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CONTROLLABILITY RESULTS IN α-NORM FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

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Abstract. This work concerns the controllability in the α-norm for some partial functional integrodifferential equations with infinite delay in Banach spaces. We give sufficient conditions ensuring the controllability by assuming that the undelayed part admits a resolvent operator in the sense of Grimmer [7] and that the delayed part is continuous with respect to the fractional power of the generator. The results are obtained using the Schauder's fixed-point theorem.

Keywords: controllability; partial functional integrodifferenttial; analytic semigroup; resolvent operator; Schauder fixed point theorem.

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1. Introduction

In this work, we study the controllability in the α-norm for the following partial functional integrodifferential integrodifferential equation

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$$
\begin{cases} x'(t) = -Ax(t) + \int_0^t B(t-s)x(s)ds + f(t,x_t) + Cu(t) \text{ for } t \geq 0 \\ x_0 = \varphi \in \mathscr{B}_\alpha, \end{cases} \tag{1.1}
$$

where $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators, on a Banach space X, $\mathscr{B}_\alpha$ defined by

$$
\mathscr{B}_\alpha = \{\varphi \in \mathscr{B};\ \varphi(\theta) \in D(A^\alpha) \text{ for } \theta < 0 \text{ and } A^\alpha \varphi \in \mathscr{B}\} \text{ with } \|\varphi\|_{\mathscr{B}_\alpha} = \|A^\alpha \varphi\|_{\mathscr{B}},
$$

is a subset of $\mathscr{B}$, where $\mathscr{B}$ is a Banach space of functions mapping from $]-\infty,0]$ into X and satisfying some axioms that will be introduced later.
For $0 < \alpha < 1$, A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be describe later.
For $x \in \mathscr{B}_\alpha$ and $t \in [0,b]$, x_t denotes the history function of $\mathscr{B}_\alpha$ defined by

$$
x_t(\theta) = x(t+\theta) \text{ for } \theta \leq 0,
$$

$f : \mathbb{R}^+ \times \mathscr{B}_\alpha \to X$ is a continuous function. For $t \geq 0$, $B(t)$ is closed linear operator with domain $D(B) \supset D(A)$. The control $u(.)$ belongs to $L^2(J,U)$ which is a Banach space of admissible controls, where U is a Banach space and $J = [0,b]$. The operator C belongs $\mathscr{L}(U,X)$, the Banach space of bounded linear operator from U in to X. function.

A dynamical system is a system that evolves over time through the iterated application of an of an underlying dynamical rule. It is a mathematical model usually constructed to study a physical phenomenon that evolves over time. that evolves in time. This model usually consists mainly of ordinary differential equations, partial differential equations, or functional differential equations that describe the which describe the evolution of the process under study in mathematical terms. mathematical terms. Controllability plays an essential role in the development of modern mathematical control systems. It has many important applications not only in theory, but also in areas such as industrial and chemical process control, reactor control, and chemical process control, reactor control, control of bulk electrical power systems, aerospace engineering, and, more recently, quantum systems theory. For all these reasons some recent contribution have been made by many authors, see for instance [5, 10, 11, 15, 16, 17] and the references therein.

In [6] the authors proved the controlability of an integrodifferential system with nonlocal conditions by making use of the measure of noncompactness and the Mönch fixed-point Theorem.

In [20] Issa Zabsonre considered the following impulsive differential system with infnite delay

$$
\begin{cases}
u'(t) = Au(t) + f(t, u_t) + Bv(t), \text{ for } t \in J = [0, b], t \neq t_k, & k = 1, 2, \ldots, m, \\
\\
\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\
\\
u_0 = \varphi \in \mathscr{B}_{h_\alpha}.
\end{cases}
\tag{1.2}
$$

Using Schauder's fixed-point theorem, he proved the controllability of the solution in the α-norm for the equation (1.2). To do this, the author assumed that the linear part generates a compact analytic semigroup on a Banach space X and that the delayed part is continuous with respect to the fractional power of the generator.

Recently, in [13] Djendode Mbainadji considered the following impulsive partial functional differential system with infnite delay

$$
\begin{cases}
x'(t) = -Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) \\
\\
\qquad\qquad +Cu(t), \text{ for } t \in J = [0, b], t \neq t_k, \quad k = 1, 2, \ldots, m, \\
\\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \qquad k = 1, 2, \ldots, m, \\
\\
x_0 = \varphi \in \mathscr{B}_{h_\alpha},
\end{cases}
\tag{1.3}
$$

By assuming that the undelayed part admits a resolvent operator in the sense of Grimmer and that the delayed part is continuous with respect to the fractional power of the generator. The author obtained the controllability results by using the Schauder fixed-point theorem.

To establish our results, let's remember that.

In [8], Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameters formula for the solution.

In [19], Desch et al. proved the equivalence of the compactness of the resolvent operator and that of the operator semigroup.

To the best of the author knowledge, the controllability in the α-norm of the equation (1.1), has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work. which is the mean motivation of this paper.

The organisation of this work as follows: In section 2, we recall some preliminary results about analytic semigroups and fractional power associated to its generator will be used throughout this work and some useful results on the analytic resolvent operator. This allows us to define the mild solution of Eq. (1.1). In section 3, we study the controllability of Eq.(1.1). In section 4 we give an exemple to illustrate this work.

2. Preliminary Results

Let $(X, \|.\|)$ be a Banach space and α such that $0 < \alpha < 1$, let be a constant such that $0 < \alpha < 1$ and let $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t \geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α. For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. [2] *Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have*

(i) $T(t) : X \to D(A^\alpha)$ for every $t > 0$,

(ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$.

(iii) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \text{ for } t > 0,$$

(iv) if $0 < \alpha \leq \beta < 1$, *then* $D(A^{\beta}) \hookrightarrow D(A^{\alpha})$,

(v) there exists $N_{\alpha} > 0$ *such that*

$$\|(T(t)-I)A^{-\alpha}\| \leq N_{\alpha} t^{\alpha} \text{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t)dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$.
Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

We also collect some basic results about resolvent operators. Consider the following linear nonhomogeneous equation

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds \ \text{ for } t \geq 0 \\ \\ v(0) = v_0 \in X, \end{cases} \tag{2.1}$$

where A and $B(t)$ are closed linear operators on a Banach space X.

Definition 2.2. [7] *A resolvent operator for equation* (2.1) *is a bounded operator valued function* $R(t) \in \mathscr{B}(X)$ *for* $t \geq 0$ *such that*

(a) $R(0) = I$ *and* $|R(t)| \leq Ne^{\beta t}$ *for some constants* N *and* β.

(b) *For all* $x \in X$, $R(t)x$ *is strongly continuous for* $t \geq 0$.

(c) $R(t) \in \mathscr{B}(D(A))$ *for* $t \geq 0$. *For* $x \in Y$, $R(.)x \in C^1([0,+\infty[;X) \cap C([0,+\infty[;D(A))$ *and*

$$\begin{aligned} R'(t)x &= -AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= -R(t)Ax + \int_0^t R(t-s)B(s)xds \text{ for } t \geq 0. \end{aligned}$$

The resolvent operator will play an interesting role to investigate the existence of solution of Eq. (2.1). The existence of an analytic resolvent operator has been discussed in [7] under the following assumptions. The notation f^* denotes the Laplace transform of f.

$(\mathbf{V}_1)$: $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. $(B(t))_{t\geq 0}$ is a closed operator on

X with domain at least $D(A)$ a.e $t \geq 0$ with $B(t)x$ strongly measurable for each $x \in D(A)$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for $b \in L^1_{loc}(0,\infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

$(\mathbf{V}_2)$: $\rho(\lambda) = (\lambda I + A - B^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for $\lambda \in \Lambda = \{\lambda \in \mathbb{C} : \; |arg(\lambda)| < \frac{\pi}{2} + \delta\}$, where $0 < \delta < \frac{\pi}{2}$. In Λ if $|\lambda| \geq \varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leq \frac{M}{|\lambda|}$.

$(\mathbf{V}_3)$: $A\rho(\lambda) \in \mathscr{L}(X)$ for $\lambda \in \Lambda$ and is analytic from Λ to $\mathscr{L}(X)$. $B^*(\lambda) \in \mathscr{L}(Y,X)$ and $B^*(\lambda)\rho(\lambda) \in \mathscr{L}(Y,X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ so that for $x \in Y$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$ $\|A\rho(\lambda)x\| + \|B^*(\lambda)\rho(\lambda)x\| < \frac{M}{|\lambda|}\|x\|$ and $\|B^*(\lambda)\| \to 0$ as $|\lambda| \to +\infty$ in Λ. In addition, $\|A\rho(\lambda)x\| \leq M|\lambda|^n$ for some $n > 0$, $\lambda \in \Lambda$ with , $|\lambda| \geq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in Y such that $A(D)$ and $B^*(\lambda)(D)$ are contained in Y and $\|B^*(\lambda)x\|$ is bounded for each $x \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$.

Moreover, the resolvent operator is given by $R(0) = I$ and

$$R(t)x = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda t}(\lambda I - A - B^*(\lambda))^{-1} x d\lambda, \text{ for } t > 0,$$

where

$$\gamma = \gamma_1 \cup \gamma_2 \cap \gamma_3,$$

with

$$\gamma_1 = \{re^{i\varphi} : r \geq 1\}, \gamma_2 = \{e^{i\theta} : -\varphi \leq \theta \leq \varepsilon\}, \gamma_3 = \{re^{-i\varphi} : r \geq 1\},$$

where $\frac{\pi}{2} < \varphi < \frac{\pi}{2} + \delta$ and $\delta > 0$.

Furthermore, for $a > 0$ there exist $N, K_\alpha > 0$ such that

$$\|R(t)\| \leq N \text{ and } \|A^\alpha R(t)\| \leq K_\alpha t^{-\alpha}, \; 0 < t \leq a, \; 0 < \alpha < 1.$$

Remark 2.3. [12] *Generally, the commutative between A^α and $R(t)$ is not true. But if we assume that*

$$B^*(\lambda)A^{-\alpha}x = A^{-\alpha}B^*(\lambda)x, \textit{ for any } x \in D(A),$$

then the commutativity holds. In fact for $x \in D(A^\alpha)$, we have $A^\alpha\rho(\lambda)x = \rho(\lambda)A^\alpha x$.

Hence

$$A^\alpha R(t) = R(t)A^\alpha.$$

Note that this commutativity can be realized in a number of situations. For example, let $B(t) = \beta(t)A$, where β is a scalar function on $]0,+\infty[$, then the linear problem 2.1 becomes:

$$\begin{cases} v'(t) = Av(t) + \int_0^t \beta(t-s)Av(s)ds \ \text{ for } t \geq 0 \\ \\ v(0) = v_0 \in X. \end{cases}$$

If we assume the following conditions:

($\mathbf{V'}_1$): A generates an analytic semigroup on X. In particular

$$\wedge_1 = \{\lambda \in \mathbb{C} : |\arg\lambda| < (\frac{\pi}{2}) + \delta_1\},\ 0 < \delta_1 < \frac{\pi}{2},$$

is contained in the resolvent set of A and $\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}$ on λ_1 for some constant $M > 0$, the scalar function $\beta(.)$ in $L^1(0,+\infty)$ with $\beta^*(\lambda)$ is absolutely convergent for $Re\lambda > 0$, where $\beta^*(\lambda)$ denotes Laplace transform of $\beta(t)$.

($\mathbf{V'}_2$): There exists $\wedge = \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta_2\}$, $0 < \delta_2 < \frac{\pi}{2}$, so that $\lambda \in \wedge$ implies $g_1(\lambda) = 1 + \beta^*(\lambda) \neq 0$. Furthermore, $\lambda g_1^{-1}(\lambda) \in \wedge_1$ for $\lambda \in \wedge$.

($\mathbf{V'}_3$): In $\wedge$, $\beta^*(\lambda) \longrightarrow 0$ as $|\lambda| \longrightarrow +\infty$.

Then according to [7], the afore-mentioned conditions $(\mathbf{V'}_1)-(\mathbf{V'}_3)$ are fulfilled, and hence the resolvent operator $R(t)$ is analytic and $A^\alpha R(t)x = R(t)A^\alpha x$ for any $x \in D(A^\alpha)$.

The basic theory for the existence of resolvent operator is given in [7, 8, 19].

We make the following hypothesis

($\mathbf{H}_0$) The semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.

Theorem 2.4. [19] *Assume that* $(\boldsymbol{V}_1)$-$(\boldsymbol{V}_3)$ *and* $(\boldsymbol{H}_0)$ *hold. Then the corresponding resolvent operator* $(R(t))_{t\geq 0}$ *of Eq.* (2.1) *is also compact for* $t > 0$.

Now consider the following system:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t [B_1(t-s) - B_2(t-s)]x(s)ds \text{ for } t \geq 0 \\ \\ x(0) = x_0 \in X. \end{cases} \tag{2.2}$$

where $B_1(t)$ and $B_2(t)$ are closed linear operators in X and satisfy $(\mathbf{V}_3)$. Then we have the following Lemma and corollary coming from [19].

Lemma 2.5. [19] *(Perturbation result). Assume that* $(\boldsymbol{V}_1)$, $(\boldsymbol{V}_2)$ *and* $(\boldsymbol{V}_3)$ *hold and Let* $(R_{B_1}(t))_{t\geq 0}$ *be a resolvent operator of Eq.* (2.1) *and* $(R_{B_1+B_2}(t))_{t\geq 0}$ *be a resolvent operator of Eq.*(2.5). *Then*

$$R_{B_1+B_2}(t)x - R_{B_1}(t)x = \int_0^t R_{B_1}(t-s)Q(s)xds$$

where the operator Q is defined by

$$Q(t)x = \int_0^t B_2'(t-s)\Big(\int_0^s R_{B_1+B_2}(\tau)xd\tau\Big)ds + B_2(0)\int_0^t R_{B_1+B_2}(s)xds,$$

Q is uniformly bounded on bounded intervals, and for each $x\in X$, $Q(.)x$ *belongs to* $C([0,\infty);X)$.

Corollary 2.6. [19] *Let A be a closed, densely defined linear operator in X,* $B(t)=0$ *for all* $t\geq 0$, *and* $(R(t))_{t\geq 0}$ *be a resolvent operator for Eq.* (2.1). *Then* $(R(t))_{t\geq 0}$ *is a* C_0*-semigroup with infinitesimal generator A.*

To establish, the following next result, the authors made the following hypothesis.

$(\mathbf{H}_1)$: $B(t)\in\mathscr{L}(X_\beta,X)$ for some $0<\beta<1$, a.e $t\geq 0$ and $\|B(t)x\|\leq b(t)|x|_\beta$ for $x\in X_\beta$, with $b\in L^q_{loc}(0,\infty)$ where $q>\frac{1}{(1-\beta)}$.

Theorem 2.7. [3] *Assume that* $(\boldsymbol{V}_1)$-$(\boldsymbol{V}_3)$ *and* $(\boldsymbol{H}_1)$ *hold. Then for any* $a>0$, *there exists a positive constant* $\overline{M}=\overline{M}(a)$ *such that for* $x\in X$ *we have*

$$\Big\|A^\beta\Big[R(t+h)x - R(t)R(h)x\Big]\Big\| \leq \overline{M}\int_0^t \frac{ds}{(t-s)^\beta}ds \, for\, 0\leq h<t\leq a.$$

Definition 2.8. *A mild solution of Eq.*(1.1) *is a function* $x\in C(]-\infty,b],X_\alpha)$ *satisfiying the relation*

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)[f(s,x_s)+Cu(s)]ds \, for\, t\in J, \\ \\ \varphi(t)\, for\, -\infty\leq t\leq 0. \end{cases} \tag{2.3}$$

Definition 2.9. *Equation* (1.1) *is said to be controllable on the interval J, if for every* $\varphi\in\mathscr{C}_\alpha$ *and* $x_1\in X$, *there exists a control* $u\in L^2(J,U)$ *such that a mild solution x of Eq.*(1.1) *satisfies the condition* $x(b)=x_1$.

3. CONTROLLABILITY RESULTS

In this section, we give sufficient conditions ensuring the controllability of Eq. (1.1). For that goal, we need to assume that

$(\mathscr{B},|.|_{\mathscr{B}})$ is a normed linear space of functions mapping $]-\infty,0]$ into X and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.),M(.):\mathbb{R}^+\to\mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma\in\mathbb{R}$ and $a>0$, if $x:]-\infty,a]\to X$, $x_\sigma\in\mathscr{B}$, and $x(.)$ is continuous on $[\sigma,\sigma+a]$, then for every $t\in[\sigma,\sigma+a]$ the following conditions hold

(i) $x_t\in\mathscr{B}$,

(ii) $|x(t)|\le H|x_t|_{\mathscr{B}}$, which is equivalent to $|\varphi(0)|\le H|\varphi|_{\mathscr{B}}$ for every $\varphi\in\mathscr{B}$

(iii) $|x_t|_{\mathscr{B}}\le K(t-\sigma)\sup\limits_{\sigma\le s\le t}|x(s)|+M(t-\sigma)|x_\sigma|_{\mathscr{B}}$.

$(\mathbf{A}_2)$ For the function $x(.)$ in $(\mathbf{A}_1)$, $t\mapsto x_t$ is a $\mathscr{B}$-valued continuous function for $t\in[\sigma,\sigma+a]$.

$(\mathbf{A}_3)$ The space $\mathscr{B}$ is a Banach space.

Lemma 3.1. [1] *Assume that* $A^{-\alpha}\varphi\in\mathscr{B}$ *for* $\varphi\in\mathscr{B}$, *where* $(A^{-\alpha}\varphi)(\theta)=A^{-\alpha}(\varphi(\theta))$ *for* $\theta\in]-\infty,0]$. *Then* $\mathscr{B}_\alpha$ *satisfies all the axiomes* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$.

$(\mathbf{H}_2)$ The linear operator: $W:L^2(J,U)\to X_\alpha$ defined by

$$Wu=\int_0^b R(b-s)Cu(s)ds$$

has an induced operator W^{-1} which takes values in $L^2(J,U)/\ker W$ and there exists positive constant $M>0$ such that $\|CW^{-1}\|\le M$.

$(\mathbf{H}_3)$ The function $f:J\times\mathscr{B}_\alpha\to X$ satisfies the following conditions.

(i) $f(.,\varphi)$ is measurable for $\varphi\in\mathscr{B}_\alpha$ and $f(t,.)$ is continuous for a.e $t\in J$,

(ii) for each positive number q, there exists a continuous nondecreasing function l_q such that

$\sup\limits_{\|\varphi\|_{\mathscr{B}_\alpha}\le r}\|f(t,\varphi)\|\le l_q(t)$ for a.e $t\in I$ and $t^{-\alpha}l_q\in L^1(J,\mathbb{R}^+)$ and

$$\liminf_{r\to+\infty}\frac{1}{q}\int_0^b\frac{l_q(s)}{(b-s)^\alpha}dt=\sigma<+\infty.$$

Let

$R_b = \sup_{t\in[0,b]} \|R(t)\|$, $K_b = \max_{0\le t\le b} K(t)$ and $M_b = \max_{0\le t\le b} M(t)$.

Theorem 3.2. *Asumme that* $(\boldsymbol{V}_1)$-$(\boldsymbol{V}_3)$ *and* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathscr{B}_\alpha$ *such that* $\varphi(0) \in \overline{D(A)}$ *and assume that*

$$\Big(1 + MR_b b\Big) K_\alpha K_b \sigma < 1.$$

Then Eq.(1.1) *is controllable on* J.

Proof. Using $(\mathbf{H}_2)$ and given an arbitrary function $x(.)$, we define the control by the following formula

$$u_x(t) = W^{-1}\Big\{x_1 - R(b)\varphi(0) - \int_0^b R(b-s) f(s,x_s)ds\Big\}(t) \text{ for } \in J.$$

Now define the following space

$$Z_b = \{x :]-\infty, b] \to X_\alpha, \text{ such that } x/_J \in C([0,b], X_\alpha) \text{ and } x_0 \in \mathscr{B}_\alpha\},$$

where $x/_J$ is the restriction of x to J.

For each $x \in Z_b$, we define its continuous extension $\widetilde{x}$ from $]-\infty, b]$ to X_α as follows

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ for } t \in [0,b], \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0], \end{cases}$$

we show using control that the operator $F : Z_b \to Z_b$ defined by

$$(Fx)(t) = R(t)\varphi(0) + \int_0^t R(t-s)[f(s,\widetilde{x}_s) + Cu_x(s)]ds \text{ for } t \in J.$$

We show using this control that te operator F has a fixed point. This fixed point is then a mild solution of Eq. (1.1). Observe that $(Fx)(b) = x_1$. This means that the control u_x steers in integrodifferential equation φ to x_1 in time b which implies the Eq.(1.1) is controllable on J.

For each $\varphi \in \mathscr{B}_\alpha$, such that $\varphi(0) \in X_\alpha$, we define the function $y \in C([0,b], X_\alpha)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y} \in C(]-\infty, b], X_\alpha)$ by

$$\widetilde{y}(t) = \begin{cases} y(t) \text{ for } t \in [0,b], \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0]. \end{cases}$$

For each $z \in C([0,b], X_\alpha)$, let $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is it extension by the zero of the function $z \in [-r, 0]$. Observe that x satisfies Eq.(2.3) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)[f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)]ds \text{ for } t \in [0,b].$$

We define

$$Z_b^0 = \{z \in Z_b : z(0) = 0\}$$

and let us pose $\|z\|_{Z_b^0} = \sup_{0 \le s \le b} |z(s)|_\alpha$. Thus $(Z_b^0, \|.\|_{Z_b^0})$ is Banach space.
Define the operator $\mathscr{K} : Z_b^0 \to Z_b^0$ by

$$(\mathscr{K}x)(t) = \int_0^t R(t-s)[f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)]ds \text{ for } t \in [0,b].$$

Note that the operator F has a fixed point if and only if $\mathscr{K}$ has one. So to prove that F has a fixed point, we only need to prove that $\mathscr{K}$ has one. For each positive number q,
let $B_q = \{z \in Z_b^0 : \|z\|_{\mathscr{B}_\alpha} \le q\}$. Then for any $z \in B_q$ and by $\mathbf{A}_1 - (ii)$, we have

$$\begin{aligned} \|z_s + y_s\|_{\mathscr{B}_\alpha} &\le \|z_s\|_{\mathscr{B}_\alpha} + \|y\|_{\mathscr{B}_\alpha} \\ &\le M_b\|z_0\|_{\mathscr{B}_\alpha} + K_b \sup_{s\in[0,b]} |z(s)|_\alpha + M_b\|y_0\|_{\mathscr{B}_\alpha} + K_b \sup_{s\in[0,b]} |y(s)|_\alpha \\ &\le K_b\|z\|_{Z_b^0} + M_b\|\varphi\|_{\mathscr{B}_\alpha} + K_b \sup_{s\in[0,b]} |R(s)\varphi(0)|_\alpha \\ &\le K_b\|z\|_{Z_b^0} + M_b\|\varphi\|_{\mathscr{B}_\alpha} + K_b R_b |\varphi(0)|_\alpha \\ &\le K_b q + M_b\|\varphi\|_{\mathscr{B}_\alpha} + K_b R_b H \|\varphi\|_{\mathscr{B}_\alpha} \\ &\le K_b q + \Big(M_b + K_b R_b H\Big)\|\varphi\|_{\mathscr{B}_\alpha} \end{aligned}$$

Thus

$$\|z_s + y_s\|_{\mathscr{B}_\alpha} \le K_b q + \Big(M_b + K_b R_b H\Big)\|\varphi\|_{\mathscr{B}_\alpha} = q' \tag{3.1}$$

We shall prove the theorem in the following steps.

Step 1:

We claim that there exists $q > \|\varphi\|_{\mathscr{B}_\alpha}$ such that $\mathscr{K}(B_q) \subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists $z^q \in B_q$, such that $\mathscr{K}(z^q) \notin B_q$ for some $\tau \in [0,b]$.

Thus we have

$$\begin{aligned} q &< |\mathscr{K}(z^q)(\tau)|_\alpha \leq |\int_0^t R(t-s)[f(s,\widetilde{z}_s^q+\widetilde{y}_s)+Cu_{z^q}(s)]|_\alpha ds \\ &< \int_0^t |R(t-s)f(s,\widetilde{z}_s^q+\widetilde{y}_s)|_\alpha ds + \int_0^t |R(t-s)Cu_{z^q}(s)|_\alpha ds \\ &< \int_0^t |R(t-s)f(s,\widetilde{z}_s^q+\widetilde{y}_s)|_\alpha ds + \Big| \int_0^t R(t-\xi)CW^{-1}\Big[x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,z_s^q)ds\Big]d\xi\Big|_\alpha. \end{aligned}$$

Then, we obtain

$$\begin{aligned} q &< K_\alpha \int_0^t \frac{l_{q'}(s)}{(t-s)^{-\alpha}}ds + MR_b b\Big[|x_1|_\alpha + R_b|\varphi(0)|_\alpha + K_\alpha \int_0^t \frac{l_{q'}(s)}{(b-s)^\alpha}ds\Big] \\ &< MR_b b\Big(|x_1|_\alpha + R_b|\varphi(0)|_\alpha\Big) + \Big(1+MR_b b\Big)K_\alpha \int_0^t \frac{l_{q'}(s)}{(b-s)^\alpha}ds \end{aligned}$$

where $R_b = \sup\{\|R(t)\| : t \in [0,b]\}$ and $q' := K_b q + q_0$ with $q_0 := \Big(M_b + K_b R_b H\Big)\|\varphi\|_{\mathscr{B}_\alpha}$.

On the other hand we shall show that function $g : t \mapsto \displaystyle\int_0^t \frac{l_{q'}(s)}{(t-s)^\alpha}ds$ is nondecreasing on $[0,b]$. In fact, let $t;t' \in [0,b]$ such that $t < t'$. Then we have

$$\begin{aligned} \int_0^t \frac{l_{q'}(s)}{(t-s)^\alpha}ds &= \int_0^t \frac{l_{q'}(t-s)}{s^\alpha}ds \\ &\leq \int_0^t \frac{l_{q'}(t'-s)}{s^\alpha}ds \\ &\leq \int_0^{t'} \frac{l_{q'}(t'-s)}{s^\alpha}ds = g(t'). \end{aligned}$$

Dividing both sides by q and noting that $q' = K_b + q_0 \to +\infty$ as $q \to +\infty$.

We have

$$1 < \frac{MK_\alpha b\Big(|x_1|_\alpha + R_b|\varphi(0)|_\alpha\Big)}{q} + \frac{\Big(1+MR_b b\Big)K_\alpha}{q}\int_0^b \frac{l_{q'}(s)}{(b-s)^\alpha}ds$$

and

$$\liminf_{q\to+\infty} \frac{1}{q}\int_0^b \frac{l_{q'}(s)}{(b-s)^\alpha}ds = \liminf_{q\to+\infty} \frac{1}{q'}\int_0^b \frac{l_{q'}(s)}{b(-s)^\alpha}ds \times \frac{q'}{q} = K_b\sigma$$

It follows that

$$1 < \left(1 + MR_b b\right) K_\alpha K_b \sigma$$

which gives contradiction. Consequently $\mathscr{K}(B_q) \subset B_q$ for each $q > 0$.

Step: 2 $\mathscr{K} : Z_b^0 \to Z_b^0$ is continuous. In fact let $\mathscr{K} = \mathscr{K}_1 + \mathscr{K}_2$, where

$$(\mathscr{K}_1 z)(t) = \int_0^t R(t-s) f(s, \widetilde{z}_s + \widetilde{y}_s) ds.$$

$$(\mathscr{K}_2 z)(t) = \int_0^t R(t-s) C u_z(s) ds$$

Let $(z^n)_{n\geq 1} \in Z_b^0$ with $z^n \to z$ in Z_b^0, there exists a number $q > 0$ such that $\|z^n(t)\|_{\mathscr{B}_\alpha} \leq q$ for all n and a.e $t \in I$. So $z^n, z \in B_q$. Using By equation (3.1),
$\|z_t^n + y_t\|_{\mathscr{B}_\alpha} \leq q'$ for $t \in I$. By $(\mathbf{H}_3) - (i)$ we have $f(t, \widetilde{z}_t^n + \widetilde{y}_t) \to f(t, \widetilde{z}_t + \widetilde{y}_t)$ for each $t \in [0, b]$. And by $(\mathbf{H}_3) - (ii)$ we have

$$\|f(t, \widetilde{z}_t^n + \widetilde{y}_t) - f(t, \widetilde{z}_t + \widetilde{y}_t)\| \leq 2 l_{q'}(t).$$

It follows that by dominated convergence theorem

$$\begin{aligned} |(\mathscr{K}_1 z^n)(t) - (\mathscr{K}_1 z)(t)|_\alpha &\leq \int_0^t \|A^\alpha R(t-s)[f(s, \widetilde{z}_s^n + \widetilde{y}_s) - f(s, \widetilde{z}_s + \widetilde{y}_s)]\| ds \\ &\leq K_\alpha \int_0^b \frac{\|f(s, \widetilde{z}_s^n + \widetilde{y}_s) - f(s, \widetilde{z}_s + \widetilde{y}_s)\|}{(b-s)^\alpha} \to 0 \text{ as } n \to \infty, \end{aligned}$$

by dominated convergence theorem. Also we have

$$\begin{aligned} |(\mathscr{K}_2 z^n)(t) - (\mathscr{K}_2 z)(t)|_\alpha &\leq \int_0^t \|A^\alpha R(t-s) C[u_{z^n}(s) - u_z(s)]\| ds \\ &\leq K_\alpha \int_0^b \frac{\|f(t, \widetilde{z}_s^n) - f(s, \widetilde{z}_t)\|}{(b-s)^\alpha} ds \to 0 \text{ as } n \to \infty, \end{aligned}$$

by dominated theorem and yield the continuity of $\mathscr{K}_2$. Thus

$$|(\mathscr{K} z^n)(t) - (\mathscr{K} z)(t)|_\alpha \leq |(\mathscr{K}_1 z^n)(t) - (\mathscr{K}_1 z)(t)|_\alpha + |(\mathscr{K}_2 z^n)(t) - (\mathscr{K}_2 z)(t)|_\alpha \to 0 \text{ as } n \to +\infty.$$

Hence $\mathscr{K}$ is continuous on Z_b^0.

Step:3 The set $\{(\mathscr{K})z(t) : z \in Z_b^0\}$ is relatively compact for each $t \in]0,b]$. Let $t \in]0,b]$ be fixed and $\gamma > 0$ be such that $\alpha < \gamma < 1$. Then it follows that

$$\begin{aligned}\|A^\gamma(\mathscr{K}z)(t)\| &\leq \int_0^t \|A^\gamma R(t-s)[f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)]\|ds \\ &\leq MR_b b\Big(|x_1|_\alpha + R_b|\varphi(0)|_\alpha\Big) + \Big(1+MR_b b\Big)K_\gamma \int_0^b \frac{l_{q'}(s)}{(b-s)^\alpha}ds < \infty.\end{aligned}$$

Then for $t \in]0,b]$ fixed, the set $\{A^\gamma(\mathscr{K})z(t) : z \in Z_b^0\}$ is bounded in X. By $(\mathbf{H}_0)$ we deduce that $A^{-\gamma} : X \to X_\alpha$ is compact. It follows that the set $\{(\mathscr{K})z(t) : z \in Z_b^0\}$ is relatively compact for each $t \in]0,b]$ in X_α.

Step:4: The set $\{(\mathscr{K})z(t) : z \in Z_b^0\}$ is an equicontinuous family of functions. Let $\tau_1, \tau_2 \in [0,b]$ such that $0 \leq \tau_1 < \tau_2 \leq b$ and set $\tau_2 = \tau_1 + \varepsilon$.

$$\begin{aligned}&(\mathscr{K}z)(\tau_2) - (\mathscr{K}z)(\tau_1) \\ = &\int_0^{\tau_2} R(\tau_2-s)[f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)]ds - \int_0^{\tau_1} R(\tau_1-s)[f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)]ds \\ = &\int_0^{\tau_1}[R(\tau_2-s)-R(\tau_1-s)]f(s,\widetilde{z}_s+\widetilde{y}_s)ds + \int_{\tau_1}^{\tau_2} R(\tau_2-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds \\ &+\int_0^{\tau_1}[R(\tau_2-s)-R(\tau_1-s)]Cu_z(s)ds + \int_{\tau_1}^{\tau_2} R(\tau_2-s)Cu_z(s)ds \\ = &\int_0^{\tau_1}[R(\tau_1+\varepsilon-s)-R(\tau_1-s)R(\varepsilon)]f(s,\widetilde{z}_s+\widetilde{y}_s)ds \\ &+\int_{\tau_1}^{\tau_1+\varepsilon} R(\tau_1+\varepsilon-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds + [R(\varepsilon)-I]\times\int_0^{\tau_1} R(\tau_1-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds \\ &+\int_0^{\tau_1}[R(\tau_1+\varepsilon-s)-R(\tau_1-s)R(\varepsilon)]Cu_z(s)ds \\ &+[R(\varepsilon)-I]\times\int_0^{\tau_1} R(\tau_1-s)Cu_z(s)ds + \int_{\tau_1}^{\tau_2} R(\tau_2-s)Cu_z(s)ds.\end{aligned}$$

Using Proposition 2.7

$$\begin{aligned}&|(\mathscr{K}z)(\tau_2) - (\mathscr{K}z)(\tau_1)|_\alpha \\ \leq\ &\overline{M}\int_0^{\tau_1}\Big(\int_0^\varepsilon \frac{d\xi}{\xi^\alpha}\Big)\|f(\widetilde{z}_s+\widetilde{y}_s)\|ds + \int_{\tau_1}^{\tau_2}|R(\tau+\varepsilon-s)f(\widetilde{z}_s+\widetilde{y}_s)|_\alpha ds \\ &+\Big\|(R(\varepsilon)-I)\int_0^{\tau_1} A^\alpha R(\tau_1-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds\Big\| \\ &+\overline{M}M\int_0^{\tau_1}\Big[\int_0^\varepsilon \frac{d\xi}{\xi^\alpha}\Big|x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\widetilde{z}_s)ds\Big|_\alpha\Big]d\sigma\end{aligned}$$

$$+\|R(\varepsilon)-I\|\times\int_{\tau_1}^{\tau_1+\varepsilon}\Big|R(\tau+\varepsilon-\xi)CW^{-1}\Big[x_1-R(b)\varphi(0)$$

$$-\int_0^b R(b-s)f(s,\widetilde{z}_s)ds\Big|_\alpha\Big]d\xi+\Big[\int_{\tau_1}^{\tau_1+\varepsilon}\Big|R(\tau_1+\varepsilon-\xi)CW^{-1}\Big(x_1-R(b)\varphi(0)$$

$$-\int_0^b R(b-s)f(s,\widetilde{z}_s)\Big)ds\Big|_\alpha\Big]d\xi$$

Thus we have

$$\begin{aligned}|(\mathscr{K}z)(\tau_2)-(\mathscr{K}z)(\tau_1)|_\alpha \;\leq\;& \overline{M}\int_0^\varepsilon\frac{d\xi}{\xi^\alpha}\int_0^{\tau_1}l_{q'}(s)ds+K_\alpha\int_{\tau_1}^{\tau_1+\varepsilon}\frac{l_{q'}(s)}{(\tau_1+\varepsilon-s)^\alpha}ds\\
&+\|(R(\varepsilon)-I)\int_0^{\tau_1}A^\alpha R(\tau_1-s)f(s,\widetilde{z}_s+\widetilde{y}_s)\|ds\\
&+\Big(\int_0^\varepsilon\frac{d\xi}{\xi^\alpha}\Big)\Big[\overline{M}Mb\Big(|x_1|_\alpha+R_b|\varphi(0)|_\alpha+K_\alpha\int_0^b\frac{l_{q'}(s)}{(b-s)^\alpha}ds\Big)\Big]\\
&+\|R(\varepsilon)-I\|\times\Big[MR_bb\Big(|x_1|_\alpha+R_b|\varphi(0)|_\alpha\Big)+K_\alpha\int_0^b\frac{l_{q'}(s)}{(b-s)^\alpha}ds\Big]\\
&+\Big[MR_bb\Big(|x_1|_\alpha+R_b|\varphi(0)|_\alpha+K_\alpha\int_0^b\frac{l_{q'}(s)}{(b-s)^\alpha}ds\Big]\varepsilon.\end{aligned}\tag{3.2}$$

On the other hand let $t\in]0,b]$ fixed and $\gamma>0$ such that $\alpha<\gamma<1$, we have

$$\|A^\gamma\int_0^t R(t-s)f(t,\widetilde{z}_t+\widetilde{y}_t)ds\|\leq K_\gamma\int_0^b\frac{l_{q'}(s)}{(b-s)^\alpha}ds$$

Then for each $t\in]0,b]$ fixed the set

$$\{A^\gamma\int_0^t R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds : x\in B_q\},$$

is bounded in X. By $(\mathbf{H}_0)$, we can see that $A^{-\gamma}:X\to X_\alpha$ is compact. Consquently

$$\Big\{\int_0^t R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds : x\in B_q\}$$

is relatively compact set in X_α. Then there exists a compact set Γ in X such that

$$\int_0^t A^\gamma R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds\in\Gamma \text{ for } x\in B_q.$$

On the other hand, by Banach-Steinhaus theorem, we have

$$\left\| \big(R(\varepsilon)-I\big)\int_0^{\tau_1} A^{\alpha}R(\tau_1-s)f(s,\widetilde{z}_s+\widetilde{y}_s)ds\right\| \to 0 \text{ as } \varepsilon \to 0.$$

By the continuity of $(R(t))_{t\geq 0})$ in the operator-norm toplogy, the dominated convergence theorem, we conclude that the right hand side of the above inequality 3.2 tends to zero and independent of z as $\varepsilon \to 0$. Thus $\mathscr{K}$ maps Z_b^0 into an equicontinuous family of functions.
The equicontinuities for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.
So from the above **step 1** to **step 4** and the Ascoli-Arzela theorem, we can conclude that $\mathscr{K}$: $Z_b^0 \to Z_b^0$ is completely continuous. Hence by the Schauder fixed point theorem, $\mathscr{K}$ has at least one fixed point z in B_q. Then $x = z+y$ is a fixed point of F in B_q and thus Eq.(1.1) is controllable on J.

4. Application

For illustration, we propose to study the existence of solutions for the following model

$$(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) - \displaystyle\int_0^t b(t-s)\dfrac{\partial^2}{\partial x^2}z(s,x)ds + Cu(t) \\ \qquad +te^{-|t|}\displaystyle\int_{-\infty}^0 g(\theta,\dfrac{\partial}{\partial x}z(t+\theta,x))d\theta \text{ for } t\geq 0 \text{ and } x\in[0,\pi] \\ z(t,0)=z(t,\pi)=0 \text{ for } t\geq 0 \\ z(\theta,x)=\varphi(\theta)(x) \text{ for } \theta \in]-\infty,0] \text{ and } x\in[0,\pi], \end{cases}$$

where $g : \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$ is lipschitzian with respect to the second argument and $b : \mathbb{R}^+ \to \mathbb{R}$ is an appropriate functions. To rewrite equation (4.1) in the abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^{\pi} |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $A : X \to X$ be defined by

$$\begin{cases} D(A) = H^2(0,\pi)\cap H_0^1(0,\pi) \\ Ay = -y''. \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 : \ n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns), \ s \in [0,\pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2(y,e_n)e_n, \ y \in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X : \ \sum_{n=1}^{+\infty} n(y,e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,e_n)e_n, \ y \in D(A).$$

Lemma 4.1. [18] *If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and*

$$\|y\| = \|y'\| = \|A^{\frac{1}{2}}y\|.$$

It is well known that $-A$ is the generator of a compact analytic semigroup semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x,e_n)e_n, \ x \in X.$$

Here we choose $\alpha = \dfrac{1}{2}$. Let $\gamma > 0$, we define the phase space

$$\mathscr{B} = C_\gamma = \{\varphi \in C(]-\infty,0];X) : \ \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exist in } X\},$$

with the norm

$$\|\varphi\|_\gamma = e^{\gamma\theta}\sup_{\theta\leq 0}\|\varphi(\theta)\|, \text{ for } \varphi \in C_\gamma$$

This space satisfies axioms $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$. The norm in $\mathscr{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathscr{B}_{\frac{1}{2}}} = e^{\gamma\theta}\sup_{\theta\leq 0}\|A^{\frac{1}{2}}\varphi(\theta)\| = e^{\gamma\theta}\sup_{\theta\leq 0}\sqrt{\int_0^\pi \Big(\frac{\partial}{\partial x}(\varphi)(\theta)(x)\Big)^2 dx}.$$

Let $h(t)=e^t$ for $t\in\mathbb{R}^-$ and define $\|\varphi||_{\mathscr{B}_{h\alpha}}$ by

$$\|\varphi||_{\mathscr{B}_{h\alpha}}=\int_{-\infty}^{0}h(s)\sup_{s\leq\theta\leq 0}|\varphi(\theta)|ds.$$

We define the functions f by

$$f(t,\varphi)(\xi)=te^{-|t|}\int_{-\infty}^{0}g(\theta,z(t+\theta,\xi))d\theta \text{ for } \xi\in[0,\pi] \text{ and } t\in[0,b].$$

and let $B:D(A)\to X$ be defined by

$$B(t)(y)=b(t)Ay\ for\ t\geq 0$$

For $t\geq 0$ and $\xi\in[0,\pi]$, let us pose $v(t)=z(t,\xi)$. Then

$$v'(t)=\frac{\partial}{\partial\xi}z(t,\xi),\ \frac{\partial^2}{\partial\xi^2}z(t,\xi)=-Av(t)$$

and

$$\int_0^t b(t-s)\frac{\partial^2}{\partial\xi^2}z(s,\xi)ds=-\int_0^t b(t-s)Av(s)ds=-\int_0^t B(t-s)v(s)ds.$$

Let us pose $v(t)=z(t,x)$. Then equation (4.1) takes the following abstract form

$$\text{(4.2)}\qquad\begin{cases}\dfrac{d}{dt}v(t)=-Av(t)+\displaystyle\int_0^t B(t-s)v(s)ds+Cu(t)+f(t,v_t) \text{ for } t\geq 0\\[2ex] v_0=\varphi\in\mathscr{B}_{\frac{1}{2}}.\end{cases}$$

Now we assume the following hypothesis

($\mathbf{H}_4$) The scalar function $b(.)\in L^2_{loc}(0,\infty)$ satisfying $b(\lambda)=1+b^*(\lambda)\neq 0$ and $\lambda b^{-1}(\lambda)\in\Lambda$ for $\lambda\in\Lambda$. Further $b^*(\lambda)\to 0$ as $|\lambda|\to+\infty$, for $\lambda\in\Lambda$ and $(b^*(\lambda))^{-1}=\circ(|\lambda|^n)$.

($\mathbf{H}_5$) there exists a function $k\in L^1(\mathbb{R}^-,\mathbb{R}^+)$ such that for $\theta\leq 0$ and $x,y\in\mathbb{R}$

$$|g(\theta,x)-g(\theta,y)|\leq k(\theta)|x-y|.$$

($\mathbf{H}_6$) $e^{-2\mu}k\in L^2(\mathbb{R}^-)$.

Let $C:U\to X, U\subset[0,+\infty[$ be linear operator defined for all $x\in X$ by

$$Cv(t)=\int_0^t e^{-\frac{s}{2}}v(s)ds,\ \text{for } t\in[0,b].$$

Then using Holder inequality, we can see that $|Cv(t)| \leq \|v\|_{L^2([0,b],\mathbb{R})}$, which implies that B is a bounded linear operator. Let the operator $W: L^2(J,U) \to X$ be defined by

$$Wu = \int_0^b R(b-s)Cv(s)ds.$$

Assuming that W satisfies $(\mathbf{H}_2)$. By Hölder inegality and Lemma 4.1, we have

$$\begin{aligned}
\|f(t,\varphi)\|^2 &= \int_0^\pi |te^{-|t|}\int_{-\infty}^0 k(\theta)\frac{\partial}{\partial x}\varphi(\theta)(\xi)d\theta|^2 d\xi \\
&= t^2e^{-2|t|}\int_0^\pi \Big|\int_{-\infty}^0 k(\theta)e^{-2\mu\theta}e^{2\mu\theta}\frac{\partial}{\partial x}\varphi(\theta)(\xi)d\theta\Big|^2 d\xi \\
&\leq t^2e^{-2|t|}\int_0^\pi \Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)\Big(\int_{-\infty}^0 e^{-4\mu\theta}\big|\frac{\partial}{\partial x}\varphi(\theta)(\xi)\big|^2 d\theta\Big)d\xi \\
&\leq \Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\int_0^\pi \Big(\int_{-\infty}^0 e^{-4\mu\theta}\big|\frac{\partial}{\partial x}\varphi(\theta)(\xi)\big|^2 d\theta\Big)d\xi \\
&\leq \Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\int_0^\pi \Big(\int_{-\infty}^0 e^{-2\mu\theta}e^{-2\mu\theta}\big|\frac{\partial}{\partial x}\varphi(\theta)(\xi)\big|^2 d\theta\Big)d\xi \\
&\leq \Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\int_0^\pi \sup_{\theta\leq 0} e^{-2\mu\theta}\big|\frac{\partial}{\partial x}\varphi(\theta)(\xi)\big|^2\Big(\int_{-\infty}^0 e^{-2\mu\theta}d\theta\Big)d\xi \\
&\leq \frac{1}{2\mu}\Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\sup_{\theta\leq 0} e^{-2\mu\theta}\int_0^\pi \big|\frac{\partial}{\partial x}\varphi(\theta)(\xi)\big|^2 d\xi \\
&\leq \frac{1}{2\mu}\Big(\int_{-\infty}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\|\varphi\|^2_{\mathscr{B}_{\frac{1}{2}}}.
\end{aligned}$$

Let $q>0$ for every $\varphi \in \mathscr{B}_{\frac{1}{2}}$ such that $\|\varphi\|_{\mathscr{B}_{\frac{1}{2}}} \leq q$, then we have

$$\sup_{\|\varphi\|\leq q} \|f(t,\varphi)\| \leq |\beta(t)|q,$$

were

$$\beta(t) = \Big[\frac{1}{2\mu}\Big(\int_{-r}^0 k^2(\theta)e^{-4\mu\theta}d\theta\Big)\Big]^{\frac{1}{2}} te^{-t} \ \text{ for } t\in[0,b].$$

So f satisfies $(\mathbf{H}_3)$-(i) and $(\mathbf{H}_3)$-(ii)) with $l_q(t) = |\beta(t)|q$. Moreover

$$\liminf_{q\to+\infty}\int_0^b \frac{l_q(t)}{q}dt = \|\beta\|_{L^1} = \sigma < +\infty.$$

From $(\mathbf{H}_4)$ there exists an analytic resolvent operator for equation (4.2).

Theorem 4.2. *Assume that* $(\boldsymbol{H}_4)$-$(\boldsymbol{H}_6)$ *hold. If*

$$(1+MR_bb)K_\alpha K_b\sigma < 1,$$

then equation (4.2) *is controllable on* J.

CONCLUSION

In this paper, we have shown the controllability of some partial functional integrodifferential equation with infinite delay in Banach spaces by using Schauder's fixed-point theorem . We achieved this by assuming that the linear part generates a compact analytic semigroup and that the delayed part is continuous with the respect to the fractional power of the generator.
The next challenge is to study the controllability of stochastic partial functional integrodifferential equation in the α- norm. It is well known that the stochastic modeling is crucial for many fields such as physics, engineering, economics, and social sciences.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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