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FULLY-DISCRETE H^1-GALERKIN MIXED FINITE ELEMENT METHOD FOR PSEUDO-PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we discuss a fully-discrete H^1-Galerkin mixed finite element method for pseudo-parabolic integro-differential equations in one dimensional case. Optimal order error estimates for the scalar unknown and its gradient in L^2-norms and H^1-norms are obtained.

Keywords: pseudo-parabolic integro-differential equations, fully-discrete, H^1-Galerkin mixed finite element method, error estimate.

2000 AMS Subject Classification: 65N30

1. Introduction

In this paper, we consider the following pseudo-parabolic integro-differential equations in one dimensional case

$$\begin{cases} u_t = \left(a(x,t)u_{xt} + b(x,t)u_x + \int_0^t c(x,t,\tau)u_x(x,\tau)ds\right)_x + f(x,t), & (x,t) \in [0,1] \times J, \\ u(0,t) = u(1,t) = 0, & t \in J, \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{1}$$

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where $J = (0, T]$ is the time interval with $0 < T < \infty$. f, u_0 are given functions, and we make the following assumptions for the coefficients of equation (1): $0 < a_0 \leq a(x,t), a_t \leq a_1, |b(x,t)| \leq b_1, |c(x,t,\tau)| \leq c_1$, where a_0, a_1, b_1, c_1 are all positive constants.

Evolution integro-differential equations are a class of important evolution partial differential equations, and have a lot of applications in many physical problems, such as the transport of reactive and passive contaminates in aquifers. For this class of equations, the existence of solutions has been studied, such as in [1], [2]. In recent years, the numerical methods to solve these equations are also proposed, for example, finite element method in [3] and adaptive least-squares mixed finite element method in [4].

Standard mixed finite element methods were developed and analyzed in [6], [7], [8] for elliptic equations, [9] for parabolic equations, and [10], [11] for wave equations. In general, the LBB stability condition is required for the mixed finite element method, which restricts the choice of finite element spaces. Recently, in order to overcome this difficulty, an H^1-Galerkin mixed finite element method was proposed in [12] for parabolic problems. The proposed method is a non-symmetric version of least square method. It has been proved that the H^1-Galerkin mixed finite element method has the same rate of convergence as standard mixed finite element method.

In this paper, we construct a fully-discrete H^1-Galerkin mixed finite element method to the equations (1). By introducing a flux q, we split (1) into a system of two equations and then apply the H^1-Galerkin mixed finite element method. For the temporal discretization, we consider the backward Euler method, which is first order accurate in time. Optimal error estimates for the scalar unknown u and its flux q in L^2-norm and H^1-norm are achieved.

Throughout the paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev space on Ω with a norm $\|\cdot\|_{m,q}$ and a semi-norm $|\cdot|_{m,q}$. For $q = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and for $m = 0$, we denote $\|\cdot\| = \|\cdot\|_0$. Moreover, the inner products in $L^2(\Omega)$ are indicated by $(\cdot,\cdot)$. Let X be a Banach space and $\phi(t) : [0,T] \longmapsto X$, we set

$$\|\phi\|^2_{L^2(X)} = \int_0^T \|\phi(s)\|^2_X ds, \qquad \|\phi\|_{L^\infty(X)} = ess \sup_{0\leq t\leq T} \|\phi\|_X.$$

In addition, C denotes a generic constant independent of the spatial mesh parameter h and time discretization parameter $\triangle t$, and ε denotes an arbitrarily small positive constant.

2. The Fully discrete H^1-Galerkin Mixed Finite Element Approximation Scheme

For simplicity, we set $a := a(x,t), \quad b := b(x,t), c := c(x,t,\tau)$ and $q = a(x,t)u_{tx} + b(x,t)u_x + \int_0^t c(x,t,\tau)u_x(x,\tau)d\tau$, so equation (1) can be rewritten as

$$\begin{cases} (a) \;\; u_t - q_x = f(x,t), & (x,t) \in [0,1] \times (0,T], \\ (b) \;\; a(x,t)u_{tx} + b(x,t)u_x + \int_0^t c(x,t,\tau)u_x(x,\tau)d\tau = q, & (x,t) \in [0,1] \times (0,T] \\ (c) \;\; u(0,t) = u(1,t) = 0, & t \in (0,T], \\ (d) \;\; u(x,0) = u_0(x), & x \in [0,1]. \end{cases} \tag{2}$$

Let $w \in H^1$, multiplying (2a) by $w_x \in H^1$ and integrating on interval $[0,1]$ we obtain

$$(u_t, w_x) - (q_x, w_x) = (f, w_x), \quad w \in H^1.$$

Let $v \in H_0^1$, multiplying (2b) by $v_x \in H_0^1$ and integrating on interval $[0,1]$ we obtain

$$(au_{tx}, v_x) + (bu_x, v_x) + \int_0^t (cu_x, v_x)d\tau = (q, v_x), \quad v \in H_0^1.$$

Note that $u_t(0) = u_t(1) = 0$ and $q = a(x,t)u_{tx} + b(x,t)u_x + \int_0^t c(x,t,\tau)u_x(x,\tau)d\tau$. Then by Green formula we have

$$(\alpha q, w) + (q_x, w_x) = (\beta u_x, w) + \int_0^t (\gamma u_x, w)d\tau - (f, w_x), \quad w \in H^1,$$

where $\alpha = \frac{1}{a}, \beta = \frac{b}{a}, \gamma = \frac{c}{a}$.

Then the weak form of H^1-Galerkin mixed finite element method for (2) can be defined by

$$\begin{cases} (au_{tx}, v_x) + (bu_x, v_x) + \int_0^t (cu_x, v_x)d\tau = (q, v_x), & v \in H_0^1, \\ (\alpha q, w) + (q_x, w_x) = (\beta u_x, w) + \int_0^t (\gamma u_x, w)d\tau - (f, w_x) & , w \in H^1. \end{cases} \tag{3}$$

Let V_h, W_h be finite dimensional subspaces of H_0^1 and H^1, respectively, with the following approximation properties:

$$\inf_{v_h \in V_h} \{\|v - v_h\|_{L^p} + h\|v - v_h\|_{W^{1,p}}\} \le Ch^{k+1}\|v\|_{W^{k+1,p}}, \quad v \in H_0^1 \cap W^{k+1,p},$$

$$\inf_{w_h \in W_h}\{\|w-w_h\|_{L^p}+h\|w-w_h\|_{W^{1,p}}\} \le Ch^{r+1}\|w|_{W^{r+1,p}}, \quad w \in W^{r+1,p},$$

where $1 \le p \le \infty, k, r$ are integers.

For the temporal discretization, we consider the backward Euler method, which is first order accurate in time. Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a given partition of the time interval $[0,T]$ with step length $\Delta t = \frac{T}{N}$, for some positive integer N. Define $t^n = n\triangle t$, $\phi^n = \phi(t^n), \bar{\partial}_t\phi^n = (\phi^n - \phi^{n-1})/\triangle t$ for a smooth function ϕ. Let U^n and Q^n be the approximation of u and q at $t = t^n$, then the fully discrete H^1-Galerkin mixed finite element approximation scheme of (3) is to find $\{U^n, Q^n\} \in V_h \times W_h$ such that

$$\begin{cases} (a_n\bar{\partial}_t U_x^n, v_{hx}) + (b_n U_x^n, v_{hx}) + \triangle t \sum\limits_{j=0}^{n-1}(c_{nj}U_x^j, v_{hx}) = (Q^n, v_{hx}), & v_h \in V_h, \\ (\alpha_n Q^n, w_h) + (Q_x^n, w_{hx}) = (\beta_n U_x^n, w_h) + \triangle t \sum\limits_{j=0}^{n-1}(\gamma_{nj}U_x^j, w_h) - (f^n, w_{hx}), & w_h \in W_h, \end{cases} \tag{4}$$

where $a_n = a(t^n), b_n = b(t^n), c_{nj} = c(t_n, t_j), \alpha_n = \alpha(t^n), \beta_n = \beta(t^n), \gamma_{nj} = \gamma(t_n, t_j)$.

For our error estimates, we introduce the following projections.

(i) From [15], we define the Soblev-Volterra projection: to find $\tilde{u}_h \in V_h$ such that:

$$(a(u_t - \tilde{u}_{ht})_x + b(u_x - \tilde{u}_{hx}) + \int_0^t c(u_x - \tilde{u}_{hx})d\tau, v_{hx}) = 0, \tag{5}$$

which satisfy the following estimate,

$$\| u - \tilde{u}_h \| + h \| (u - \tilde{u}_h)_x \| \le Mh^{k+1}(\| u \|_{k+1} + \int_0^t \| u \|_{k+1} d\tau). \tag{6}$$

(ii) Following [16], we define an elliptic projection $\tilde{q}_h \in W_h$, such that:

$$((q - \tilde{q}_h)_x, w_{hx}) = 0, \qquad \forall w_h \in W_h, \tag{7}$$

and also we have

$$\| q - \tilde{q}_h \| + h \| (q - \tilde{q}_h)_x \| \le Mh^{r+1} \| q \|_{r+1} . \tag{8}$$

3. Convergence Analysis

Let

$$u(t^n) - U^n = u(t^n) - \tilde{u}_h(t^n) + \tilde{u}_h(t^n) - U^n = \eta^n + \zeta^n$$

$$q(t^n) - Q^n = q(t^n) - \tilde{q}_h(t^n) + \tilde{q}_h(t^n) - Q^n = \rho^n + \xi^n$$

Set $t = t^n$ in (2) and combine (3), (5), (7), we can get the following error equations,

$$\begin{cases} (a)\ (a_n\bar{\partial}_t\zeta_x^n, v_{hx}) + (b_n\zeta_x^n, v_{hx}) + \triangle t\sum\limits_{j=0}^{n-1}(c_{nj}\zeta_x^j, v_{hx}) = (\rho^n + \xi^n, v_{hx}) + \epsilon_1^n(v_h) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (a_n\tau^n, v_{hx}), \\ (b)\ (\alpha_n\xi_n, w_h) + (\xi_x^n, w_{hx}) = -(\beta_n\eta_x^n, w_{hx}) - (\beta_{xn}\eta^n, w_h) + (\beta_n\zeta_x^n, w_h) \\ \qquad\qquad\qquad\qquad + \triangle t\sum\limits_{j=0}^{n-1}[(\gamma_{n,j}\rho^j, w_{hx}) + (\gamma_{xnj}\rho^j, w_h)] \\ \qquad\qquad\qquad\qquad + \triangle t\sum\limits_{j=0}^{n-1}(\gamma_{nj}\xi_x^j, w_h) + \epsilon_2^n(w_h), \end{cases} \tag{9}$$

where

$$\tau^n = \bar{\partial}_t\tilde{u}_{hx}(t^n) - \tilde{u}_{hxt}(t^n),$$

$$\epsilon_1^n = \int_0^{t^n}(c_n\tilde{u}_{hx}, v_{hx})d\tau - \triangle t\sum_{j=0}^{n-1}(c_{nj}\tilde{u}_{hx}(t_j), v_{hx}),$$

$$\epsilon_2^n = \int_0^{t^n}(\gamma_n u_x(t^n), w_h)d\tau - \triangle t\sum_{j=0}^{n-1}(\gamma_{nj}u_x(t_j), w_h).$$

Since the estimates of η^n and ρ^n can be found out easily from (6) and (8) at $t = t^n$, it is enough to estimate ζ^n and ξ^n.

Theorem 3.1. *Assume that $U^0 = \tilde{u}_h(0), Q^0 = \tilde{q}_h(0)$ and $0 \le J \le N$. Then there exists a positive constant C independent of h and Δt such that for $j = 0, 1$ the following estimate holds*

$$\begin{aligned} & \| u^J - U^J \|_j + \| q^J - Q^J \|_j \\ \le\ & Ch^{\min\{k+1-j, r+1-j\}}(\| u \|_{L^\infty(H^{k+1})} + \| q \|_{L^\infty(H^{r+1})})\ \cdot \\ +\ & C\triangle t(\| u \|_{L^2(H^1)} + \| u_t \|_{L^2(H^1)} + \| u_{tt} \|_{L^2(H^1)}) \end{aligned} \tag{3.1}$$

Proof. Choose $v_h = \zeta^n$ in (9a) to obtain for $n = 0, 1, \cdots, N$,

$$(a_n\bar{\partial}_t\zeta_x^n, \zeta_x^n) + (b_n\zeta_x^n, \zeta_x^n) + \triangle t\sum_{j=0}^{n-1}(c_{nj}\zeta_x^j, \zeta_x^n) = (\rho^n + \xi^n, \zeta_x^n) + \epsilon_1^n + (a_n\tau^n, \zeta_x^n). \tag{10}$$

Note that $a \geq a_0, |b| \leq b_1, |c| \leq c_1$ and a_0, b_1, c_1 are positive constants, then by Hölder inequality and ε inequality we have

$$\begin{aligned}&(a_n\bar{\partial}_t\zeta_x^n, \zeta_x^n) + (b_n\zeta_x^n, \zeta_x^n) + \triangle t\sum_{j=0}^{n-1}(c_{nj}\zeta_x^j, \zeta_x^n)\\ &\geq \tfrac{1}{2}a_0 \parallel \bar{\partial}_t\zeta_x^n \parallel^2 - b_1 \parallel \zeta_x^n \parallel^2 - c_1\triangle t\sum_{j=0}^{n-1} \parallel \zeta_x^n \parallel^2 - c_1T\varepsilon \parallel \zeta_x^n \parallel^2,\end{aligned} \tag{11}$$

and

$$\begin{aligned}&(\rho^n + \xi^n, \zeta_x^n) + \epsilon_1^n + (a_n\tau^n, \zeta_x^n)\\ &\leq C(\parallel \rho^n \parallel^2 + \parallel \xi^n \parallel^2 + \parallel \epsilon_1^n \parallel^2 + \parallel \tau^n \parallel^2) + \varepsilon \parallel \zeta_x^n \parallel^2 .\end{aligned} \tag{12}$$

Combine (10)-(12), we can get

$$\begin{aligned}\tfrac{1}{2}a_0 \parallel \bar{\partial}_t\zeta_x^n \parallel^2 \leq{}& c_1\triangle t\sum_{j=0}^{n-1} \parallel \zeta_x^n \parallel^2 + C(\parallel \rho^n \parallel^2 + \parallel \xi^n \parallel^2 + \parallel \epsilon_1^n \parallel^2 + \parallel \tau^n \parallel^2)\\ &+(b_1 + \varepsilon + c_1T\varepsilon) \parallel \zeta_x^n \parallel^2 .\end{aligned}$$

By Taylor formula we can derive

$$\begin{aligned}\parallel \varepsilon_1^n \parallel^2 &= \parallel \int_0^{t^n}(c_n\tilde{u}_{hx}, v_{hx})d\tau - \triangle t\sum_{j=0}^{n-1}(c_{nj}\tilde{u}_{hx}(t_j), v_{hx}) \parallel^2\\ &\leq C(\triangle t)^2\int_0^{t^n}\{\parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2\}d\tau,\end{aligned}$$

and

$$\parallel \tau^n \parallel^2 \leq C\triangle t\int_{t^{n-1}}^{t^n} \parallel \tilde{u}_{hxtt} \parallel^2 d\tau.$$

Therefore, we have

$$\begin{aligned}\tfrac{1}{2}a_0\bar{\partial}_t \parallel \zeta_x^n \parallel^2 &\leq c_1\triangle t\sum_{j=0}^{n-1} \parallel \zeta_x^n \parallel^2 + C(\parallel \rho^n \parallel^2 + \parallel \xi^n \parallel^2)\\ &+ C(\triangle t)^2\int_0^{t^n}\{\parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2\}d\tau + C\triangle t\int_{t^{n-1}}^{t^n} \parallel \tilde{u}_{hxtt} \parallel^2 d\tau\\ &+ (b_1 + \varepsilon + c_1T\varepsilon) \parallel \zeta_x^n \parallel^2,\end{aligned}$$

i.e.,

$$\begin{aligned}\tfrac{1}{2}a_0\frac{\|\zeta_x^n\|^2 - \|\zeta_x^{n-1}\|^2}{\triangle t} &\leq c_1\triangle t\sum_{j=0}^{n-1} \parallel \zeta_x^n \parallel^2 + C(\parallel \rho^n \parallel^2 + \parallel \xi^n \parallel^2)\\ &+ C(\triangle t)^2\int_0^{t^n}\{\parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2\}d\tau + C\triangle t\int_{t^{n-1}}^{t^n} \parallel \tilde{u}_{hxtt} \parallel^2 d\tau\\ &+ (b_1 + \varepsilon + c_1T\varepsilon) \parallel \zeta_x^n \parallel^2 .\end{aligned}$$

Then, multiplying by $\triangle t$ and summing from $n = 1, 2, \cdots, J$, we conclude that

$$\begin{aligned}
\tfrac{1}{2}a_0 \parallel \zeta_x^n \parallel^2 &\leq c_1(\triangle t)^2 \sum_{n=1}^{J}\sum_{j=0}^{n-1} \parallel \zeta_x^n \parallel^2 + C\sum_{n=1}^{J}(\parallel \rho^n \parallel^2 + \parallel \xi^n \parallel^2) \\
&+ CJ(\triangle t)^3 \int_0^T \{\parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2\} d\tau + C(\triangle t)^2 \int_0^T \parallel \tilde{u}_{hxtt} \parallel^2 d\tau \\
&+ (b_1 + \varepsilon + c_1 T\varepsilon)\triangle t \sum_{n=1}^{J} \parallel \zeta_x^n \parallel^2 .
\end{aligned}$$

Further, we have

$$\begin{aligned}
\tfrac{1}{2}a_0 \parallel \zeta_x^n \parallel^2 &\leq C(\parallel \rho \parallel^2_{L^\infty(L^2)} + \triangle t \sum_{n=1}^{J} \parallel \xi^n \parallel^2) + C(\triangle t)^2 \int_0^T \{\parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2\} d\tau \\
&+ C(\triangle t)^2 \int_0^T \parallel \tilde{u}_{hxtt} \parallel^2 d\tau + (b_1 + \varepsilon + c_1 T\varepsilon)\triangle t \parallel \zeta_x^n \parallel^2 \\
&+ (b_1 + \varepsilon + c_1 T\varepsilon + c_1 \triangle t T) \sum_{n=1}^{J-1} \parallel \zeta_x^n \parallel^2,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&[\tfrac{1}{2}a_0 - (b_1 + \varepsilon + c_1 T\varepsilon)\triangle t] \parallel \zeta_x^n \parallel^2 \\
\leq\ & C(\parallel \rho \parallel^2_{L^\infty(L^2)} + \triangle t \sum_{n=1}^{J} \parallel \xi^n \parallel^2) + C(\triangle t)^2 \int_0^T \{\parallel \tilde{u}_{hx} \parallel^2 \\
+\ & \parallel \tilde{u}_{hxt} \parallel^2\} d\tau + C(\triangle t)^2 \int_0^T \parallel \tilde{u}_{hxtt} \parallel^2 d\tau \\
+\ & (b_1 + \varepsilon + c_1 T\varepsilon + c_1 \triangle t T) \sum_{n=1}^{J-1} \parallel \zeta_x^n \parallel^2 .
\end{aligned}$$

Choosing $\triangle t$ such that $\frac{1}{2}a_0 - (b_1 + \varepsilon + c_1 T\varepsilon)\triangle t > 0$ and using discrete Gronwall's lemma we can derive that

$$\begin{aligned}
\parallel \zeta_x^n \parallel^2 &\leq C(\parallel \rho \parallel^2_{L^\infty(L^2)} + \triangle t \sum_{n=1}^{J} \parallel \xi^n \parallel^2) + C(\triangle t)^2 \int_0^T \{\parallel \tilde{u}_{hx} \parallel^2 \\
&+ \parallel \tilde{u}_{hxt} \parallel^2\} d\tau + C(\triangle t)^2 \int_0^T \parallel \tilde{u}_{hxtt} \parallel^2 d\tau.
\end{aligned} \tag{13}$$

Choose $w_h = \xi^n$ in (9b) to obtain

$$\begin{aligned}
&(\alpha_n \xi_n, \xi^n) + (\xi_x^n, \xi_x^n) = -(\beta_n \eta_x^n, \xi_x^n) - (\beta_{xn}\eta^n, \xi^n) + (\beta_n \zeta_x^n, \xi^n) \\
&+ \triangle t \sum_{j=0}^{n-1} [(\gamma_{n,j}\rho^j, \xi_x^n) + (\gamma_{xnj}\rho^j, \xi^n)] + \triangle t \sum_{j=0}^{n-1} (\gamma_{nj}\xi_x^j, \xi^n) + \epsilon_2^n(\xi^n),
\end{aligned} \tag{14}$$

For the left side of (14), we can get

$$(\alpha_n \xi_n, \xi^n) + (\xi_x^n, \xi_x^n) \geq \alpha_0 \parallel \xi^n \parallel^2 + \parallel \xi_x^n \parallel^2 . \tag{15}$$

Combine (15) and the right side of (14), we can derive the following estimates

$$\begin{aligned}\alpha_0 \parallel \xi^n \parallel^2 + \parallel \xi_x^n \parallel^2 &\leq C(\parallel \eta^n \parallel^2 + \parallel \zeta_x^n \parallel^2 + \varepsilon(\parallel \xi_x^n \parallel^2 + \parallel \xi^n \parallel^2)\\ &+ C\triangle t \sum_{n=1}^{J} \parallel \rho^j \parallel^2 + c_1 \triangle t(\parallel \xi_x^n \parallel^2 + \parallel \xi^n \parallel^2)\\ &+ c_1 \triangle t \sum_{j=0}^{n-1} \parallel \xi_x^j \parallel^2 + \parallel \epsilon_2^n \parallel^2,\end{aligned}$$

where

$$\begin{aligned}\parallel \epsilon_2^n \parallel^2 &= \parallel \int_0^{t^n} (\gamma_n u_x(t^n), w_h) d\tau - \triangle t \sum_{j=0}^{n-1} (\gamma_{nj} u_x(t_j), w_h) \parallel^2\\ &\leq C(\triangle t)^2 \int_0^{t^n} (\parallel u_x \parallel^2 + \parallel u_{xt} \parallel^2) ds\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_0 \parallel \xi^n \parallel^2 + \parallel \xi_x^n \parallel^2 &\leq C(\parallel \eta^n \parallel^2 + \parallel \rho \parallel^2_{L^\infty(L^2)}) + C\triangle t \sum_{n=1}^{J} \parallel \xi^n \parallel^2\\ &+ C(\triangle t)^2 \int_0^{t^n} (\parallel u_x \parallel^2 + \parallel u_{xt} \parallel^2 + \parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2 + \parallel \tilde{u}_{hxtt} \parallel^2) ds\\ &+ c_1 \triangle t \sum_{j=1}^{n-1} \parallel \xi_x^j \parallel^2 .\end{aligned}$$

Let $\bar{\alpha}_0 = \min\{\alpha_0, 1\}$ such that

$$\begin{aligned}\bar{\alpha}_0 \parallel \xi^n \parallel_1^2 &\leq C(\parallel \eta^n \parallel^2 + \parallel \rho \parallel^2_{L^\infty(L^2)}) + C\triangle t \sum_{n=1}^{J} \parallel \xi^n \parallel^2\\ &+ C(\triangle t)^2 \int_0^{t^n} (\parallel u_x \parallel^2 + \parallel u_{xt} \parallel^2 + \parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2 + \parallel \tilde{u}_{hxtt} \parallel^2) ds\\ &+ C\triangle t \sum_{j=1}^{n-1} \parallel \xi^j \parallel_1^2 .\end{aligned}$$

For sufficiently small $\triangle t$ and by discrete Gronwall's lemma, we obtain that

$$\begin{aligned}\parallel \xi^n \parallel_1^2 &\leq C(\parallel \eta^n \parallel^2 + \parallel \rho \parallel^2_{L^\infty(L^2)})\\ &+ C(\triangle t)^2 \int_0^{t^n} (\parallel u_x \parallel^2 + \parallel u_{xt} \parallel^2 + \parallel \tilde{u}_{hx} \parallel^2 + \parallel \tilde{u}_{hxt} \parallel^2 + \parallel \tilde{u}_{hxtt} \parallel^2) ds.\end{aligned}$$

Therefore, we can get

$$\begin{aligned}\parallel \xi^n \parallel_1 &\leq Ch^{\min\{k+1, r+1\}}(\parallel u \parallel_{L^\infty(H^{k+1})} + \parallel q \parallel_{L^\infty(H^{r+1})})\\ &+ C\triangle t(\parallel u \parallel_{L^2(H^1)} + \parallel u_t \parallel_{L^2(H^1)} + \parallel u_{tt} \parallel_{L^2(H^1)})\end{aligned} \tag{16}$$

and

$$\begin{aligned}\parallel \zeta_x^n \parallel &\leq Ch^{\min\{k+1, r+1\}}(\parallel u \parallel_{L^\infty(H^{k+1})} + \parallel q \parallel_{L^\infty(H^{r+1})})\\ &+ C\triangle t(\parallel u \parallel_{L^2(H^1)} + \parallel u_t \parallel_{L^2(H^1)} + \parallel u_{tt} \parallel_{L^2(H^1)}).\end{aligned} \tag{17}$$

Combining (16), (17), and the estimates of η^n and ρ^n, by the triangle inequality we can complete the proof.

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