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APPROXIMATION OF A FUNCTION BY THE EXTENDED CHEBYSHEV WAVELET METHOD OF FIRST KIND AND ITS APPLICATIONS

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Abstract. In this paper, we have first introduced the Extended Chebyshev wavelet of first kind. Then, it is used to calculate the approximation error of a function having its first and second derivatives bounded. Also, we have solved various differential equations like the Hermite differential equation of order zero, nonlinear Riccati differential equation, and differential equation corresponding to radioactive decay using this wavelet technique to show the usefulness of this method.

Keywords: extended Chebyshev wavelet; operational matrix of integration; function with bounded derivative; Chebyshev polynomial.

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1. Introduction

After the development of differential and integral equations, many phenomena and problems in real world are described easily and suitably by differential equations and integral equations. Therefore, the study of differential and integral equations and to find methods to solve them are very fruitful in applications.There are many such equations which are not solved easily by

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traditional methods, so to find the solution of these problems, we use approximation theory, a very important branch of mathematics.

Over the last few years, numerous straightforward and precise methods are established on orthogonal basis functions, including wavelets, which have been used to approximate the solutions of differential and integral equations.The leading benefit of using orthogonal basis is that it reduces the problem into a solvable system of algebraic equations. There are so many families of orthogonal functions which can be used in these methods, but sometimes difficult to choose the most suitable one.

The development of wavelet theory has been taken place very quickly in recent years. This is actually the generalization of Fourier analysis. Authors like Chen([3],[4]), Cohen[6], Coifman[7], Daubechies([8],[9],[10],[11],[12]), Debnath[13], Donoho[14], Horng[15], Lippus[19], Mallat[20], Meyer[21],
Wojtaszczyk[27], Zhou[28] have discussed the various applications of wavelet methods.

Wavelet theory has become more prestigious in approximation theory and harmonic analysis due to the well localized behaviour of wavelets. In the function spaces like Lebesgue, Sobolov, Besov ([16]) etc. different kinds of unequivocal bases have been obtained to discuss the behaviour of different types of signals in these spaces. Due to this reason, the wavelet theory becomes a powerful approximation tool in approximation theory as well as in other areas of mathematical analysis.

The term wavelet simply means small waves(the waves used in Fourier analysis are not much small waves), i.e. wavelets are oscillations that decay quickly. This is the reason why wavelets are used widely in signal analysis, image processing etc. . In signal processing, we have to isolate signal discontinuities, for which we would prefer to have some very short basis functions. At the same time, if we want to get detailed frequency analysis, we would prefer to have some very long basis functions. To accomplish these requirements, we need to have a tool of short high frequency basis functions and long low frequency one.This tool is exactly what you get with wavelet transform.

Since any function in $L^2(\mathbb{R})$ that satisfies the admissibility condition $\int\limits_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ be a mother wavelet. So, by the use of dilation and translation processes, we can find infinite number of wavelets, whereas in the Fourier analysis, we only use sine and cosine functions to do so.

Adibi et al. [1] proposed a method based on the Chebyshev wavelet for the solution of Fredholm integral equation of first kind. Celik[2] found the solution of differential equations by Chebyshev wavelet collocation method, Lal and Sharma([17],[25]) used the Chebyshev wavelet of first kind to find estimation error for a function belonging to Lipschitz class and solve some differential equations by using Legendre wavelet operational matrix of integration. Working in this direction, in this paper, we have first defined the extended Chebyshev wavelet of first kind and some approximation theorems for bounded derivatives of a function have been proved, and then it is used to solve some special kinds of differential equations. The solution is obtained for different values of μ and it is compared with the exact solution, which shows the validity and usefulness of the paper.

This paper is organized as follows. Section 1 is Introductory. In section 2, the extended Chebyshev wavelet and wavelet approximation have been introduced. In section 3, the theorems and their proofs have been discussed. In section 4, the operational matrix of integration for $\mu = 2,3$ have been calculated. In section 5, numerical verification has been given for the extended Chebyshev wavelet of first kind. In section 6, three differential equations, namely the radio decay problem, the Hermite differential equation of order zero, and the nonlinear Riccati differential equation have been solved by this wavelet method and comparison with exact solution have been discussed. Lastly, in section 7, some conclusions have been given.

2. PRELIMINARIES

2.1. The extended Chebyshev wavelet of first kind and its properties.

. Wavelets are a family of functions constructed from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter α and translation parameter β vary continuously, we have the following family of continuous wavelets as

$$\psi_{\alpha,\beta}(t) = |\alpha|^{\frac{-1}{2}} \psi\left(\frac{t-\beta}{\alpha}\right), \qquad \alpha,\beta \in \mathbb{R}, \alpha \neq 0 \quad \text{[Chui[5]]}. \tag{1}$$

Let $\mu \geq 2$ be a positive integer, the extended Chebyshev wavelet on the interval $[0,1)$ is represented by $\psi^{(\mu)}_{n,m}$ and defined by

$$\psi^{(\mu)}_{n,m}(t) = \begin{cases} \left(\frac{\mu^k}{\pi}\right)^{\frac{1}{2}} \tilde{T}_m(2\mu^k t - 2n + 1), & \frac{n-1}{\mu^k} \leq t < \frac{n}{\mu^k}; \\ 0, & otherwise, \end{cases} \tag{2}$$

where,

$$\tilde{T}_m(t) = \begin{cases} \sqrt{2}T_m(t), & m = 0; \\ 2T_m(t), & m \geq 1, \end{cases} \tag{3}$$

$n = 1,2,...,\mu^k$, m is degree of the Chebyshev polynomial, $k = 0,1,2,3,...,M$ is a positive integer ≥ 2, and t is normalized time. In the above definition, the polynomials $T_m(t)$, $m = 0,1,...,$ are the Chebyshev polynomials[24] of first kind of degree m and given by

$$T_m(t) = cos(m\theta), \tag{4}$$

in which $\theta = arccos(t)$. The Chebyshev polynomials are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval [-1, 1] and satisfy the following recurrence relation:

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), m = 1,2,3,... \quad \text{with} \quad T_0(t) = 1, \quad \text{and} \quad T_1(t) = t. \tag{5}$$

The set of the extended Chebyshev wavelets is orthonormal with respect to weight function $w^{(\mu)}_{n,k}(t) = \frac{1}{\sqrt{1-(2\mu^k t-2n+1)^2}}$.

2.2. Extended Chebyshev Wavelet Expansion. A function $f \in L^2[0,1]$ is expanded in terms of extended Chebyshev wavelet series as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t), \tag{6}$$

where $c^{(\mu)}_{n,m} = \langle f, \psi^{(\mu)}_{n,m} \rangle_{w^{(\mu)}_{n,k}}$ on $L^2[0,1]$ (Razzaghi[23]).

If the above infinite series is truncated, then its $(\mu^k, M)^{th}$ partial sums $S_{\mu^k,M}(t)$ can be written as

$$S_{\mu^k,M}(t) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t) = C^T \psi^{(\mu)}(t), \tag{7}$$

where

$$C = \left[c^{(\mu)}_{1,0}, c^{(\mu)}_{1,1}, ..., c^{(\mu)}_{1,M-1}, c^{(\mu)}_{2,0}, ..., c^{(\mu)}_{2,M-1}, ..., c^{(\mu)}_{\mu^k,0}, ..., c^{(\mu)}_{\mu^k,M-1}\right]^T$$

and

$$\psi^{(\mu)}(t) = \left[\psi_{1,0}^{(\mu)}, \psi_{1,1}^{(\mu)}, \ldots, \psi_{1,M-1}^{(\mu)}, \psi_{2,0}^{(\mu)}, \ldots, \psi_{2,M-1}^{(\mu)}, \ldots, \psi_{\mu^k,0}^{(\mu)}, \ldots, \psi_{\mu^k,M-1}^{(\mu)}\right]^T. \tag{8}$$

2.3. Approximation to functions by the partial sums of the first kind extended Chebyshev wavelet series. For a given funcion $f \in L^2[0,1]$, the deviation $\delta(f, S_{\mu^k,M})$ of $(\mu^k, M)^{th}$ partial sums $S_{\mu^k,M}$ from f is defined by the formula

$$\delta(f, S_{\mu^k,M}) = \max_{x\in[0,1]} |f(x) - S_{\mu^k,M}|.$$

The lower bound of the number $\delta(f, S_{\mu^k,M})$ for all

$$(S_{\mu^k,M}f)(t) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{n,m}^{\mu} \psi_{n,m}^{\mu}(t)$$

is denoted by $E_{\mu^k,M}(f)$, is defined by

$$E_{\mu^k,M} = \inf_{S_{\mu^k,M}} ||f - S_{\mu^k,M}||_2.$$

$E_{\mu^k,M}$ is called the best approximation of f if $E_{\mu^k,M}(f) \to 0$ as $\mu^k \to \infty,\ M \to \infty$(Zygmund p.114 [29]).

3. Main Results

Theorem 3.1 Let $f \in L^2[0,1]$ be a function such that $|f'(t)| \leq M_1$ for some $0 < M_1 < \infty, \forall t \in [0,1]$. If the extended Chebyshev wavelet expansion of f is given by equation (6) and (μ^k, M) partial sum $S_{\mu^k,M}(t)$ of the above series(6) is defined by equation (7) then the approximation error $E_{\mu^k,M}^{(1)}$ of f by $S_{\mu^k,M}$ is given by

$$E_{\mu^k,M}^{(1)} = inf||f - S_{\mu^k,M}||_2 = O\left(\left(\frac{1}{\mu^k}\right)\left(\frac{1}{M^{(\frac{1}{2})}}\right)\right), \quad M \geq 1. \tag{9}$$

Theorem 3.2 Let $f \in L^2[0,1]$ be a function such that $|f''(t)| \leq M_2$ for some $0 < M_2 < \infty, \forall t \in [0,1]$. If the extended Chebyshev wavelet expansion of f is and (μ^k, M) partial sum $S_{\mu^k,M}(t)$ of the above series (6) is as defined in section (2), then the approximation error $E_{\mu^k,M}^{(2)}$ of f by

$S_{\mu^k,M}$ is given by

$$E^{(2)}_{\mu^k,M} = inf||f - S_{\mu^k,M}||_2 = \begin{cases} O\left(\left(\frac{1}{\mu^{2k}}\right)\left(\frac{1}{M^{\left(\frac{3}{2}\right)}}\right)\right), & M \geq 3; \\ O\left(\frac{1}{\mu^k}\right), & M = 1. \end{cases} \tag{10}$$

3.1. Proof of Theorem 3.1. Let $f \in L^2[0,1]$, then extended Chebyshev wavelet series of f is given by (6).

Now, for $m \geq 1$

$$\begin{aligned}
c^{(\mu)}_{n,m} &= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(t)\psi^{(\mu)}_{n,m}(t)w^{(\mu)}_{n,k}(t)\,dt \\
&= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(t)\left(\frac{\mu^k}{\pi}\right)^{\frac{1}{2}} \tilde{T}_m(2\mu^k t - 2n + 1)\frac{1}{\sqrt{1-(2\mu^k t - 2n+1)^2}}\,dt \\
&= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(t)\left(\frac{\mu^k}{\pi}\right)^{\frac{1}{2}} 2T_m(2\mu^k t - 2n + 1)\frac{1}{\sqrt{1-(2\mu^k t - 2n+1)^2}}\,dt \\
&= \left(\frac{1}{\pi\mu^k}\right)^{\frac{1}{2}} \int_{-1}^{1} f\left(\frac{u+2n-1}{2\mu^k}\right) T_m(u)\frac{1}{\sqrt{1-u^2}}\,du \qquad \left(2\mu^k t - 2n + 1 = u\right) \\
&= \left(\frac{1}{\pi\mu^k}\right)^{\frac{1}{2}} \int_{0}^{\pi} f\left(\frac{cos\theta+2n-1}{2\mu^k}\right) cosm\theta\,d\theta \qquad (u = cos\theta) \\
&= \left(\frac{1}{\pi\mu^k}\right)^{\frac{1}{2}} \left[f\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\frac{sinm\theta}{m}\right]_0^{\pi} \\
&\quad -\left(\frac{1}{\pi\mu^k}\right)^{\frac{1}{2}} \int_{0}^{\pi} f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\frac{sinm\theta}{m}\frac{(-sin\theta)}{2\mu^k}\,d\theta \\
&= \left(\frac{1}{2m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right) \int_{0}^{\pi} f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right) sinm\theta sin\theta\,d\theta \\
&= \left(\frac{1}{2m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right) \int_{0}^{\pi} f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\left(\frac{1}{2}\right)\left[cos(m-1)\theta - cos(m+1)\theta\right]d\theta. \qquad (11)
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|c^{(\mu)}_{n,m}\right| &= \left|\left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right) \int_{0}^{\pi} f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\left[cos(m-1)\theta - cos(m+1)\theta\right]d\theta\right| \\
&\leq \left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right) \int_{0}^{\pi} \left|f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\right| \left|\left[cos(m-1)\theta - cos(m+1)\theta\right]\right|d\theta
\end{aligned}$$

$$\leq \left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right)\int_0^\pi \left|f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\right|[|cos(m-1)\theta|+|cos(m+1)\theta|]\,d\theta$$

$$\leq \left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right)[M_1\pi+M_1\pi]$$

$$= \left(\frac{\pi^{\frac{1}{2}}}{2m}\right)\left(\frac{M_1}{\mu^{\frac{3k}{2}}}\right).$$

Now,

$$(12)\quad ||f-S_{\mu^k,M}||_2^2 = \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty}\left|c_{n,m}^{(\mu)}\right|^2$$

$$\leq \frac{\pi M_1^2}{4}\left(\frac{1}{\mu^{3k}}\right)\mu^k\sum_{m=M}^{\infty}\frac{1}{m^2}$$

$$||f-S_{\mu^k,M}||_2^2 \leq \frac{\pi M_1^2}{4}\left(\frac{1}{\mu^{2k}}\right)\left(\frac{1}{M}\right). \qquad \left(\because \sum_{m=M}^{\infty}\frac{1}{m^2}\leq\frac{1}{M},\ M>1 \text{ or } M\geq 2\right)$$

or

$$(13)\qquad ||f-S_{\mu^k,M}||_2 \leq \frac{\sqrt{\pi}M_1}{2}\left(\frac{1}{\mu^k}\right)\left(\frac{1}{M}\right)^{\frac{1}{2}},$$

and so

$$E_{\mu^k,M}^{(1)} = O\left(\left(\frac{1}{\mu^k}\right)\left(\frac{1}{M^{(\frac{1}{2})}}\right)\right),\ M\geq 2.$$

3.2. Proof of Theorem 3.2. From equation(11), we have

$$c_{n,m}^{(\mu)} = \left(\frac{1}{2m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right)\int_0^\pi f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\left(\frac{1}{2}\right)[cos(m-1)\theta-cos(m+1)\theta]\,d\theta$$

$$= \left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right)\left[\int_0^\pi f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)cos(m-1)\theta\,d\theta - \int_0^\pi f'\left(\frac{cos\theta+2n-1}{2\mu^k}\right)cos(m+1)\theta\,d\theta\right]$$

$$= \left(\frac{1}{4m\pi^{\frac{1}{2}}\mu^{\frac{3k}{2}}}\right)\left[-\int_0^\pi f''\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\left(\frac{sin(m-1)\theta}{m-1}\right)\left(\frac{-sin\theta}{2\mu^k}\right)d\theta + \int_0^\pi f''\left(\frac{cos\theta+2n-1}{2\mu^k}\right)\left(\frac{sin(m+1)\theta}{m+1}\right)\left(\frac{-sin\theta}{2\mu^k}\right)d\theta\right]$$

$$= \left(\frac{1}{8m(m-1)\pi^{\frac{1}{2}}\mu^{\frac{5k}{2}}}\right)\int_0^\pi f''\left(\frac{cos\theta+2n-1}{2\mu^k}\right)(cos(m-2)\theta-cosm\theta)\,d\theta$$

$$+\left(\frac{1}{8m(m+1)\pi^{\frac{1}{2}}\mu^{\frac{5k}{2}}}\right)\int_0^{\pi} f''\left(\frac{cos\theta+2n-1}{2\mu^k}\right)(cosm\theta-cos(m+2)\theta)\,d\theta.$$

Then,

$$\begin{aligned}\left|c_{n,m}^{(\mu)}\right| &\leq \left(\frac{1}{8m(m-1)\pi^{\frac{1}{2}}\mu^{\frac{5k}{2}}}\right)M_2\,(2\pi)+\left(\frac{1}{8m(m+1)\pi^{\frac{1}{2}}\mu^{\frac{5k}{2}}}\right)M_2\,(2\pi)\\ &\leq \left(\frac{M_2\pi^{\frac{1}{2}}}{4(m-1)^2\mu^{\frac{5k}{2}}}\right)+\left(\frac{M_2\pi^{\frac{1}{2}}}{4(m-1)^2\mu^{\frac{5k}{2}}}\right)\\ &\leq \left(\frac{M_2\pi^{\frac{1}{2}}}{2(m-1)^2\mu^{\frac{5k}{2}}}\right).\end{aligned}$$

Now, using equation (12) along with the above relation, we get

$$||f-S_{\mu^k,M}||_2 = O\left(\left(\frac{1}{\mu^{2k}}\right)\left(\frac{1}{(M-1)^{\left(\frac{3}{2}\right)}}\right)\right), M\geq 3.$$

Thus,

$$E_{\mu^k,M}^{(1)} = O\left(\left(\frac{1}{\mu^{2k}}\right)\left(\frac{1}{(M-1)^{\left(\frac{3}{2}\right)}}\right)\right), M\geq 3.$$

3.3. Proof of Theorems (3.1) and (3.2) for $M=1$.

Let

$$e_{n,0}^{(\mu)} = c_{n,0}^{(\mu)}\psi_{n,0}^{(\mu)} - f(t).$$

$$\therefore\ ||e_{n,0}^{(\mu)}||_2^2 = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}}\left(c_{n,0}^{(\mu)}\psi_{n,0}^{(\mu)}-f(t)\right)^2\frac{1}{\sqrt{1-(2\mu^k t-2n+1)^2}}\,dt.$$

By applying mean value theorem, we get

$$\begin{aligned}||e_{n,0}^{(\mu)}||_2^2 &= \left(c_{n,0}^{(\mu)}\psi_{n,0}^{(\mu)}-f(u_n)\right)^2\int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}}\frac{1}{\sqrt{1-(2\mu^k t-2n+1)^2}}\,dt, \qquad u_n\in\left(\frac{n-1}{\mu^k},\frac{n}{\mu^k}\right)\\ &= \left(c_{n,0}^{(\mu)}\psi_{n,0}^{(\mu)}-f(u_n)\right)^2\left(\frac{\pi}{2\mu^k}\right). \qquad (14)\end{aligned}$$

So,

$$c_{n,0}^{(\mu)} = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(t)\psi_{n,0}^{(\mu)}\frac{1}{\sqrt{1-(2\mu^k t-2n+1)^2}}\,dt.$$

By applying mean value theorem, we get

$$\begin{aligned} c_{n,0}^{(\mu)} &= f(v_n)\frac{2\mu^k}{\pi}\int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} \frac{1}{\sqrt{1-(2\mu^k t-2n+1)^2}}dt, \qquad v_n \in \left(\frac{n-1}{\mu^k},\frac{n}{\mu^k}\right) \\ &= f(v_n)\left(\frac{2\mu^k}{\pi}\right)\left(\frac{\pi}{2\mu^k}\right) \\ &= f(v_n)\left(\sqrt{\frac{\pi}{2\mu^k}}\right). \end{aligned} \tag{15}$$

Using equation (15) in (14), we get

$$\begin{aligned} ||e_{n,0}^{(\mu)}||_2^2 &= \left(f(v_n)\left(\sqrt{\frac{\pi}{2\mu^k}}\right)\left(\sqrt{\frac{2\mu^k}{\pi}}\right)-f(u_n)\right)^2\left(\frac{\pi}{2\mu^k}\right) \\ &= (f(v_n)-f(u_n))^2\left(\frac{\pi}{2\mu^k}\right). \end{aligned} \tag{16}$$

Since first derivative of function f is bounded so that f belongs to the Lipschitz class.

Therefore,

$$\begin{aligned} |f(u_n)-f(v_n)| &\leq N_1|(u_n-v_n)| \\ &\leq \frac{N_1}{\mu^k} \qquad \left(\because u_n \& v_n \in \left(\frac{n-1}{\mu^k},\frac{n}{\mu^k}\right)\right). \end{aligned} \tag{17}$$

Using equation (17) in (16), we get

$$\begin{aligned} ||e_{n,0}^{(\mu)}||_2^2 &\leq \left(\frac{N_1^2}{\mu^{2k}}\right)\left(\frac{\pi}{2\mu^k}\right) \\ &= \frac{N_2^2}{\mu^{3k}} \qquad \left(N_2 = N_1\sqrt{\frac{\pi}{2}}\right). \end{aligned} \tag{18}$$

$$\begin{aligned} \therefore\ E_{\mu^k,1} &= \sqrt{\sum_{n=1}^{\mu^k}||e_{n,0}^{(\mu)}||_2^2} \\ &\leq \frac{N_1}{\mu^k}. \end{aligned}$$

Therefore,

$$E_{\mu^k,1} = O\left(\frac{1}{\mu^k}\right). \tag{19}$$

4. Extended Chebyshev Wavelets of First Kind Operational Matrix of Integration

In this section, the operational matrix of integration $P^{(\mu)}$ for extended Chebyshev wavelet of first kind is introduced.

4.1. Operational Matrix of Integration for $\mu = 2$. We find matrix $P^{(2)}$ with $\mu = 2$, $M = 8$ and $k = 1$. The sixteen basis functions are given by

$$(20)\quad \begin{cases} \psi^{(2)}_{1,0}(t) = \frac{2}{\sqrt{\pi}}, \\ \psi^{(2)}_{1,1}(t) = 2\sqrt{\frac{2}{\pi}}(4t-1), \\ \psi^{(2)}_{1,2}(t) = 2\sqrt{\frac{2}{\pi}}\left[2(4t-1)^2-1\right], \\ \psi^{(2)}_{1,3}(t) = 2\sqrt{\frac{2}{\pi}}\left[4(4t-1)^3-3(4t-1)\right], \\ \psi^{(2)}_{1,4}(t) = 2\sqrt{\frac{2}{\pi}}\left[8(4t-1)^4-8(4t-1)^2+1\right], \\ \psi^{(2)}_{1,5}(t) = 2\sqrt{\frac{2}{\pi}}\left[16(4t-1)^5-20(4t-1)^3+5(4t-1)\right], \\ \psi^{(2)}_{1,6}(t) = 2\sqrt{\frac{2}{\pi}}\left[32(4t-1)^6-48(4t-1)^4+18(4t-1)^2-1\right], \\ \psi^{(2)}_{1,7}(t) = 2\sqrt{\frac{2}{\pi}}\left[64(4t-1)^7-112(4t-1)^5+56(4t-1)^3-7(4t-1)\right] \end{cases} \quad 0 \le t < \frac{1}{2},$$

$$(21)\quad \begin{cases} \psi^{(2)}_{1,0}(t) = \frac{2}{\sqrt{\pi}}, \\ \psi^{(2)}_{1,1}(t) = 2\sqrt{\frac{2}{\pi}}(4t-3), \\ \psi^{(2)}_{1,2}(t) = 2\sqrt{\frac{2}{\pi}}\left[2(4t-3)^2-1\right], \\ \psi^{(2)}_{1,3}(t) = 2\sqrt{\frac{2}{\pi}}\left[4(4t-3)^3-3(4t-3)\right], \\ \psi^{(2)}_{1,4}(t) = 2\sqrt{\frac{2}{\pi}}\left[8(4t-3)^4-8(4t-3)^2+1\right], \\ \psi^{(2)}_{1,5}(t) = 2\sqrt{\frac{2}{\pi}}\left[16(4t-3)^5-20(4t-3)^3+5(4t-3)\right], \\ \psi^{(2)}_{1,6}(t) = 2\sqrt{\frac{2}{\pi}}\left[32(4t-3)^6-48(4t-3)^4+18(4t-3)^2-1\right], \\ \psi^{(2)}_{1,7}(t) = 2\sqrt{\frac{2}{\pi}}\left[64(4t-3)^7-112(4t-3)^5+56(4t-3)^3-7(4t-3)\right] \end{cases} \quad \frac{1}{2} \le t < 1.$$

Integrating $(20),(21)$ from 0 to t and expanding in terms of $\psi^{(2)}_{n,m}(t)$, we get

$$\begin{aligned}
\int_0^t \psi^{(2)}_{1,0}(x)\,dx &= \begin{cases} \frac{2t}{\sqrt{\pi}}, & 0 \le t < \frac{1}{2}; \\ \frac{1}{\sqrt{\pi}}, & \frac{1}{2} \le t < 1, \end{cases} \\
&= \frac{1}{4}\psi^{(2)}_{1,0}(t) + \frac{1}{4\sqrt{2}}\psi^{(2)}_{1,1}(t) + \frac{1}{2}\psi^{(2)}_{2,0}(t) \\
&= \left[\frac{1}{4}, \frac{1}{4\sqrt{2}}, 0,0,0,0,0,0, \frac{1}{2}, 0,0,0,0,0,0,0\right]\psi^{(2)}(t). \\
\int_0^t \psi^{(2)}_{1,1}(x)\,dx &= \begin{cases} -2\sqrt{\frac{2}{\pi}}(t-2t^2), & 0 \le t < \frac{1}{2}; \\ 0, & \frac{1}{2} \le t < 1, \end{cases}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{8\sqrt{2}}\psi^{(2)}_{1,0}(t) + \frac{1}{16}\psi^{(2)}_{1,2}(t) \\
&= \left[-\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0,0,0,0,0,0,0,0,0,0,0,0,0\right]\psi^{(2)}(t). \\
\int_0^t \psi^{(2)}_{1,2}(x)\,dx &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(t - 8t^2 + \frac{32}{3}t^3), & 0 \le t < \frac{1}{2}; \\ \frac{\sqrt{\frac{2}{\pi}}}{3}, & \frac{1}{2} \le t < 1, \end{cases} \\
&= -\frac{1}{6\sqrt{2}}\psi^{(2)}_{1,0}(t) - \frac{1}{8}\psi^{(2)}_{1,1}(t) + \frac{1}{24}\psi^{(2)}_{1,3}(t) - \frac{1}{3\sqrt{2}}\psi^{(2)}_{2,0}(t) \\
&= \left[\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, \frac{1}{24}, 0,0,0,0, -\frac{1}{3\sqrt{2}}, 0,0,0,0,0,0,0\right]\psi^{(2)}(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_0^t \psi^{(2)}_{1,3}(x)\,dx &= \left[\frac{1}{16\sqrt{2}}, 0, -\frac{1}{16}, 0, \frac{1}{32}, 0,0,0,0,0,0,0,0,0,0,0\right]\psi^{(2)}(t). \\
\int_0^t \psi^{(2)}_{1,4}(x)\,dx &= \left[-\frac{1}{30\sqrt{2}}, 0,0, -\frac{1}{24}, 0, \frac{1}{40}, 0,0, -\frac{1}{30\sqrt{2}}, 0,0,0,0,0,0,0\right]\psi^{(2)}(t). \\
\int_0^t \psi^{(2)}_{1,5}(x)\,dx &= \left[\frac{1}{48\sqrt{2}}, 0,0,0, -\frac{1}{32}, 0, \frac{1}{48}, 0,0,0,0,0,0,0,0,0\right]\psi^{(2)}(t). \\
\int_0^t \psi^{(2)}_{1,6}(x)\,dx &= \left[-\frac{1}{70\sqrt{2}}, 0,0,0,0, -\frac{1}{40}, 0, \frac{1}{56}, -\frac{1}{35\sqrt{2}}, 0,0,0,0,0,0,0\right]\psi^{(2)}(t).
\end{aligned}$$

(22)

$$\int_0^t \psi_{1,7}^{(2)}(x)\,dx = \left[\frac{1}{96\sqrt{2}},0,0,0,0,0,-\frac{1}{48},0,0,0,0,0,0,0,0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,0}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,\frac{1}{4},\frac{1}{4\sqrt{2}},0,0,0,0,0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,1}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,-\frac{1}{8\sqrt{2}},0,\frac{1}{16},0,0,0,0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,2}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,-\frac{1}{8\sqrt{2}},0,\frac{1}{16},0,0,0,0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,3}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,\frac{1}{16\sqrt{2}},0,-\frac{1}{16},0,\frac{1}{32},0,0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,4}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,-\frac{1}{30\sqrt{2}},0,0,-\frac{1}{24},0,\frac{1}{40},0,0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,5}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,\frac{1}{48\sqrt{2}},0,0,0,-\frac{1}{32},0,\frac{1}{48},0\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,6}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,-\frac{1}{70\sqrt{2}},0,0,0,0,-\frac{1}{40},0,\frac{1}{56}\right]\psi^{(2)}(t).$$

$$\int_0^t \psi_{2,7}^{(2)}(x)\,dx = \left[0,0,0,0,0,0,0,0,\frac{1}{96\sqrt{2}},0,0,0,0,0,-\frac{1}{48},0\right]\psi^{(2)}(t).$$

and hence the operational matrix of integration of the extended Chebyshev wavelet of first kind for $\mu = 2$ is,

$$P^{(2)} = \begin{bmatrix} A_{8\times 8}^{(2)} & B_{8\times 8}^{(2)} \\ 0_{8\times 8} & A_{8\times 8}^{(2)} \end{bmatrix},$$

where A and B are 8×8 matrices given by

$$A^{(2)}_{8\times 8} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6\sqrt{2}} & -\frac{1}{8} & 0 & \frac{1}{24} & 0 & 0 & 0 & 0 \\ \frac{1}{16\sqrt{2}} & 0 & -\frac{1}{16} & 0 & \frac{1}{32} & 0 & 0 & 0 \\ -\frac{1}{30\sqrt{2}} & 0 & 0 & -\frac{1}{24} & 0 & \frac{1}{40} & 0 & 0 \\ \frac{1}{48\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{1}{48} & 0 \\ -\frac{1}{70\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{40} & 0 & \frac{1}{56} \\ \frac{1}{96\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & 0 \end{bmatrix},$$

and

$$B^{(2)}_{8\times 8} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{35\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next, by using the same process as above for $\mu = 2$, we find the operational matrix of integration for $\mu = 3$ as

$$P^{(3)} = \begin{bmatrix} A^{(3)}_{8\times 8} & B^{(3)}_{8\times 8} & B^{(3)}_{8\times 8} \\ 0_{8\times 8} & A^{(3)}_{8\times 8} & B^{(3)}_{8\times 8} \\ 0_{8\times 8} & 0_{8\times 8} & A^{(3)}_{8\times 8} \end{bmatrix},$$

where A and B are 8×8 matrices given by

$$A^{(3)}_{8\times 8}=\begin{bmatrix} \frac{1}{6} & \frac{1}{6\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12\sqrt{2}} & 0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{9\sqrt{2}} & -\frac{1}{12} & 0 & \frac{1}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{24\sqrt{2}} & 0 & -\frac{1}{24} & 0 & \frac{1}{48} & 0 & 0 & 0 \\ -\frac{1}{45\sqrt{2}} & 0 & 0 & -\frac{1}{36} & 0 & \frac{1}{60} & 0 & 0 \\ \frac{1}{72\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{48} & 0 & \frac{1}{72} & 0 \\ -\frac{1}{105\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{60} & 0 & \frac{1}{84} \\ \frac{1}{144\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{72} & 0 \end{bmatrix},$$

and

$$B^{(3)}_{8\times 8}=\begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{105} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. NUMERICAL EXAMPLE OF APPROXIMATION BY EXTENDED CHEBYSHEV WAVELET OF FIRST KIND

Define a function $f(t)$ as

$$f(t)=\begin{cases} 3t^{12}+2t^{11}+t^{10}, & \forall t\in[0,1]; \\ 0, & otherwise. \end{cases} \tag{23}$$

Now, we have calculated the approximation $S_{\mu^k,M}$ for this choosen function $f(t)$ for $\mu=2,3$ &5, $k=0,1$ and $M=4$ and they are plotted in the figures 1 to 3. From the figures, we can see that as we increase the value of μ from 2 to 5, graph of the exact function and approximation $S_{\mu^k,M}$ come very close to each other. This shows the usefulness of taking the positive integer μ as a parameter in the definition of the extended Chebyshev wavelet.

$S_{\mu^k,M}$ are given as

$$S_{2^1,4}(t)=\begin{cases} 0.0004026709993309124\left(\frac{2}{\sqrt{\pi}}\right)+0.0005213222465007394\left(2\sqrt{\frac{2}{\pi}}\right)(4t-1) \\ +0.0003994981634050595\left(2\sqrt{\frac{2}{\pi}}\right)\left[2(4t-1)^2-1\right] \\ +0.0002553357377997655\left(2\sqrt{\frac{2}{\pi}}\right)\left[4(4t-1)^3-3(4t-1)\right], & 0\le t<\frac{1}{2}, \\ 1.282221204791795\left(\frac{2}{\sqrt{\pi}}\right)+1.4988006171097665\left(2\sqrt{\frac{2}{\pi}}\right)(4t-3) \\ +0.8624086255193825\left(2\sqrt{\frac{2}{\pi}}\right)\left[2(4t-3)^2-1\right] \\ +0.3555346563281834\left(2\sqrt{\frac{2}{\pi}}\right)\left[4(4t-3)^3-3(4t-3)\right], & \frac{1}{2}\le t<1, \\ 0, & otherwise. \end{cases}$$

$$S_{3^1,4}(t)=\begin{cases} 4.191556693639439(10^{-6})\left(\sqrt{\frac{6}{\pi}}\right)+5.416612457498025(10^{-6})\left(2\sqrt{\frac{3}{\pi}}\right)(6t-1) \\ +4.127408868320194(10^{-6})\left(2\sqrt{\frac{3}{\pi}}\right)\left[2(6t-1)^2-1\right] \\ +2.6122165202388637(10^{-6})\left(2\sqrt{\frac{3}{\pi}}\right)\left[4(6t-1)^3-3(6t-1)\right], & 0\le t<\frac{1}{3}, \\ 0.011224242767965797\left(\sqrt{\frac{6}{\pi}}\right)+0.013056926261291471\left(2\sqrt{\frac{3}{\pi}}\right)(6t-3) \\ +0.007418840195179225\left(2\sqrt{\frac{3}{\pi}}\right)\left[2(6t-3)^2-1\right] \\ +0.0030026512406400307\left(2\sqrt{\frac{3}{\pi}}\right)\left[4(6t-3)^3-3(6t-3)\right], & \frac{1}{3}\le t<\frac{2}{3}, \\ 1.3284273192458527\left(\sqrt{\frac{6}{\pi}}\right)+1.3500693320408341\left(2\sqrt{\frac{3}{\pi}}\right)(6t-5) \\ +0.5788688746822768\left(2\sqrt{\frac{3}{\pi}}\right)\left[2(6t-5)^2-1\right] \\ +0.1639264084874334\left(2\sqrt{\frac{3}{\pi}}\right)\left[4(6t-5)^3-3(6t-5)\right], & \frac{2}{3}\le t<1, \\ 0, & otherwise. \end{cases}$$

$$S_{5^1,4}(t)=\begin{cases}1.5084233918571205(10^{-8})\left(\sqrt{\frac{10}{\pi}}\right)+1.945635145858096(10^{-8})\left(2\sqrt{\frac{5}{\pi}}\right)(10t-1)\\ +1.47402814831035(10^{-8})\left(2\sqrt{\frac{5}{\pi}}\right)\left[2(10t-1)^2-1\right]\\ +9.235116139022133(10^{-9})\left(2\sqrt{\frac{10}{\pi}}\right)\left[4(10t-1)^3-3(10t-1)\right], & 0\le t<\frac{1}{5},\\ 0.000033191046388326036\left(\sqrt{\frac{10}{\pi}}\right)+0.000038365414382791 01\left(2\sqrt{\frac{5}{\pi}}\right)(10t-3)\\ +0.000021435395631356222\left(2\sqrt{\frac{5}{\pi}}\right)\left[2(10t-3)^2-1\right]\\ +8.459832396322085(10^{-6})\left(2\sqrt{\frac{5}{\pi}}\right)\left[4(10t-3)^3-3(10t-3)\right], & \frac{1}{5}\le t<\frac{2}{5},\\ 0.0034562972667185705\left(\sqrt{\frac{10}{\pi}}\right)+0.0034713816393338423\left(2\sqrt{\frac{5}{\pi}}\right)(10t-5)\\ +0.0014555178555358695\left(2\sqrt{\frac{5}{\pi}}\right)\left[2(10t-5)^2-1\right]\\ +0.00040067671684113277\left(2\sqrt{\frac{5}{\pi}}\right)\left[4(10t-5)^3-3(10t-5)\right], & \frac{2}{5}\le t<\frac{3}{5},\\ 0.10021210497603478\left(\sqrt{\frac{10}{\pi}}\right)+0.08621692253987188\left(2\sqrt{\frac{5}{\pi}}\right)(10t-7)\\ +0.028383433223664274\left(2\sqrt{\frac{5}{\pi}}\right)\left[2(10t-7)^2-1\right]\\ +0.005940326112916907\left(2\sqrt{\frac{5}{\pi}}\right)\left[4(10t-7)^3-3(10t-7)\right], & \frac{3}{5}\le t<\frac{4}{5},\\ 1.4121148871293914\left(\sqrt{\frac{10}{\pi}}\right)+1.0450497377240309\left(2\sqrt{\frac{5}{\pi}}\right)(10t-9)\\ +0.281186391605936\left(2\sqrt{\frac{5}{\pi}}\right)\left[2(10t-9)^2-1\right]\\ +0.04732253976621698\left(2\sqrt{\frac{5}{\pi}}\right)\left[4(10t-9)^3-3(10t-9)\right], & \frac{4}{5}\le t<1,\\ 0, & \textit{otherwise}.\end{cases}$$

The graphs of $S_{\mu^k,M}$ and $f(t)$ has been plotted for $\mu=2,3\ \&5\quad M=4$ and $k=1$ in figures $(1),(2),(3)$

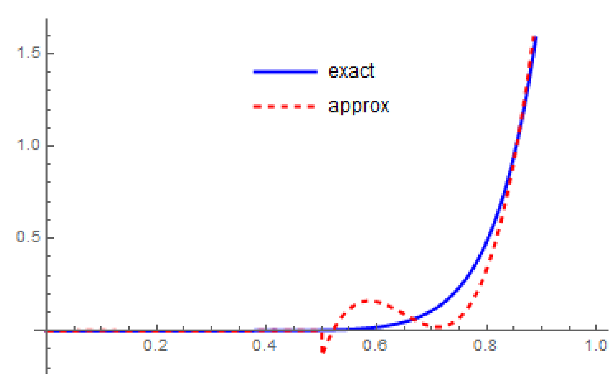

FIGURE 1. Comparison between the exact and approximate graph of function (23) for $\mu=2$.

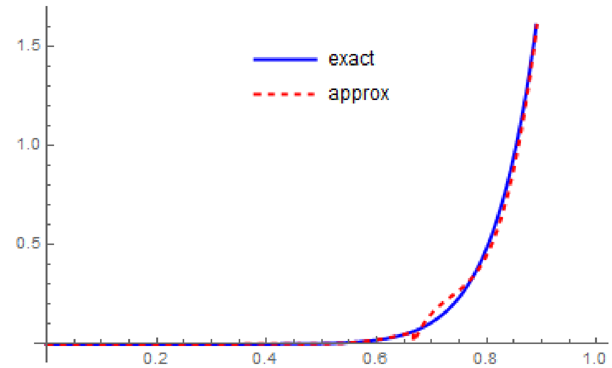

FIGURE 2. Comparison between the exact and approximate graph of function (23) for $\mu = 3$.

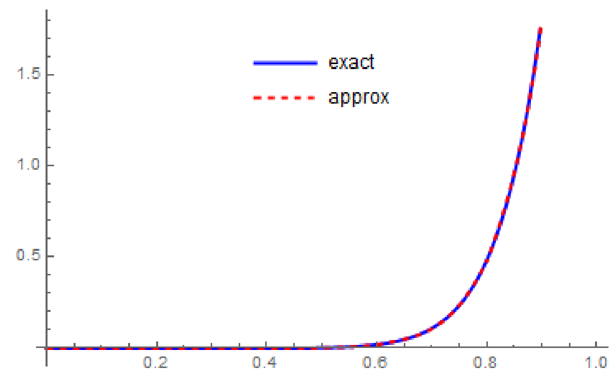

FIGURE 3. Comparison between the exact and approximate graph of function (23) for $\mu = 5$.

Lastly, in this section, approximation of the function $f(t)$ calculated in the paper Nigam et al.[30] and the function $f(t)$ are plotted in the figures 4 and 5. If we compare the figures 4 & 5 to the figures 2 & 3, we observe that the approximation errors in the figures 4 and 5 are larger than the approximation errors in the figures 2 & 3. This shows that the approximation obtained in our paper is better than the approximation calculated in Nigam et al. [23].

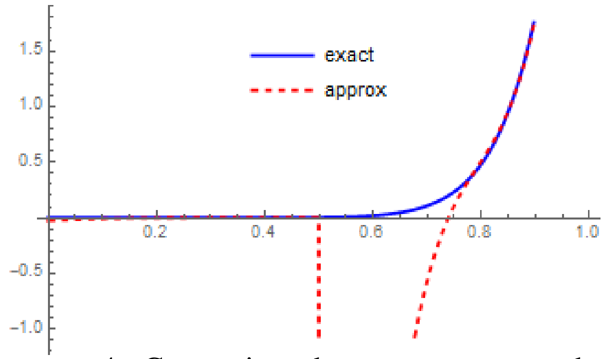

FIGURE 4. Comparison between exact graph of the function (23) and approximation calculated in Nigam et al.[30] by the third kind Chebyshev wavelet.

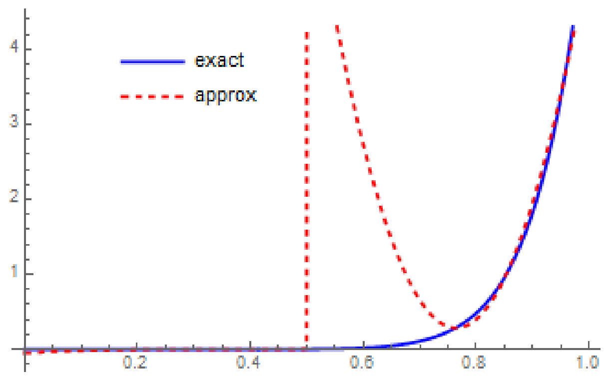

FIGURE 5. Comparison between exact graph of the function (23) and the approximation calculated in Nigam et al.[30] by the fourth kind Chebyshev wavelet..

6. APPLICATIONS OF EXTENDED CHEBYSHEV WAVELET OF FIRST KIND EXPANSION

6.1. Application of the extended Chebyshev wavelet of first kind in Real-World problem.

We have used the method of approximation by extended Chebyshev wavelet of first kind to solve the differential equations related to the following real world problem.

6.1.1. *Radioactive Decay.* Radioactivity is one of the effects of disruption in the nucleus of a radioactive substance.

Let $m(t)$ denotes the mass of a radioactive object at any time t, then

$$\frac{dm}{dt} = -k_1 m(t), m(0) = m_0. \text{ (Lal et al.[18])} \tag{24}$$

where k_1 is a decay constant and m_0 be the initial mass. Let us consider $k_1 = 2$, and $m_0 = 2$, then the equation(24) is converted to

$$\frac{dm}{dt} = -2m(t), \quad m(0) = 2. \tag{25}$$

$$\psi^{(\mu)}(t) = \left[\psi_{1,0}^{(\mu)}, \psi_{1,1}^{(\mu)}, ..., \psi_{1,M-1}^{(\mu)}, \psi_{2,0}^{(\mu)}, ..., \psi_{2,M-1}^{(\mu)}, ..., \psi_{\mu^k,0}^{(\mu)}, ..., \psi_{\mu^k,M-1}^{(\mu)}\right]^T. \tag{26}$$

Here $\psi^{(\mu)}(t)$ be $(\mu^k M) \times 1$ column vector, the integration of above vector $\psi^{(\mu)}(t)$ is given as :

$$\int_0^t \psi^{(\mu)}(x)\, dx = P^{(\mu)} \psi^{(\mu)}(t). \tag{27}$$

Here $P^{(\mu)}$ is the operational matrix of integration of extended Chebyshev wavelet of first kind.

$$\text{Let} \quad m(t) = N^T \psi^{(\mu)}(t), \tag{28}$$

$$\text{where} \quad N(t) = \left[n_{1,0}^{(\mu)}, n_{1,1}^{(\mu)}, ..., n_{1,M-1}^{(\mu)}, n_{2,0}^{(\mu)}, ..., n_{2,M-1}^{(\mu)}, ..., n_{\mu^k,0}^{(\mu)}, ..., n_{\mu^k,M-1}^{(\mu)}\right]^T$$

is an unknown vector.

Integrating equation (25), we get

$$m(t) - m(0) = -2\int_0^t m(t)\, dt.$$

$$\text{Let} \quad 1 = D^{(\mu)^T} \psi^{(\mu)}(t), \tag{29}$$

where D is a column vector of $(\mu^k M) \times 1$

$$\begin{aligned} N^T \psi(t) - 2D^{(\mu)^T} \psi^{(\mu)}(t) &= -2\int_0^t N^T \psi^{(\mu)}(t)\, dt \\ &= -2N^T \int_0^t \psi^{(\mu)}(t)\, dt \\ &= -2N^T P^{(\mu)} \psi^{(\mu)}(t) \end{aligned}$$

Therefore,

$$(I+2P^{(\mu)})N^T\psi^{(\mu)}(t) \;=\; 2D^{(\mu)^T}\psi^{(\mu)}(t).$$

or

$$(I+2P^{(\mu)^T})N\psi^{(\mu)^T}(t) \;=\; 2D^{(\mu)}\psi^{(\mu)^T}(t),$$

and so

$$N \;=\; \left(I+2P^{(\mu)^T}\right)^{-1}2D^{(\mu)}. \tag{30}$$

Here I is the identity matrix of order $(\mu^k M)$. Equation (30) denotes the set of $(\mu^k M)$ algebraic equations which can be solved for N. Now comparison between exact solution $[m(t)=2e^{-2t}]$ and approximate solution of equation (25) for $\mu=2,3$ is given in the tables $(1)\&(2)$

t	ECW of first kind Solution for $\mu=2$, $k=1$, $M=8$	Exact Solution	Absolute error
0.0	1.999999999073624	2.000000000000000	0.000000000926376
0.1	1.637461505803944	1.637461506155936	0.000000000351992
0.2	1.340640092046318	1.340640092071278	0.00000000002496
0.3	1.097623272278750	1.097623272188053	0.000000000090697
0.4	0.898657927796311	0.898657928234443	0.000000000438132
0.5	0.735758881991905	0.735758882342884	0.000000000350979
0.6	0.602388423686564	0.602388423824404	0.00000000013784
0.7	0.493193927867203	0.493193927883213	0.00000000001601
0.8	0.403793036017086	0.403793035989310	0.000000000027776
0.9	0.330597776277417	0.330597776443173	0.000000000165756

TABLE 1. Comparison between approximate solution and exact solution for $\mu=2$, $k=1$, $M=8$ of equation (25) for some values of t.

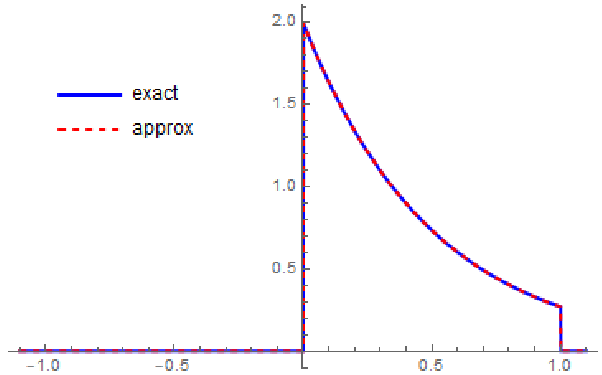

FIGURE 6. Comparison between the exact and approximate solution of eq.(25) for $\mu = 2$.

t	ECW of first kind Solution for $\mu = 3$, $k = 1$, $M = 8$	Exact Solution	Absolute error
0.0	1.999999999929182	2.000000000000000	0.000000000070818
0.1	1.637461506174692	1.637461506155936	0.000000000018756
0.2	1.340640092055606	1.340640092071278	0.000000000015672
0.3	1.097623272155381	1.097623272188053	0.000000000032672
0.4	0.898657928224875	0.898657928234443	0.000000000009568
0.5	0.735758882320238	0.735758882342884	0.000000000022646
0.6	0.602388423813791	0.602388423824404	0.000000000010613
0.7	0.493193927884127	0.493193927883213	0.000000000000914
0.8	0.403793035994103	0.403793035989310	0.0000000000047931
0.9	0.330597776458123	0.330597776443173	0.00000000001495

TABLE 2. Comparison between approximate solution and exact solution for $\mu = 3$, $k = 1$, $M = 8$ of equation (25) for some values of t.

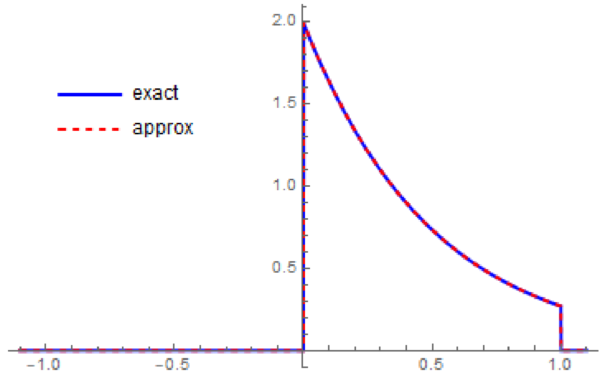

FIGURE 7. Comparison between the exact and approximate solution of eq.(25) for $\mu = 3$.

6.2. Hermite Differential Equation of Order Zero.

$$y'' - 2ty' = 0, \quad y(0) = y'(0) = 1. \quad \text{(Lal et al. [18])} \tag{31}$$

$$\text{Let} \quad y'' = L^T \psi^{(\mu)} \tag{32}$$

$$\text{where} \quad L(t) = \left[l^{(\mu)}_{1,0}, l^{(\mu)}_{1,1}, \ldots, l^{(\mu)}_{1,M-1}, l^{(\mu)}_{2,0}, \ldots, l^{(\mu)}_{2,M-1}, \ldots, l^{(\mu)}_{\mu^k,0}, \ldots, l^{(\mu)}_{\mu^k,M-1} \right]^T$$

is an unknown vector.

On integration equation (32), we get

$$\begin{aligned}
\int_0^t y''(x)\,dx &= \int_0^t L^T \psi^{(\mu)}(x)\,dx \\
y'(t) - y'(0) &= L^T \int_0^t \psi^{(\mu)}(x) \\
y'(t) - D^{(\mu)^T} \psi(t) &= L^T P^{(\mu)} \psi^{(\mu)}(t)
\end{aligned}$$

$$\text{or,} \quad y'(t) = D^{(\mu)^T} \psi(t) + L^T P^{(\mu)} \psi^{(\mu)}(t) \tag{33}$$

again integrating equation (33), we get

$$y(t) - D^{(\mu)^T} \psi^{(\mu)}(t) = D^{(\mu)^T} P^{(\mu)} \psi^{(\mu)}(t) + L^T P^{(\mu)^2} \psi^{(\mu)}(t)$$

$$y(t) = D^{(\mu)^T} \psi^{(\mu)}(t) + D^{(\mu)^T} P^{(\mu)} \psi(t) + L^T P^{(\mu)^2} \psi^{(\mu)}(t) \tag{34}$$

Using equations (32) and (33) in equation (31), we get

$$L^T \psi^{(\mu)} - 2t\left(\psi^{(\mu)^T} P^T L + \psi^{(\mu)^T} D^{(\mu)}\right) = 0. \tag{35}$$

$$\text{Let} \quad t = e^{(\mu)^T} \psi^{(\mu)}, \tag{36}$$

where $e^{(\mu)}$ is a column vector of $(\mu^k M) \times 1$.

Using equation (36) in (35), we get

$$L^T \psi^{(\mu)} - 2e^{(\mu)^T} \psi^{(\mu)}\left(\psi^{(\mu)^T} P^{(\mu)^T} L + \psi^{(\mu)^T} D^{(\mu)}\right) = 0$$

or

$$L^T \psi^{(\mu)} - 2e^{(\mu)^T} \psi^{(\mu)} \psi^{(\mu)^T} P^{(\mu)^T} L - 2e^{(\mu)^T} \psi^{(\mu)} \psi^{(\mu)^T} D^{(\mu)} = 0.$$

or

$$L^T \psi^{(\mu)} - 2L^T P^{(\mu)} \psi^{(\mu)} \psi^{(\mu)^T} e^{(\mu)} - 2D^{(\mu)^T} \psi^{(\mu)^T} \psi^{(\mu)} e^{(\mu)} = 0. \tag{37}$$

Let

$$\psi^{(\mu)} \psi^{(\mu)^T} e^{(\mu)} = \tilde{e} \psi^{(\mu)}, \tag{38}$$

where $\tilde{e}$ is matrix of order $(\mu^k M) \times (\mu^k M)$.

Using the equation (38) in (37), we get

$$L^T \psi^{(\mu)} - 2L^T P^{(\mu)} \tilde{e} \psi^{(\mu)} - 2D^{(\mu)^T} \tilde{e} \psi^{(\mu)} = 0$$

or

$$L^T - 2L^T P^{(\mu)} \tilde{e} - 2D^{(\mu)^T} \tilde{e} = 0$$

or

$$L^T - 2L^T P^{(\mu)} \tilde{e} = 2D^{(\mu)^T} \tilde{e}$$

or

$$L^T\left(I - 2P^{(\mu)} \tilde{e}\right) = 2D^{(\mu)^T} \tilde{e}$$

or

$$L^T = \left(I - 2P^{(\mu)}\tilde{e}\right)^{-1} 2D^{(\mu)^T}\tilde{e}. \tag{39}$$

Here I is the identity matrix of order $(\mu^k M)$. Equation (39) denotes the set of $(\mu^k M)$ algebraic equations which can be solved for L. Now comparison between exact solution $y(t) = 1 + \int_0^t e^{x^2}\,dx$ and approximate solution of equation (31) for $\mu = 2, 3$ is given in tables (3)&(4)

t	Solution of Hermite DE of order zero by ECW of first kind for $\mu = 2, k = 1, M = 8$	Exact Solution	Absolute error
0.0	0.999999998445	1.000000000000	0.000000001555
0.1	1.100334334907	1.100334335617	0.00000000071
0.2	1.202698974194	1.202698973606	0.000000000588
0.3	1.309248299493	1.309248299298	0.000000000195
0.4	1.422397588617	1.422397588523	0.000000000094
0.5	1.544987087838	1.544987103541	0.000000015703
0.6	1.680492056612	1.680492063901	0.000000007289
0.7	1.833304057375	1.833304052364	0.000000005011
0.8	2.009120712592	2.009120717607	0.000000005015
0.9	2.215498510877	2.215498512631	0.000000001754
1.0	2.462651731325	2.462651745907	0.000000014582

TABLE 3. Comparison between approximate solution and exact solution of Hermite DE of order zero for $\mu = 2$, $k = 1$, $M = 8$ of equation (31) for some values of t.

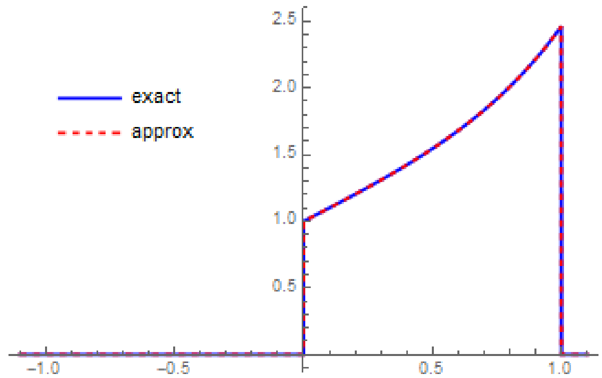

FIGURE 8. Comparison between the exact and approximate solution of eq.(31) for $\mu = 2$.

t	Solution of Hermite DE of order zero by ECW of first kind for $\mu = 3, k = 1, M = 8$	Exact Solution	Absolute error
0.0	0.999999999940	1.000000000000	0.00000000006
0.1	1.100334335713	1.100334335617	0.000000000096
0.2	1.202698973754	1.202698973606	0.000000000148
0.3	1.309248299535	1.309248299298	0.000000000237
0.4	1.422397588670	1.422397588523	0.000000000147
0.5	1.544987103696	1.544987103541	0.000000000155
0.6	1.680492064327	1.680492063901	0.000000000426
0.7	1.833304052167	1.833304052364	0.000000000197
0.8	2.009120712056	2.009120717607	0.000000005551
0.9	2.215498513806	2.215498512631	0.000000001175
1.0	2.462651742729	2.462651745907	0.000000003178

TABLE 4. Comparison between approximate solution and exact solution of Hermite DE of order zero for $\mu = 3$, $k = 1$, $M = 8$ of equation (31) for some values of t.

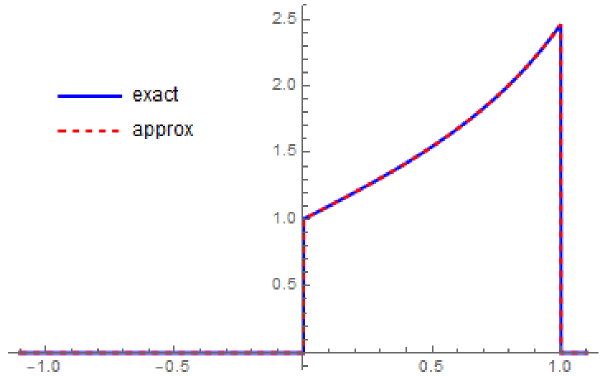

FIGURE 9. Comparison between the exact and approximate solution of eq.(31) for $\mu = 3$.

6.3. Nonlinear Riccati Differential Equation. Consider the following nonlinear Riccati differential equation (Giklou et.al[26]) with the indicating initial condition

$$y'(x) = 1 + x^2 - y^2(x), \quad y(0) = 1. \tag{40}$$

Here, we have solved this differential equation by operational matrix of integration of the extended Chebyshev wavelet of first kind.

Let

$$y'(x) = R^T \psi^{(\mu)}(x), \tag{41}$$

where

$$R(t) = \left[r^{(\mu)}_{1,0}, r^{(\mu)}_{1,1}, \ldots, r^{(\mu)}_{1,M-1}, r^{(\mu)}_{2,0}, \ldots, r^{(\mu)}_{2,M-1}, \ldots, r^{(\mu)}_{\mu^k,0}, \ldots, r^{(\mu)}_{\mu^k,M-1}\right]^T$$

is an unknown vector.

On integration equation (41), we get

$$\begin{aligned} \int_0^x y'(t)\,dt &= \int_0^x R^T \psi^{(\mu)}(t)\,dt \\ y(x) - y(0) &= R^T \int_0^x \psi^{(\mu)}(t) \\ y(x) &= R^T \int_0^x \psi^{(\mu)}(t)dt + y(0). \end{aligned}$$

Using equations (27) and (29), we get

$$y(x) = R^T P^{(\mu)} \psi^{(\mu)}(x) + {D^{(\mu)}}^T \psi^{(\mu)}(x). \tag{42}$$

Let

$$x^2 = {f^{(\mu)}}^T \psi^{(\mu)}(x). \tag{43}$$

Where $f^{(\mu)}$ is a column vector of $(\mu^k M) \times 1$, which is calculated by expanding x^2 in terms of $\psi^{(\mu)}_{n,m}(x)$.

Using equations (41), (29), (42) and (43) in (40), we get

$$\begin{aligned} R^T \psi^{(\mu)} &= {D^{(\mu)}}^T \psi^{(\mu)} + {f^{(\mu)}}^T \psi^{(\mu)} - \left(R^T P^{(\mu)} \psi^{(\mu)} + {D^{(\mu)}}^T \psi^{(\mu)}\right)\left({\psi^{(\mu)}}^T {P^{(\mu)}}^T R + {\psi^{(\mu)}}^T D^{(\mu)}\right) \\ &= {D^{(\mu)}}^T \psi^{(\mu)} + {f^{(\mu)}}^T \psi^{(\mu)} - R^T P^{(\mu)} \psi^{(\mu)} {\psi^{(\mu)}}^T {P^{(\mu)}}^T R - R^T P^{(\mu)} \psi^{(\mu)} {\psi^{(\mu)}}^T D^{(\mu)} \\ &\quad - {D^{(\mu)}}^T \psi^{(\mu)} {\psi^{(\mu)}}^T {P^{(\mu)}}^T R - {D^{(\mu)}}^T \psi^{(\mu)} {\psi^{(\mu)}}^T D^{(\mu)} \\ &= {D^{(\mu)}}^T \psi^{(\mu)} + {f^{(\mu)}}^T \psi^{(\mu)} - R^T P^{(\mu)} \tilde{E} \psi^{(\mu)} - 2{D^{(\mu)}}^T \tilde{E}^T \psi^{(\mu)} - {D^{(\mu)}}^T \tilde{D}^T \psi^{(\mu)}, \end{aligned}$$

where $\tilde{E}^T \psi^{(\mu)} = \psi^{(\mu)} {\psi^{(\mu)}}^T {P^{(\mu)}}^T R$ and $\tilde{D}^T \psi^{(\mu)} = \psi^{(\mu)} {\psi^{(\mu)}}^T D^{(\mu)}$.

Therefore

$$R^T + R^T P^{(\mu)} \tilde{E} + 2{D^{(\mu)}}^T \tilde{E}^T + {D^{(\mu)}}^T \tilde{D}^T - {D^{(\mu)}}^T - {f^{(\mu)}}^T = 0. \tag{44}$$

Equation (44) represents system of non-linear equations which can be solved for R. Now comparison between exact solution $y(x) = x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2}\,dt}$ and approximate solution of equation (40) for $\mu = 2$ and $\mu = 3$ is given in tables (5)&(6).

t	Solution of Riccati DE by ECW of first kind for $\mu = 2$, $k = 1$, $M = 8$	Exact Solution	Absolute error
0.0	0.999999999648	1.000000000000	0.000000000352
0.1	1.000317310222	1.000317310391	0.000000000169
0.2	1.002419825401	1.002419825609	0.000000000208
0.3	1.007794588173	1.007794588232	0.000000000059
0.4	1.017650878648	1.017650879015	0.000000000367
0.5	1.032957576247	1.032957576168	0.000000000079
0.6	1.054466810033	1.054466810047	0.000000000014
0.7	1.082727481297	1.082727481456	0.000000000159
0.8	1.118092545421	1.118092545487	0.000000000066
0.9	1.160723972797	1.160723972793	0.000000000004
1.0	1.210599014894	1.210599014669	0.000000000225

TABLE 5. Comparison between approximate solution and exact solution of Riccati DE for $\mu = 2$, $k = 1$, $M = 8$ of equation (40) for some values of t.

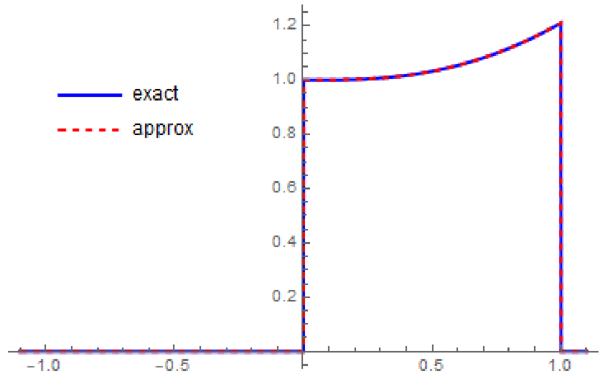

FIGURE 10. Comparison between the exact and approximate solution of eq.(40) for $\mu = 2$.

t	Solution of Riccati DE by ECW of first kind for $\mu = 3, k = 1, M = 8$	Exact Solution	Absolute error
0.0	0.999999999959	1.000000000000	0.000000000041
0.1	1.000317310320	1.000317310391	0.000000000071
0.2	1.002419825473	1.002419825609	0.000000000136
0.3	1.007794588064	1.007794588232	0.000000000168
0.4	1.017650878869	1.017650879015	0.000000000146
0.5	1.032957576031	1.032957576168	0.000000000137
0.6	1.054466809926	1.054466810047	0.000000000121
0.7	1.082727481369	1.082727481456	0.000000000087
0.8	1.118092545414	1.118092545487	0.000000000073
0.9	1.160723972727	1.160723972793	0.000000000066
1.0	1.210599014687	1.210599014669	0.000000000018

TABLE 6. Comparison between approximate solution and exact solution of Riccati DE for $\mu = 3$, $k = 1$, $M = 8$ of equation (40) for some values of t.

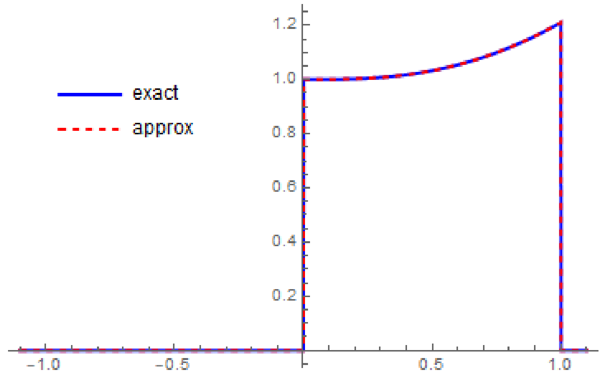

FIGURE 11. Comparison between the exact and approximate solution of eq.(40) for $\mu = 3$.

7. CONCLUSIONS

By theorems (**3.1**) and (**3.2**), it is clear that $E^{(1)}_{\mu^k,M}$ & $E^{(2)}_{\mu^k,M} \to 0$ as $\mu, k, M \to \infty$. Therefore, these are the best possible approximation (Zygmund [29]) in wavelet analysis.

In this paper, we have first defined the extended Chebyshev wavelet and it is used to prove some approximation theorems in which approximation errors depends on μ. Further, the calculated approximations are justified by an example given in section(5). In this example, approximation $S_{\mu^k,M}$ is calculated for $\mu = 2,3,5$ for the function $f(t)$ considered in that section and then this approximation $S_{\mu^k,M}$ and the function $f(t)$ for $\mu = 2,3,5$ are plotted in figures [1, 2, 3]. From the figures [1, 2, 3], we observe that as we increase the value of μ from 2 to 5, the calculated approximations become closer and closer to the function $f(t)$.

After this, we have used these approximations to solve the Hermite differential equation of order zero, non-linear Ricatti differential, and the differential equation related to the radioactive decay problem. In tables (1) to (6), the exact solutions are compared with the approximate solutions for $\mu = 2,3$, and we see that the approximate solution for $\mu = 3$ is more closer to the exact solution as compared to the approximate solution for $\mu = 2$. This shows the importance and applicability for taking the parameter μ in the definition of the extended Chebyshev wavelet.

From our discussion, it follows that if we take $\mu \geq 4$ for the numerical solutions of these differential equations, then the approximate solution will be better than the approximate solution for $\mu = 2$ and $\mu = 3$.

Data Availability Statement

The data from the previous studies are used to support the findings of the paper and they are cited at the relevant places in the paper according as the reference list of the article.

Author's Contribution

Both the authors jointly worked on the results, and they read and approved the final manuscript.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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