceil \frac{n}{6} \rceil$ if $n \geq 6k+2$.

Now if $n = 6k+5$, then $D = \{v_{6i+2}, v_{6i+4} : i = 0,1,2,...,k-1\} \cup \{v_{6k+2}, v_{6k+4}\}$ is a disjunctive dominating set of $S'(P_n)$. As the number of vertices in this set is $2k+2 = 2\lceil \frac{n}{6} \rceil$, we get $\gamma_2^d(S'(P_n)) \leq 2\lceil \frac{n}{6} \rceil$ if $n = 6k+5$. Hence $\gamma_2^d(S'(P_n)) \leq 2\lceil \frac{n}{6} \rceil$ if $n \leq 6k+5$.

Thus $\gamma_2^d(S'(P_n)) = 2\lceil \frac{n}{6} \rceil$ if $n = 6k+2, 6k+3, 6k+4$ and $6k+5$.

By summing up the cases (i), (ii),(iii) and (iv) we see that if $n \geq 2$,

$$\gamma_2^d(S'(P_n)) = \begin{cases} 2\lceil \frac{n-1}{6} \rceil + 1 & \text{if } n \equiv 0,1\ (mod\ 6) \\ 2\lceil \frac{n}{6} \rceil & \text{if } n \equiv 2,3,4,5\ (mod\ 6) \end{cases}$$

$\square$

Theorem 2.5. For any integer $n \geq 3$,

$$\gamma_2^d(S'(C_n)) = \begin{cases} 2\lceil \frac{n}{6} \rceil - 1 & \text{if } n \equiv 1,2 \ (mod\ 6) \\ \\ 2\lceil \frac{n}{6} \rceil & \text{otherwise} \end{cases}$$

Proof. Let $\{v_1, v_2, v_3, ..., v_n\}$ be the vertices of C_n. If $n = 3,4,5,6$, then $D = \{v_1, v_3\}$ is a disjunctive dominating set of $S'(C_n)$. As there is no universal vertex, we also see that $\gamma_2^d(S'(C_n)) \geq 2$. Thus $\gamma_2^d(S'(C_n)) = 2$ if $n = 3,4,5,6$. γ_2^d-set of $S'(C_6)$ is depicted in Figure 3.

Now let $n \geq 7$ and let $V_i = \{v_i, v_i'\}$ for $i \in \{1,2,...,n\}$. It can be noted here that a vertex in V_i

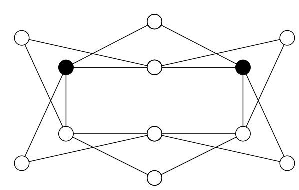

FIGURE 3. γ_2^d set of $S'(C_6)$

contributes only half towards the disjunctive domination of a vertex in V_{i+2} and V_{i-2}. It has no contribution towards the disjunctive domination of a vertex in V_{i+3} and V_{i-3}.

It can also be noted here that a vertex $x \in D \cap V_i$ contributes at most half towards the disjunctive domination of the other vertex in V_i. So D must contain one more vertex in the first or second neighborhood it. Hence corresponding to each vertex $x \in D \cap V_i$, there exist a vertex $y \in D$ such that $d(x,y) \leq 2$.

Let $n \equiv 0 (mod\ 6)$**.** If $n = 6k$, then $D = \{v_1, v_3, v_7, v_9, v_{13}, v_{15}, ..., v_{6k-5}, v_{6k-3}\} = \{v_{6j-5}, v_{6j-3} : j = 1,2,...,k\}$ is a disjunctive dominating set of $S'(C_{6k})$ of cardinality $2k$. Hence if $n = 6k$, then $\gamma_2^d(S'(C_n)) \leq 2k = 2\lceil \frac{n}{6} \rceil$.

The reverse inequality can be seen as follows. Let D be any disjunctive dominating set of the graph $G = S'(C_n)$. Consider any subset $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$ of the vertex set of G for any six consecutive indices $i, i+1, i+2, i+3, i+4, i+5$. For the disjunctive domination of $V_{i+2} \cup V_{i+3}$ it is clear that D must contain at least two vertices from $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$. As this is true for any set of six consecutive sets V_i in G we get $|D| \geq 2k$. Hence if

$n=6k$, then $\gamma_2^d(S'(C_n))=2k=2\lceil\frac{n}{6}\rceil$.

γ_2^d-set of $S'(C_{12})$ is depicted in Figure 4.

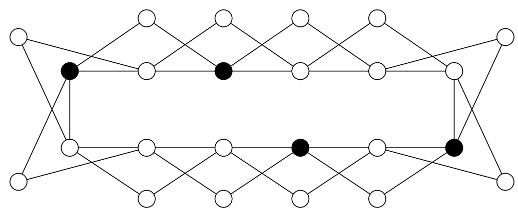

FIGURE 4. γ_2^d set of $S'(C_{12})$

Let $n\geq 7$ **and** $n\equiv 1,2(mod\ 6)$**.** Let $n=6k+1$ and let D be any disjunctive dominating set of $S'(C_n)$. Let $x\in D$. Without loss of generality we can assume that $x\in V_1$. Then there exists another vertex $y\in D$ such that $d(x,y)\leq 2$. Hence either $|D\cap(V_{n-1}\cup V_n\cup V_1)|\geq 2$ or $|D\cap(V_1\cup V_2\cup V_3)|\geq 2$. Let $|D\cap(V_1\cup V_2\cup V_3)|\geq 2$. If $n\geq 7$ and $|D\cap(V_1\cup V_2\cup V_3)|=2$, then they will contribute at most half towards the disjunctive domination of vertices in V_5 and V_6. Hence D must contain at least one more vertex from their first or second neighborhood. Thus $|D\cap(V_1\cup V_2\cup ...\cup V_7)|\geq 3$. Now from the remaining set of vertices in $S'(C_n)$, D must contain at least two vertices corresponding to every set of six consecutive V_i's in $\{V_8,V_9,...,V_{6k+1}\}$. Thus $|D|\geq 3+2(k-1)=2k+1=2\lceil\frac{n}{6}\rceil-1$.

Hence $\gamma_2^d(S'(C_n))\geq 2\lceil\frac{n}{6}\rceil-1$ if $n\geq 6k+1$.

On the other hand, $D=\{v_{6j-5},v_{6j-3}:j=1,2,...,k\}\cup\{v_{6k+1}\}$ is a disjunctive dominating set of $S'(C_n)$ if $n=6k+1$, $6k+2$. As the number of vertices in this set is $2k+1$ we get $\gamma_2^d(S'(C_n))\leq 2k+1=2\lceil\frac{n}{6}\rceil-1$.

Thus $\gamma_2^d(S'(C_n))=2\lceil\frac{n}{6}\rceil-1$ if $n=6k+1$ or $6k+2$.

Let $n\equiv 3,4,5(mod\ 6)$**.** Let $n=6k+3$ and let D be any disjunctive dominating set of $S'(C_n)$. For the disjunctive domination of vertices in $V_1\cup V_2\cup...\cup V_{6k}$ at least 2k vertices are needed. Two vertices that contribute to the disjunctive domination of V_{i+2} and V_{i+3} from six consecutive sets $V_i,V_{i+1,...,}V_{i+5}$ provide at most half to the disjunctive domination of a vertex outside this set. Hence for the disjunctive domination of vertices in $V_{6k+1}\cup V_{6k+2}\cup V_{6k+3}$ at least two more vertices are needed in D. Thus $|D|\geq 2k+2=2\lceil\frac{n}{6}\rceil$. Hence $\gamma_2^d(S'(C_n))\geq 2\lceil\frac{n}{6}\rceil$ if $n\geq 6k+3$.

On the other hand, $D = \{v_{6j-5}, v_{6j-3} : j = 1,2,...,k\} \cup \{v_{6k+1}, v_{6k+3}\}$ is a disjunctive dominating set of $S'(C_n)$ of cardinality $2k+2$ if $n = 6k+3, 6k+4, 6k+5$.
Thus $\gamma_2^d(S'(C_n)) = 2k+2 = 2\lceil \frac{n}{6} \rceil$ for $n \equiv 3,4,5 (mod\ 6)$.

By summing up all the above results we see that, if $n \geq 3$ then,

$$\gamma_2^d(S'(C_n)) = \begin{cases} 2\lceil \frac{n}{6} \rceil - 1 & \text{if } n \equiv 1,2 \ (mod\ 6) \\ \\ 2\lceil \frac{n}{6} \rceil & \text{otherwise} \end{cases}$$

□

3. Conclusion

In this article, we investigated the disjunctive domination number of splitting graphs of path and cycle graphs. By applying the splitting graph operator to these fundamental graph classes, we derived the exact values of their disjunctive domination numbers. These results provide valuable insights into how structural transformations impact domination-related parameters. Future research may explore similar analysis for other graph classes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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