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DISCRETE DYNAMICS OF AN EPIDEMIC MODEL INCORPORATING A CONTROL STRATEGY

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Abstract. This paper proposes an epidemic model that incorporates media awareness and analyses the stability and bifurcation of the discrete-time system. It has been found that while the coefficient of media awareness m has no effect on R_0, it has a considerable impact on the infectious population portion. The centre manifold theorem and bifurcation theory have been used to investigate the existence of flip bifurcation and Hopf bifurcation in the interior of R^2_+. Numerical simulations have been provided to support analytical findings. Hopf bifurcation is characterised by a smooth invariant circle that emerges from the fixed point, while flip bifurcation is characterised by a cascade of periodic doubling bifurcation in orbits of period-2, 4, 8, and chaotic orbits. The calculation of Lyapunov exponents validates the complexity of dynamical behaviour.

Keywords: epidemic model; basic reproduction number; flip bifurcation; Hopf bifurcation; chaos.

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1. Introduction

The study of epidemic propagation has become a focal point for both mathematicians and biologists. With one million individuals crossing international borders every day, the rapid spread of communicable diseases poses a significant threat to global health [1]. Health officials have a toolkit of pharmaceutical and non-pharmaceutical strategies at their disposal, including mask mandates, school closures, isolation of the infected, and encouraging people to stay home [2]. Crucially, the media plays an essential role in improving public understanding of these non-pharmaceutical interventions (NPIs), which are vital for effective epidemic management. Yet, many researchers have modeled epidemics without factoring in the influence of media awareness on public behavior and health outcomes [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Only a small number of studies have examined how media awareness can effectively assist in managing infectious diseases [13, 14, 15, 16, 17]. Moreover, some researchers argue that discrete-time models are better suited for capturing the complexities of epidemic dynamics and yielding actionable insights [18, 19, 20, 21, 22, 23]. Understanding this interplay between media awareness and epidemic management is crucial for public health initiatives and can enhance our ability to respond effectively in future outbreaks.

In the current study, an epidemic model with a control strategy is proposed and analysed using the stability theory of ordinary differential equations. The structure of this paper is as follows: An epidemic model that incorporates media awareness is described in section 2. Section 3 discusses the fixed points and stability conditions of the discrete-time system. Section 4 uses bifurcation theory to discuss the existence conditions for flip bifurcation and Hopf bifurcation in the discrete-time domain. In section 5, some numerical simulations were provided to back up our analytical solutions. Finally, conclusions are provided in section 6.

2. Formulation of Mathematical Model

Under the following presumptions, we put forth an epidemic model that takes media awareness into account as a control measure:

(i) The population $\tilde{N}(\tilde{t})$ is divided into various mutually exclusive compartments, namely, susceptible($\tilde{S}$), infected($\tilde{I}$) and recovered($\tilde{R}$).

(ii) Newborn join to susceptible class with a fixed recruitment rate Λ

(iii) The natural death rate is $\tilde{\mu}$.

(iv) Media induced effective transmission rate is proposed as $\beta(\tilde{I}) = \beta e^{-\frac{m\tilde{I}}{\tilde{N}}}$.

(v) Recovered individuals develop disease-acquired temporary immunity that wanes at rate θ.

The following is the structure of the suggested epidemic system:

$$\frac{d\tilde{S}}{d\tilde{t}} = \Lambda - \tilde{\beta} e^{-m\frac{\tilde{I}}{\tilde{N}}} \frac{\tilde{S}\tilde{I}}{\tilde{N}} + \tilde{\theta}\tilde{R} - \tilde{\mu}\tilde{S}, \tag{1}$$

$$\frac{d\tilde{I}}{d\tilde{t}} = \tilde{\beta} e^{-m\frac{\tilde{I}}{\tilde{N}}} \frac{\tilde{S}\tilde{I}}{\tilde{N}} - \tilde{\mu}\tilde{I} - \tilde{\delta}\tilde{I} - \tilde{\gamma}\tilde{I}, \tag{2}$$

$$\frac{d\tilde{R}}{d\tilde{t}} = \tilde{\gamma}\tilde{I} - \tilde{\theta}\tilde{R} - \tilde{\mu}\tilde{R}, \tag{3}$$

with preliminary settings:

$$\tilde{S}(0) = \tilde{S}_0 > 0, \tilde{I}(0) = \tilde{I}_0 > 0, \tilde{R}(0) = \tilde{R}_0 > 0 \tag{4}$$

with Table 1 listing the parameters of the suggested system.

Further $\tilde{N}(\tilde{t}) = \tilde{S}(\tilde{t}) + \tilde{I}(\tilde{t}) + \tilde{R}(\tilde{t})$. So

$$\frac{d\tilde{N}}{d\tilde{t}} = \Lambda - \tilde{\mu}\tilde{N} - \tilde{\delta}\tilde{I}. \tag{5}$$

TABLE 1. An explanation of the system's parameters (1)-(4)

The parameter	Summary	Unit
Λ	Rate of recruitment	days^{-1}
$\tilde{\delta}$	Rate of disease-induced mortality	days^{-1}
$1/\tilde{\mu}$	Mean lifespan	days
$\tilde{\beta}$	Contact rate (via media awareness in the absence of NPIs)	days^{-1}
m	Media awareness coefficient	-
$1/\tilde{\gamma}$	Average duration of infection	days
$1/\tilde{\theta}$	Average duration of disease-induced immune decline	days

The aforementioned system is non-dimensionalized using

$$S=\frac{\tilde{S}}{\tilde{N}}, I=\frac{\tilde{I}}{\tilde{N}}, R=\frac{\tilde{R}}{\tilde{N}}, N=\frac{\tilde{N}}{\tilde{N_0}}, \tilde{N_0}=\frac{\Lambda}{\tilde{\mu}}, t=\tilde{\mu}\tilde{t}.$$

Equations that have been rescaled are

$$\frac{dS}{dt} = \frac{1}{N}-\beta e^{-mI}SI+\theta R-\frac{S}{N}+\delta SI, \tag{6}$$

$$\frac{dI}{dt} = \beta e^{-mI}SI-\delta I-\gamma I-\frac{I}{N}+\delta I^2, \tag{7}$$

$$\frac{dR}{dt} = \gamma I-\theta R-\frac{R}{N}+\delta RI, \tag{8}$$

$$\frac{dN}{dt} = 1-N-\delta NI, \tag{9}$$

where

$$\beta=\frac{\tilde{\beta}}{\tilde{\mu}}, \theta=\frac{\tilde{\theta}}{\tilde{\mu}}, \gamma=\frac{\tilde{\gamma}}{\tilde{\mu}}, \delta=\frac{\tilde{\delta}}{\tilde{\mu}}$$

and under the preliminary circumstances:

$$S(0)=S_0>0, I(0)=I_0>0, R(0)=R_0>0, N(0)=N_0>0. \tag{10}$$

At the steady state, or the limiting case, $\frac{dN}{dt}=0$, which suggests $max\ N=1$. As a result, the equation system (6)-(9) reductions to

$$\frac{dS}{dt} = 1-\beta e^{-mI}SI+\theta R-S+\delta SI, \tag{11}$$

$$\frac{dI}{dt} = \beta e^{-mI}SI-\delta I-\gamma I-I+\delta I^2, \tag{12}$$

$$\frac{dR}{dt} = \gamma I-\theta R-R+\delta RI. \tag{13}$$

Here, $maxN=1$ and $S+I+R=N$. When $S+I+R=1$ is the limiting situation, the system of equations (11)-(13) further reduces to

$$\frac{dS}{dt} = 1-\beta e^{-mI}SI+\theta(1-S-I)-S+\delta SI, \tag{14}$$

$$\frac{dI}{dt} = \beta e^{-mI}SI-\delta I-\gamma I-I+\delta I^2. \tag{15}$$

For the non-dimensional system, the biologically viable zone is

$$\Omega=\{(S,I):0\le S,I\le 1\}.$$

The disease-free equilibrium of the system (14)-(15) is $A = (1,0)$. The basic reproduction number R_0 is now determined. Let $x = (S,I)$. Consequently

$$\frac{dx}{dt} = f - \nu,$$

where

$$f = \begin{pmatrix} -\beta e^{-mI} SI + \delta SI \\ \beta e^{-mI} SI + \delta I^2 \end{pmatrix}$$

and

$$\nu = \begin{pmatrix} -1 - \theta(1 - S - I) + S \\ \delta I + \gamma I + I \end{pmatrix}.$$

We have

$$F = Df|_A = \begin{pmatrix} 0 & -\beta + \delta \\ 0 & \beta \end{pmatrix}$$

and

$$V = D\nu|_A = \begin{pmatrix} \theta + 1 & \theta \\ 0 & \delta + \gamma + 1 \end{pmatrix}.$$

The model's next generation matrix is provided by

$$K = FV^{-1} = \begin{pmatrix} 0 & \frac{-\beta+\delta}{\delta+\gamma+1} \\ 0 & \frac{\beta}{\delta+\gamma+1} \end{pmatrix}.$$

The formula $R_0 = \rho(FV^{-1})$ establishes the basic reproduction number. Consequently,

$$R_0 = \frac{\beta}{\delta + \gamma + 1}.$$

Remark: Figure 1 and Figure 2 illustrate the impact of m on the infectious population fraction for varying values of m. When $\tilde{\beta} = 0.075$ and $R_0 = 0.9375 < 1$, we found that the media coefficient m has a considerable impact on the fraction of infectious population (see Figure 1). Further, when $\tilde{\beta} = 0.1$ and $R_0 = 1.25 > 1$, the media coefficient m has a considerable impact on the fraction of infectious population (see Figure 2).

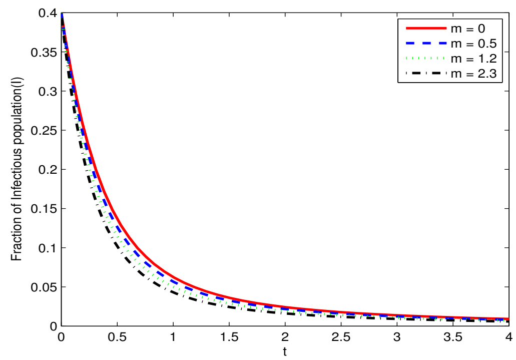


FIGURE 1. Effect of m on I for parameteric values $\tilde{\beta} = 0.075,\ \tilde{\theta} = 0.01,\ \tilde{\gamma} = 0.05,\ \tilde{\delta} = 0.01,\ \tilde{\mu} = 0.02,\ R_0 = 0.9375 < 1$.

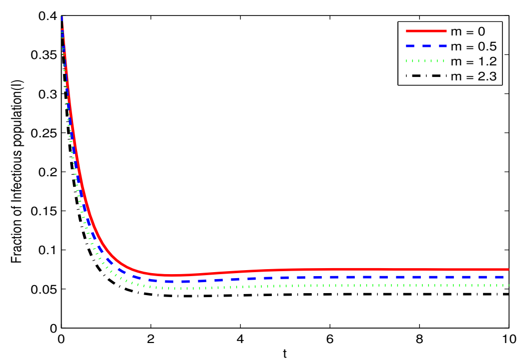


FIGURE 2. Effect of m on I for parameteric values $\tilde{\beta} = 0.1,\ \tilde{\theta} = 0.01,\ \tilde{\gamma} = 0.05,\ \tilde{\delta} = 0.01,\ \tilde{\mu} = 0.02,\ R_0 = 1.25 > 1$.

3. Discrete Dynamical Behavior of the System

The following discrete-time system is obtained by applying forward Euler's method to the system of equations (14)-(15):

$$S \rightarrow S + h\left[1 - \beta e^{-mI} SI + \theta(1 - S - I) - S + \delta SI\right], \tag{16}$$

$$I \rightarrow I + h\left[\beta e^{-mI} SI - \delta I - \gamma I - I + \delta I^2\right]. \tag{17}$$

where the step size is denoted by h. Euler's method will produce numerical solutions that are closer to the actual solution when the step size is reduced. With step size h and total number of steps M, the numerical solution to the initial-value problem derived from Euler's technique satisfies $0 < h \leq \frac{L}{M}$, where L is the interval length.

The fixed points of the system (16)-(17) are $A(1,0)$ and $B(S^*,I^*)$, where I^*,S^* satisfy

$$1-\beta e^{-mI}SI+\theta(1-S-I)-S+\delta SI = 0, \tag{18}$$

$$\beta e^{-mI}S-\delta-\gamma-1+\delta I = 0. \tag{19}$$

The jacobian matrix of system of equations (16)-(17) at the fixed point (S,I) is given by

$$J=\begin{bmatrix} 1+ha & hb \\ hc & 1+hd \end{bmatrix},$$

where

$$a=-\beta e^{-mI}I-\theta-1+\delta I,$$

$$b=-\beta e^{-mI}S+\beta m e^{-mI}SI-\theta+\delta S,$$

$$c=\beta e^{-mI}I,$$

$$d=\beta e^{-mI}S-\beta m e^{-mI}SI-\delta-\gamma-1+2\delta I.$$

The characteristic equation of the jacobian matrix at the fixed point (S,I) can be written as:

$$\lambda^2+p(S,I)\lambda+q(S,I)=0, \tag{20}$$

where

$$p(S,I) = -trJ=-2-h(a+d),$$

$$q(S,I)=detJ=1+h(a+d)+h^2(ad-bc).$$

Now, we state a Lemma as similar as in [24, 25, 26, 27]:

Lemma 1. *Let $F(\lambda)=\lambda^2+P\lambda+Q$. Suppose that $F(1)>0$; λ_1 and λ_2 are roots of $F(\lambda)=0$. Then, we have:*

(i) $|\lambda_1|<1$ *and* $|\lambda_2|<1$ *if and only if* $F(-1)>0$ *and* $Q<1$;

(ii) $|\lambda_1|<1$ *and* $|\lambda_2|>1$ *(or* $|\lambda_1|>1$ *and* $|\lambda_2|<1$*) if and only if* $F(-1)<0$;

(iii) $|\lambda_1|>1$ *and* $|\lambda_2|>1$ *if and only if* $F(-1)>0$ *and* $Q>1$;

(iv) $\lambda_1=-1$ *and* $|\lambda_2|\neq 1$ *if and only if* $F(-1)=0$ *and* $P\neq 0,\ 2$;

(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1|=|\lambda_2|=1$ *if and only if* $P^2-4Q<0$ *and* $Q=1$.

Let λ_1 and λ_2 be the roots of (20), which are known as eigen values of the fixed point (S,I). The fixed point (S,I) is a sink or locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The fixed point (S,I) is a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. The fixed point (S,I) is non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$. The fixed point (S,I) is a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

Proposition 2. *There exist different topological types of $A(1,0)$ for possible parameters.*

(i) $A(1,0)$ *is sink if* $1+\gamma+\delta > \beta$ *and* $0 < h < min\left\{\frac{2}{1+\theta}, \frac{2}{1+\gamma+\delta-\beta}\right\}$.

(ii) $A(1,0)$ *is source if* $1+\gamma+\delta > \beta$ *and* $h > max\left\{\frac{2}{1+\theta}, \frac{2}{1+\gamma+\delta-\beta}\right\}$.

(iii) $A(1,0)$ *is non-hyperbolic if* $h = \frac{2}{1+\theta}$ *or* $h = \frac{2}{1+\gamma+\delta-\beta}$ *and* $1+\gamma+\delta > \beta$.

(iv) $A(1,0)$ *is saddle for all values of the parameters, except for that values which lies in (i) to (iii).*

The term *(iii)* of proposition 2 implies that the parameters lie in the set

$$F_A = \{(\beta, m, \theta, \gamma, \delta, h),\ h = \frac{2}{1+\theta},\ h \neq \frac{2}{1+\gamma+\delta-\beta},\ 1+\gamma+\delta > \beta \text{ and } \beta, m, \theta, \gamma, \delta, h > 0\}.$$

If the term *(iii)* of proposition 2 holds, then one of the eigen values of the fixed point $A(1,0)$ is -1 and magnitude of the other is not equal to 1. The point $A(1,0)$ undergoes flip bifurcation when the parameter changes in limited neighborhood of F_A.

The characteristic equation of the jacobian matrix J of the system (16)-(17) at the fixed point $B(S^*,I^*)$ can be written as

$$\lambda^2 + p(S^*,I^*)\lambda + q(S^*,I^*) = 0, \tag{21}$$

where

$$p(S^*,I^*) = -2 - Gh,$$

$$q(S^*,I^*) = 1 + Gh + Hh^2,$$

$$G = -\beta e^{-mI^*} I^* - \theta - 1 + \delta I^* + \beta e^{-mI^*} S^* - \beta m e^{-mI^*} S^* I^* - \delta - \gamma - 1 + 2\delta I^*,$$

$$\begin{aligned} H = \ & \left[-\beta e^{-mI^*}I^* - \theta - 1 + \delta I^*\right] \\ & \times \left[\beta e^{-mI^*}S^* - \beta m e^{-mI^*}S^*I^* - \delta - \gamma - 1 + 2\delta I^*\right] \\ & -\beta e^{-mI^*}I^*[-\beta e^{-mI^*}S^* + \beta m e^{-mI^*}S^*I^* - \theta + \delta S^*]. \end{aligned}$$

Now

$$F(\lambda) = \lambda^2 - (2+Gh)\lambda + (1+Gh+Hh^2).$$

Therefore

$$F(1) = Hh^2,\ F(-1) = 4+2Gh+Hh^2.$$

Using Lemma 1, we get the following proposition:

Proposition 3. *There exist different topological types of $B(S^*,I^*)$ for all possible parameters.*

(i) $B(S^*,I^*)$ is sink if either condition (i.1) or (i.2) holds:

(i.1) $G^2-4H \geq 0$ and $0 < h < \frac{-G-\sqrt{G^2-4H}}{H}$,

(i.2) $G^2-4H < 0$ and $0 < h < -\frac{G}{H}$.

(ii) $B(S^*,I^*)$ is source if either condition (ii.1) or (ii.2) holds:

(ii.1) $G^2-4H \geq 0$ and $h > \frac{-G+\sqrt{G^2-4H}}{H}$,

(ii.2) $G^2-4H < 0$ and $h > -\frac{G}{H}$.

(iii) $B(S^*,I^*)$ is non-hyperbolic if either condition (iii.1) or (iii.2) holds:

(iii.1) $G^2-4H \geq 0$ and $h = \frac{-G\pm\sqrt{G^2-4H}}{H}$,

(iii.2) $G^2-4H < 0$ and $h = -\frac{G}{H}$.

(iv) $B(S^*,I^*)$ is saddle for all values of the parameters, except for that values which lies in (i) to (iii).

If the term *(iii.1)* of proposition 3 holds, then one of the eigen values of the fixed point $B(S^*,I^*)$ is -1 and magnitude of the other is not equal to 1. The term *(iii.1)* of proposition 3 may be written as follows:

$$\begin{aligned} F_{B1} = \ & \{(\beta,m,\theta,\gamma,\delta,h):\ h = \tfrac{-G-\sqrt{G^2-4H}}{H},\ G^2-4H \geq 0 \text{ and} \\ & \beta,m,\theta,\gamma,\delta,h > 0\}, \\ F_{B2} = \ & \{(\beta,m,\theta,\gamma,\delta,h):\ h = \tfrac{-G+\sqrt{G^2-4H}}{H},\ G^2-4H \geq 0 \text{ and} \\ & \beta,m,\theta,\gamma,\delta,h > 0\}. \end{aligned}$$

If the term *(iii.2)* of proposition 3 holds, then the eigen values of the fixed point $B(S^*, I^*)$ are a pair of conjugate complex numbers with modulus 1. The term *(iii.2)* of proposition 3 may be written as follows:

$$H_B = \{(\beta, m, \theta, \gamma, \delta, h) : h = -\frac{G}{H}, G^2 - 4H < 0 \text{ and } \beta, m, \theta, \gamma, \delta, h > 0\},$$

where

$$\beta = \frac{\tilde{\beta}}{\tilde{\mu}}, \theta = \frac{\tilde{\theta}}{\tilde{\mu}}, \gamma = \frac{\tilde{\gamma}}{\tilde{\mu}}, \delta = \frac{\tilde{\delta}}{\tilde{\mu}}.$$

4. Bifurcation Behavior

In this section, the flip bifurcation and Hopf bifurcation of the system (16)-(17) is studied at the fixed point $B(S^*, I^*)$.

4.1. Flip Bifurcation.

Consider the system (16)-(17) with arbitrary parameter $(\tilde{\beta}, m, \tilde{\theta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\mu}, h_1) \in F_{B1}$, which is described as follows:

$$S \rightarrow S + h_1 \left[1 - \beta e^{-mI} SI + \theta(1 - S - I) - S + \delta SI\right], \tag{22}$$

$$I \rightarrow I + h_1 \left[\beta e^{-mI} SI - \delta I - \gamma I - I + \delta I^2\right]. \tag{23}$$

System of equations (22)-(23) has fixed point B(S^*, I^*), where S^*, I^* satisfy (18)-(19) and

$$h_1 = \frac{-G - \sqrt{G^2 - 4H}}{H}.$$

The eigen values of $B(S^*, I^*)$ are $\lambda_1 = -1$, $\lambda_2 = 3 + Gh_1$ with $|\lambda_2| \neq 1$ by proposition 3.

Consider the perturbation of (22)-(23) as below:

$$S \rightarrow S + (h_1 + h^*) \left[1 - \beta e^{-mI} SI + \theta(1 - S - I) - S + \delta SI\right], \tag{24}$$

$$I \rightarrow I + (h_1 + h^*) \left[\beta e^{-mI} SI - \delta I - \gamma I - I + \delta I^2\right], \tag{25}$$

where $|h^*| \ll 1$ is a limited perturbation parameter.

Let $u = S - S^*$ and $v = I - I^*$.

After transformation of the fixed point $B(S^*, I^*)$ of map (24)-(25) to the point $(0,0)$, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u + a_{12}v + a_{13}uv + a_{14}v^2 + b_{11}h^*u + b_{12}h^*v \\ +b_{13}h^*uv + b_{14}h^*v^2 + O(|u|, |v|, |h^*|)^3 \\ a_{21}u + a_{22}v + a_{23}uv + a_{24}v^2 + b_{21}h^*u + b_{22}h^*v \\ +b_{23}h^*uv + b_{24}h^*v^2 + O(|u|, |v|, |h^*|)^3 \end{pmatrix}, \tag{26}$$

where

$$a_{11} = 1 - \frac{h_1}{S^*}(1 + \theta - \theta I^*),$$

$$a_{12} = h_1\left[m(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*) - \frac{1}{I^*}(1 + \theta - \theta S^* - S^*)\right],$$

$$a_{13} = h_1\left[\frac{m}{S^*}(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*) - \frac{1}{S^* I^*}(1 + \theta - \theta S^* - \theta I^* - S^*)\right],$$

$$a_{14} = h_1\left(\frac{m}{I^*} - \frac{m^2}{2}\right)(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*),$$

$$b_{11} = -\frac{1}{S^*}(1 + \theta - \theta I^*),$$

$$b_{12} = \left[m(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*) - \frac{1}{I^*}(1 + \theta - \theta S^* - S^*)\right],$$

$$b_{13} = \left[\frac{m}{S^*}(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*) - \frac{1}{S^* I^*}(1 + \theta - \theta S^* - \theta I^* - S^*)\right],$$

$$b_{14} = \left(\frac{m}{I^*} - \frac{m^2}{2}\right)(1 + \theta - \theta S^* - \theta I^* - S^* + \delta S^* I^*),$$

$$a_{21} = \frac{h_1}{S^*}\left(I^* + \gamma I^* + \delta I^* - \delta I^{*2}\right), \tag{27}$$

$$a_{22} = 1 + h_1\delta I^* - mh_1(I^* + \gamma I^* + \delta I^* - \delta I^{*2}),$$

$$a_{23} = \frac{h_1}{S^*}\left(\frac{1}{I^*} - m\right)(I^* + \gamma I^* + \delta I^* - \delta I^{*2}),$$

$$a_{24} = h_1\delta + h_1 m\left(\frac{m}{2} - \frac{1}{I^*}\right)(I^* + \gamma I^* + \delta I^* - \delta I^{*2}),$$

$$b_{21} = \frac{1}{S^*}\left(I^* + \gamma I^* + \delta I^* - \delta I^{*2}\right),$$

$$b_{22} = \delta I^* - m(I^* + \gamma I^* + \delta I^* - \delta I^{*2}),$$

$$b_{23} = \frac{1}{S^*}\left(\frac{1}{I^*} - m\right)(I^* + \gamma I^* + \delta I^* - \delta I^{*2}),$$

$$b_{24} = \delta + m\left(\frac{m}{2} - \frac{1}{I^*}\right)(I^* + \gamma I^* + \delta I^* - \delta I^{*2}).$$

Consider the following translation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix},$$

where

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}.$$

Taking T^{-1} on both sides of equation (26), we get

$$\begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix} + \begin{pmatrix} f(u,v,h^*) \\ g(u,v,h^*) \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} f(u,v,h^*) = \; & \tfrac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]uv}{a_{12}(\lambda_2+1)} + \tfrac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}]v^2}{a_{12}(\lambda_2+1)} \\ & + \tfrac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]h^*u}{a_{12}(\lambda_2+1)} + \tfrac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}]h^*v}{a_{12}(\lambda_2+1)} \\ & + \tfrac{[b_{13}(\lambda_2 - a_{11}) - a_{12}b_{23}]h^*uv}{a_{12}(\lambda_2+1)} + \tfrac{[b_{14}(\lambda_2 - a_{11}) - a_{12}b_{24}]h^*v^2}{a_{12}(\lambda_2+1)} \\ & + O(|u|,|v|,|h^*|)^3, \end{aligned}$$

$$\begin{aligned} g(u,v,\delta^*) = \; & \tfrac{[a_{13}(1 + a_{11}) + a_{12}a_{23}]uv}{a_{12}(\lambda_2+1)} + \tfrac{[a_{14}(1 + a_{11}) + a_{12}a_{24}]v^2}{a_{12}(\lambda_2+1)} \\ & + \tfrac{[b_{11}(1 + a_{11}) + a_{12}b_{21}]h^*u}{a_{12}(\lambda_2+1)} + \tfrac{[b_{12}(1 + a_{11}) + a_{12}b_{22}]h^*v}{a_{12}(\lambda_2+1)} \\ & + \tfrac{[b_{13}(1 + a_{11}) + a_{12}b_{23}]h^*uv}{a_{12}(\lambda_2+1)} + \tfrac{[b_{14}(1 + a_{11}) + a_{12}b_{24}]h^*v^2}{a_{12}(\lambda_2+1)} \\ & + O(|u|,|v|,|h^*|)^3, \end{aligned}$$

$$u = a_{12}(\bar{S} + \bar{I}),$$

$$v = -(1 + a_{11})\bar{S} + (\lambda_2 - a_{11})\bar{I}.$$

Applying center manifold theorem to equation (28) at the origin in the limited neighborhood of $h^* = 0$. The center manifold $W^c(0,0)$ can be approximately presented as:

$$W^c(0,0) = \left\{ (\bar{S},\bar{I}) : \bar{I} = a_0 h^* + a_1 \bar{S}^2 + a_2 \bar{S} h^* + a_3 h^{*2} + O\left((|\bar{S}| + |h^*|)^3 \right) \right\},$$

where $O\left((|\bar{S}| + |h^*|)^3 \right)$ is a function with at least third order in variables $(\bar{S},\ h^*)$.

By simple calculations for center manifold, we have

$$a_0 = 0,$$

$$a_1 = \frac{[a_{13}(1+a_{11})+a_{12}a_{23}]a_{12}(1+a_{11})-[a_{14}(1+a_{11})+a_{12}a_{24}](1+a_{11})^2}{a_{12}(\lambda_2^2-1)},$$
$$a_2 = \frac{-[b_{11}(1+a_{11})+a_{12}b_{21}]a_{12}+[b_{12}(1+a_{11})+a_{12}b_{22}](1+a_{11})}{a_{12}(\lambda_2+1)^2},$$
$$a_3 = 0.$$

Now, consider the map restricted to the center manifold $W^c(0,0)$ as below:

$$K : \bar{S} \to -\bar{S} + k_1\bar{S}^2 + k_2\bar{S}h^* + k_3\bar{S}^2h^* + k_4\bar{S}h^{*2} + k_5\bar{S}^3 + O\left((|\bar{S}|+|h^*|)^4\right),$$

where

$$k_1 = -\frac{[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}](1+a_{11})}{(\lambda_2+1)} + \frac{[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}](1+a_{11})^2}{a_{12}(\lambda_2+1)},$$

$$k_2 = \frac{[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]}{(\lambda_2+1)} - \frac{[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}](1+a_{11})}{a_{12}(\lambda_2+1)},$$

$$\begin{aligned} k_3 = {} & \frac{[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}](\lambda_2-1-2a_{11})a_2}{(\lambda_2+1)} + \frac{[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}][-2(1+a_{11})(\lambda-a_{11})a_2]}{a_{12}(\lambda_2+1)} \\ & + \frac{[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]a_1}{(\lambda_2+1)} + \frac{[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}](\lambda_2-a_{11})a_1}{a_{12}(\lambda_2+1)} \\ & - \frac{[b_{13}(\lambda_2-a_{11})-a_{12}b_{23}](1+a_{11})}{(\lambda_2+1)} + \frac{[b_{14}(\lambda_2-a_{11})-a_{12}b_{24}](1+a_{11})^2}{a_{12}(\lambda_2+1)}, \end{aligned}$$

$$k_4 = \frac{[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]a_2}{(\lambda_2+1)} + \frac{[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}](\lambda_2-a_{11})a_2}{a_{12}(\lambda_2+1)},$$

$$k_5 = \frac{[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}](\lambda_2-1-2a_{11})a_1}{(\lambda_2+1)} - \frac{[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}](1+a_{11})(\lambda_2-a_{11})2a_1}{a_{12}(\lambda_2+1)}.$$

According to Flip bifurcation, the discriminatory quantities γ_1 and γ_2 are given by:

$$\gamma_1 = \left(\frac{\partial^2 K}{\partial\bar{S}\partial h^*} + \frac{1}{2}\frac{\partial K}{\partial h^*}\frac{\partial^2 K}{\partial\bar{S}^2}\right)\Bigg|_{(0,0)},$$

$$\gamma_2 = \left(\frac{1}{6}\frac{\partial^3 K}{\partial\bar{S}^3} + \left(\frac{1}{2}\frac{\partial^2 K}{\partial\bar{S}^2}\right)^2\right)\Bigg|_{(0,0)}.$$

After simple calculations, we obtain $\gamma_1 = k_2$ and $\gamma_2 = k_5 + k_1^2$.

Analyzing above and the flip bifurcation conditions discussed in [28], we write the following theorem:

Theorem 4. *If $\gamma_2 \neq 0$, and the parameter h^* alters in the limiting region of the point (0,0), then the system (24)-(25) passes through flip bifurcation at the point $B(S^*, I^*)$. Also, the period-2 points that bifurcate from fixed point $B(S^*, I^*)$ are stable (resp., unstable) if $\gamma_2 > 0$ (resp., $\gamma_2 < 0$).*

4.2. Hopf Bifurcation. Consider the system (16)-(17) with arbitrary parameter $(\tilde{\beta}, m, \tilde{\theta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\mu}, h_2) \in H_B$, which is described as follows:

$$S \rightarrow S + h_2 \left[1 - \beta e^{-mI} SI + \theta(1 - S - I) - S + \delta SI\right], \tag{29}$$

$$I \rightarrow I + h_2 \left[\beta e^{-mI} SI - \delta I - \gamma I - I + \delta I^2\right]. \tag{30}$$

Equation (29)-(30) has fixed point $\mathrm{B}(S^*, I^*)$, where S^*, I^* is given by (18)-(19) and

$$h_2 = -\frac{G}{H}.$$

Consider the perturbation of (29)-(30) as follows:

$$S \rightarrow S + (h_2 + h) \left[1 - \beta e^{-mI} SI + \theta(1 - S - I) - S + \delta SI\right], \tag{31}$$

$$I \rightarrow I + (h_2 + h) \left[\beta e^{-mI} SI - \delta I - \gamma I - I + \delta I^2\right]. \tag{32}$$

where $|h| \ll 1$ is small perturbation parameter.

The characteristic equation of map (31)-(32) at $\mathrm{B}(S^*, I^*)$ is given by

$$\lambda^2 + p(h)\lambda + q(h) = 0,$$

where

$$p(h) = -2 - G(h_2 + h),$$

$$q(h) = 1 + G(h_2 + h) + H(h_2 + h)^2.$$

Since the parameter $(\tilde{\beta}, m, \tilde{\theta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\mu}, h_2) \in H_B$, the eigen values of $\mathrm{B}(S^*, I^*)$ are a pair of complex conjugate numbers $\overline{\lambda}$ and λ with modulus 1 by proposition 3, where

$$\overline{\lambda},\ \lambda = \frac{-p(h) \mp i\sqrt{4q(h) - p^2(h)}}{2}.$$

Therefore

$$\overline{\lambda},\ \lambda = 1 + \frac{G(h_2 + h)}{2} \mp \frac{i(h_2 + h)\sqrt{4H - G^2}}{2}.$$

Now we have

$$|\lambda| = (q(h))^{1/2},\ l = \left.\frac{d|\lambda|}{dh}\right|_{h=0} = -\frac{G}{2} > 0.$$

When h varies in small neighborhood of $h = 0$, then $\overline{\lambda}$, $\lambda = a \mp ib$, where

$$a = 1 + \frac{h_2 G}{2},\ b = \frac{h_2\sqrt{4H - G^2}}{2}.$$

Hopf bifurcation requires that when $h=0$, then $\overline{\lambda}^n, \lambda^n \neq 1$ $(n=1,\ 2,\ 3,\ 4)$ which is equivalent to $p(0) \neq -2,\ 0,\ 1,\ 2$.

Since the parameter $(\tilde{\beta}, m, \tilde{\theta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\mu}, h_2) \in H_B$, therefore $p(0) \neq -2,\ 2$. It is the only requirement that $p(0) \neq 0,\ 1$, which follows that

$$G^2 \neq 2H,\ 3H. \tag{33}$$

Let $u = \ S - S^*$ and $v = I - I^*$.

After transformation of the fixed point $B(S^*, I^*)$ of system (31)-(32) to the point $(0,0)$, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u + a_{12}v + a_{13}uv + a_{14}v^2 + O(|u|,|v|)^3 \\ a_{21}u + a_{22}v + a_{23}uv + a_{24}v^2 + O(|u|,|v|)^3 \end{pmatrix}, \tag{34}$$

where $a_{11},\ a_{12},\ a_{13},\ a_{14},\ a_{21},\ a_{22},\ a_{23},\ a_{24}$ are given in (27) by substituting h_2 for h_2+h.

Next, the normal form of (34) is discussed when $h=0$.

Consider the following translation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix},$$

where

$$T = \begin{pmatrix} a_{12} & 0 \\ a - a_{11} & -b \end{pmatrix}.$$

Taking T^{-1} on both sides of (34), we get

$$\begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \bar{S} \\ \bar{I} \end{pmatrix} + \begin{pmatrix} \bar{f}(\bar{S},\bar{I}) \\ \bar{g}(\bar{S},\bar{I}) \end{pmatrix},$$

where

$$\bar{f}\left(\bar{S},\bar{I}\right) = \frac{a_{13}}{a_{12}}uv + \frac{a_{14}}{a_{12}}v^2 + O(|u|,|v|)^3,$$

$$\bar{g}\left(\bar{S},\bar{I}\right) = \frac{[a_{13}(a-a_{11}) - a_{12}a_{23}]}{a_{12}b}uv + \frac{[a_{14}(a-a_{11}) - a_{12}a_{24}]}{a_{12}b}v^2 + O(|u|,|v|)^3,$$

$$u = a_{12}\bar{S},$$

and

$$v = (a - a_{11})\bar{S} - b\bar{I}.$$

Now

$$\bar{f}_{\bar{S}\bar{S}} = 2a_{13}(a-a_{11}) + \frac{2a_{14}}{a_{12}}(a-a_{11})^2, \ \ \bar{f}_{\bar{S}\bar{I}} = -a_{13}b - \frac{2a_{14}}{a_{12}}(a-a_{11})b, \ \ \bar{f}_{\bar{I}\bar{I}} = \frac{2a_{14}}{a_{12}}b^2,$$

$$\bar{f}_{\bar{S}\bar{S}\bar{S}} = 0, \ \ \bar{f}_{\bar{S}\bar{S}\bar{I}} = 0, \ \ \bar{f}_{\bar{S}\bar{I}\bar{I}} = 0, \ \ \bar{f}_{\bar{I}\bar{I}\bar{I}} = 0,$$

$$\bar{g}_{\bar{S}\bar{S}} = \frac{2(a-a_{11})}{b}[a_{13}(a-a_{11}) - a_{12}a_{23}] + \frac{2(a-a_{11})^2}{a_{12}b}[a_{14}(a-a_{11}) - a_{12}a_{24}],$$

$$\bar{g}_{\bar{S}\bar{I}} = -[a_{13}(a-a_{11}) - a_{12}a_{23}] - \frac{2(a-a_{11})}{a_{12}}[a_{14}(a-a_{11}) - a_{12}a_{24}],$$

$$\bar{g}_{\bar{I}\bar{I}} = \frac{2b}{a_{12}}[a_{14}(a-a_{11}) - a_{12}a_{24}],$$

$$\bar{g}_{\bar{S}\bar{S}\bar{S}} = 0, \ \ \bar{g}_{\bar{S}\bar{S}\bar{I}} = 0, \ \ \bar{g}_{\bar{S}\bar{I}\bar{I}} = 0, \ \ \bar{g}_{\bar{I}\bar{I}\bar{I}} = 0.$$

According to Hopf bifurcation, the discriminatory quantity s is given by

$$s = -Re\left[\frac{(1-2\overline{\lambda})\overline{\lambda}^2}{1-\lambda}\varphi_{11}\varphi_{20}\right] - \frac{1}{2}\|\varphi_{11}\|^2 - \|\varphi_{02}\|^2 + Re(\overline{\lambda}\varphi_{21}), \tag{35}$$

where

$$\varphi_{20} = \frac{1}{8}\left[(\bar{f}_{\bar{S}\bar{S}} - \bar{f}_{\bar{I}\bar{I}} + 2\bar{g}_{\bar{S}\bar{I}}) + i\left(\bar{g}_{\bar{S}\bar{S}} - \bar{g}_{\bar{I}\bar{I}} - 2\bar{f}_{\bar{S}\bar{I}}\right)\right],$$

$$\varphi_{11} = \frac{1}{4}\left[(\bar{f}_{\bar{S}\bar{S}} + \bar{f}_{\bar{I}\bar{I}}) + i(\bar{g}_{\bar{S}\bar{S}} + \bar{g}_{\bar{I}\bar{I}})\right],$$

$$\varphi_{02} = \frac{1}{8}\left[(\bar{f}_{\bar{S}\bar{S}} - \bar{f}_{\bar{I}\bar{I}} - 2\bar{g}_{\bar{S}\bar{I}}) + i\left(\bar{g}_{\bar{S}\bar{S}} - \bar{g}_{\bar{I}\bar{I}} + 2\bar{f}_{\bar{S}\bar{I}}\right)\right],$$

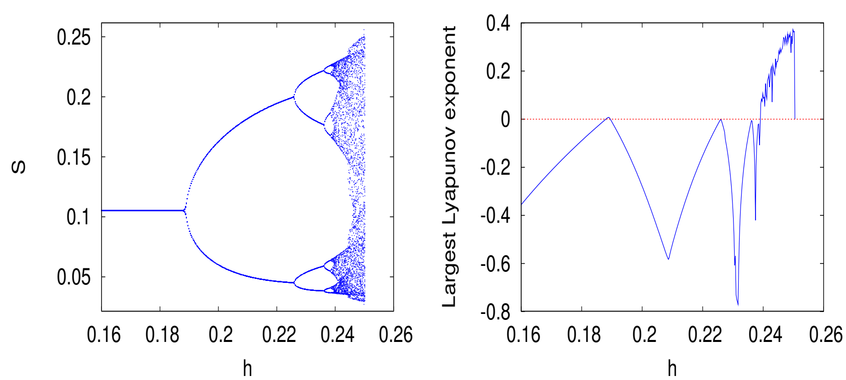
$$\varphi_{21} = \frac{1}{16}\left[(\bar{f}_{\bar{S}\bar{S}\bar{S}} + \bar{f}_{\bar{S}\bar{I}\bar{I}} + \bar{g}_{\bar{S}\bar{S}\bar{I}} + \bar{g}_{\bar{I}\bar{I}\bar{I}}) + i\left(\bar{g}_{\bar{S}\bar{S}\bar{S}} + \bar{g}_{\bar{S}\bar{I}\bar{I}} - \bar{f}_{\bar{S}\bar{S}\bar{I}} - \bar{f}_{\bar{I}\bar{I}\bar{I}}\right)\right].$$

Based on the analysis of the above conditions and the Hopf bifurcation, the following theorem is presented [28]:

Theorem 5. *If the condition (33) holds, $s \neq 0$ and the parameter h alters in the limited region of the point (0,0), then the system (31)-(32) passes through Hopf bifurcation at the point $B(S^*, I^*)$. Moreover, if $s < 0$ (resp., $s > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point $B(S^*, I^*)$ for $h > 0$ (resp., $h < 0$).*

TABLE 2. Parametric values used for the numerical simulation

Parameter	value	Reference
Λ	5.0	[15, 16]
$\tilde{\delta}$	0.01	[15, 16]
$\tilde{\mu}$	0.02	[15, 16]
$\tilde{\beta}$	[0.05,1.5]	variable
m	[0,2.3]	variable
$\tilde{\gamma}$	0.05	[15, 16]
$\tilde{\theta}$	0.01	[15, 16]



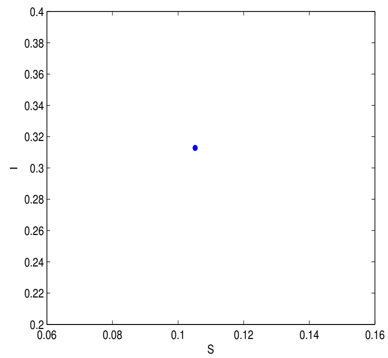
(a) (b)

FIGURE 3. (a) Bifurcation diagram of system (16)-(17) for $\tilde{\beta} = 1.5$, $m = 2.3$, $\tilde{\theta} = 0.01$, $\tilde{\gamma} = 0.05$, $\tilde{\delta} = 0.01$, $\tilde{\mu} = 0.02$ with the initial value of $(S,I) = (0.1, 0.3)$ and h covering [0.16, 0.26]. (b) Largest Lyapunov exponents related to (a).

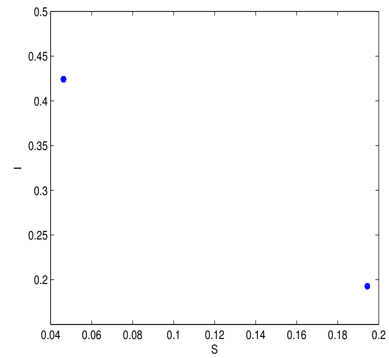
5. NUMERICAL SIMULATIONS

The system (16)-(17) is numerically simulated in order to support analytical conclusions. $m = 2.3$, $\tilde{\theta} = 0.01$, $\tilde{\gamma} = 0.05$, $\tilde{\delta} = 0.01$, $\tilde{\mu} = 0.02$ are fixed parameters and taking only $\tilde{\beta}$ as variable. The following situations are methodically discussed:

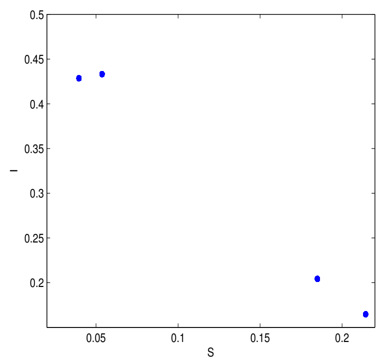
Case 1: For the model (16)-(17), the bifurcation diagram is created using $\tilde{\beta} = 1.5$. Flip bifurcation is visible at $h = 0.188987$ from the fixed point $(0.10522, 0.31278)$. According to Figure 3(a), the fixed point of the system (16)-(17) is stable for $h < 0.188987$, becomes



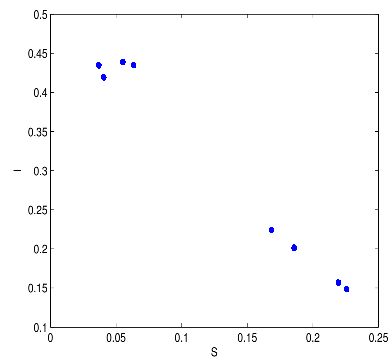
(a) $h = 0.185$



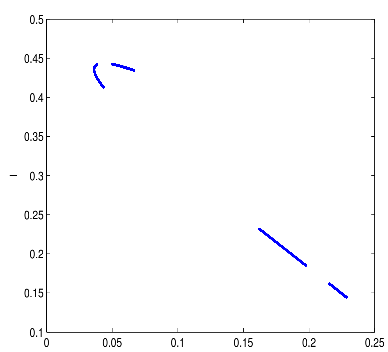
(b) $h = 0.22$



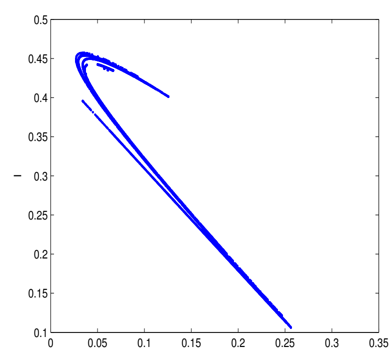
(c) $h = 0.23$



(d) $h = 0.238$



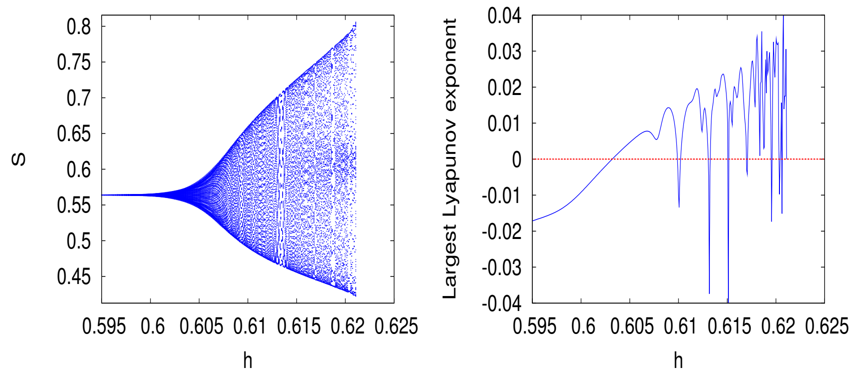
(e) $h = 0.24$



(f) $h = 0.25$

FIGURE 4. Phase portraits for several values of h from 0.185 to 0.25 related to Fig. 3 (a).

unstable at $h = 0.188987$, and exhibits period doubling bifurcation for $h > 0.188987$. The fact that $\gamma_1 = -12.6611$ and $\gamma_2 = 7.14924$ in this instance indicates that the Theorem 4 is



(a) (b)

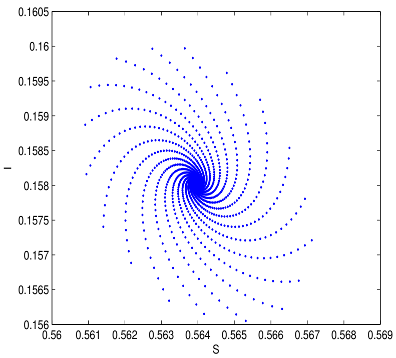
FIGURE 5. (a) Bifurcation diagram of system (16)-(17) for $\tilde{\beta} = 0.2$, $m = 2.3$, $\tilde{\theta} = 0.01$, $\tilde{\gamma} = 0.05$, $\tilde{\delta} = 0.01$, $\tilde{\mu} = 0.02$ with the initial value of $(S, I) = (0.5, 0.2)$ and h covering [0.595, 0.625]. (b) Largest Lyapunov exponents related to (a).

valid. Additionally, the phase pictures reveal chaotic sets at $h = 0.24$ and 0.25, as well as orbits of periods 2, 4, and 8 at $h = 0.22$, $h = 0.23$, and $h = 0.238$, respectively (see Figure 4). Additionally, the chaotic sets are confirmed by the positive Largest Lypunov exponents for $h = 0.24$ and 0.25 (see Figure 3(b)).

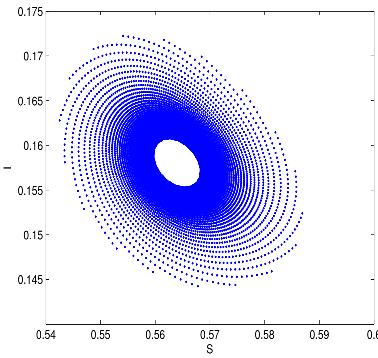
Case 2: Using $\tilde{\beta} = 0.2$, the bifurcation diagram for the model (16)-(17) is created. From the fixed point $(0.5639,\ 0.15802)$, it can be observed that the Hopf bifurcation occurs at $h = 0.60654$. The fixed point of the system (16)-(17) is stable for $h < 0.60654$, loses stability at $h = 0.60654$, and an invariant circle appears for $h > 0.60654$, according to Figure 5(a). In this case, $s = -5.9564$ demonstrates the validity of the Theorem 5. Furthermore, a smooth invariant circle bifurcates from the fixed point, and its radius grows as h increases, as seen by the phase portraits in Figure 6.

6. CONCLUSIONS

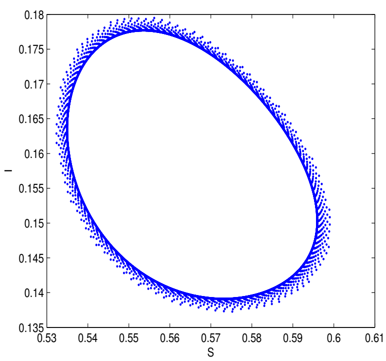
This paper presents an epidemic model that integrates media awareness, highlighting its importance in controlling disease dynamics. Notably, the coefficient of media awareness, m, does not influence the basic reproductive number R_0, ensuring that the core qualitative features of the model remain intact. However, the impact of the media coefficient on the fraction of the infectious population is significant and cannot be overlooked.



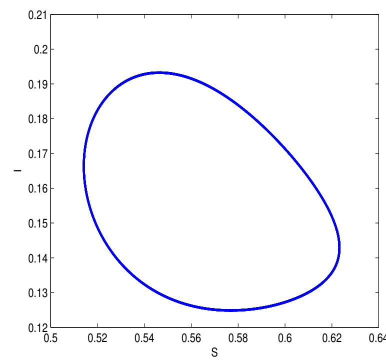
(a) $h = 0.605$



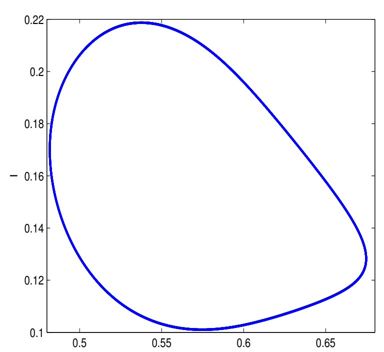
(b) $h = 0.6065$



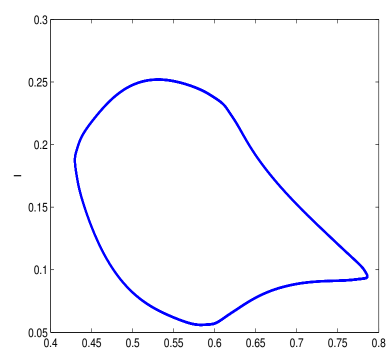
(c) $h = 0.607$



(d) $h = 0.608$



(e) $h = 0.611$



(f) $h = 0.62$

FIGURE 6. Phase portraits for several values of h from 0.605 to 0.62 related to Figure 5 (a).

The existence of bifurcation in a discrete-time system within the closed first quadrant R_+^2 is investigated. The findings reveal that the map exhibits both flip bifurcation and Hopf bifurcation at the fixed points under specific conditions, particularly when the step size h varies near F_{B1} or F_{B2} and H_B.

The numerical simulations underscore the system's complexity, showcasing a cascade of period-doubling bifurcations in orbits of periods 2, 4, and 8, alongside chaotic behavior in flip bifurcation scenarios. Moreover, smooth invariant circles are identified during the Hopf bifurcation. These results powerfully demonstrate that the infected population can coexist with the susceptible population, whether in periodic n orbits or along smooth invariant circles. Ultimately, this discrete model reveals a rich and intricate dynamical behavior that enhances our understanding of epidemic processes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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