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MAPS AND FUZZY CONNECTIONS

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Abstract. In this paper, we investigate the relations between maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices.

Keywords: complete residuated lattices; isotone (antitone) maps; residuated (dual residuated, residuated, Galois, dual Galois) connections

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1. Introduction

Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [1-3] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. Georgescue and Popescu [5,6] introduced the non-commutative fuzzy connection on generalized residuated lattice without commutative conditions. Garcia [4] investigated fuzzy connections categorically. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-3,8-10].

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In this paper, we investigate the relations between maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices. We give their examples.

Definition 1.1. [1,7] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow,^* 0, 1)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$.

Lemma 1.2.[1,7] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(3) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(6) $x \odot y = (x \rightarrow y^*)^*$ and $x \rightarrow y = y^* \rightarrow x^*$.

(7) $x \odot (x \rightarrow y) \leq y$.

(8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

(9) $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$.

Definition 1.3.[1-3] Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Remark 1.4.(1) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 1.2 (8).

(2) We denote $e_X^{-1}(x,y) = e_X(y,x)$, $(e_X)_x(y) = e_X(x,y)$ and $(e_X)_y^{-1} = e_X(x,y)$. Moreover, 1_x is a characteristic function such that $1_x(x) = 0$, $1_x(y)$, for otherwise.

Definition 1.5.[1-3] Let (X, e_X) and (Y, e_Y) be a fuzzy poset and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is called a Galois connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(x, g(y)).$$

(2)(e_X, f, g, e_Y) is called a dual Galois connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(g(y), x).$$

(3) (e_X, f, g, e_Y) is called a residuated connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(x, g(y)).$$

(4) (e_X, f, g, e_Y) is called a dual residuated connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(g(y), x).$$

(5) A map $f : (X, e_X) \to (Y, e_Y)$ is called an isotone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(x), f(z))$.

(6) A map $f : (X, e_X) \to (Y, e_Y)$ is called an antitone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(z), f(x))$.

2. Maps and fuzzy connections

Theorem 2.1. *Let (X, e_X) and (Y, e_Y) be a fuzzy poset and $f : X \to Y$ and $g : Y \to X$ maps. For each $A \in L^X$ and $B \in L^Y$, we define operations as follows:*

$$F_1(A)(y) = \bigwedge_{x \in X} (A(x) \to e_Y(y, f(x))), \quad F_2(A)(y) = \bigwedge_{x \in X} (A(x) \to e_Y(f(x), y)),$$

$$G_1(B)(x) = \bigwedge_{y \in Y} (B(y) \to e_X(x, g(y))), \quad G_2(B)(x) = \bigwedge_{y \in Y} (B(y) \to e_X(g(y), x)),$$

$$H_1(B)(x) = \bigvee_{y \in Y} (e_X(x, g(y)) \odot B(y)), \quad H_2(B)(x) = \bigvee_{y \in Y} (e_X(g(y), x) \odot B(y)),$$

$$I_1(A)(y) = \bigvee_{x\in X}(A(x)\odot e_Y(y,f(x))),\ \ I_2(A)(y) = \bigvee_{x\in X}(A(x)\odot e_Y(f(x),y)),$$

$$J_1(B)(x) = \bigwedge_{y\in Y}(e_X(x,g(y))\to B(y)),\ \ J_2(B)(x) = \bigwedge_{y\in Y}(e_X(g(y),x)\to B(y)),$$

$$K_1(A)(y) = \bigwedge_{x\in X}(e_Y(y,f(x))\to A(x)),\ \ K_2(A)(y) = \bigwedge_{x\in X}(e_Y(f(x),y)\to A(x)).$$

$$L_1(B)(x) = \bigvee_{y\in Y}(B^*(y)\odot e_X(x,g(y))),\ \ L_2(B)(x) = \bigvee_{y\in Y}(B^*(y)\odot e_X(g(y),x)),$$

$$M_1(A)(y) = \bigvee_{x\in X}(A^*(x)\odot e_Y(y,f(x))), M_2(A)(y) = \bigvee_{x\in X}(A^*(x)\odot e_Y(f(x),y)).$$

Then the following statements hold:

(1) $F_1(1_x) = (e_Y)^{-1}_{f(x)}$, $F_2(1_x) = (e_Y)_{f(x)}$, $K_1(1^*_x) = ((e_Y)^{-1}_{f(x)})^*$, $K_2(1^*_x) = (e_Y)^*_{f(x)}$, $M_1(1^*_x) = (e_Y)^{-1}_{f(x)}$, $M_2(1^*_x) = (e_Y)_{f(x)}$, $I_1(1_x) = (e_Y)^{-1}_{f(x)}$ *and* $I_2(1_x) = (e_Y)_{f(x)}$.

(2) $G_1(1_y) = (e_X)^{-1}_{g(y)}$, $G_2(1_y) = (e_X)_{g(y)}$, $H_1(1_w) = (e_X)^{-1}_{g(w)}$, $H_2(1_w) = (e_X)_{g(w)}$, $J_1(1^*_y) = ((e_X)^{-1}_{g(y)})^*$, $J_2(1^*_y) = (e_X)^*_{g(y)}$, $L_1(1^*_y) = (e_X)^{-1}_{g(y)}$ *and* $L_2(1^*_y) = (e_X)_{g(y)}$.

(3)(e_X, f, g, e_Y) *is a Galois connection iff* $(e_{L^X}, F_1, G_1, e_{L^Y})$ *is a Galois connection with antitone maps* f *and* g *iff* $(e_{L^X}, K_1, H_1, e_{L^Y})$ *is a dual residuated connection with antitone maps* f *and* g *iff* $(e_{L^X}, M_1, L_1, e_{L^Y})$ *is a dual Galois connection with antitone maps* f *and* g *iff* $(e_{L^X}, I_1, J_1, e_{L^Y})$ *is a residuated connection with antitone maps* f *and* g *.*

(4) (e_X, f, g, e_Y) *is a residuated connection iff* $(e_{L^X}, F_2, G_1, e_{L^Y})$ *is a Galois connection with isotone maps* f *and* g *iff* $(e_{L^X}, K_2, H_1, e_{L^Y})$ *is a dual residuated connection with isotone maps* f *and* g *iff* $(e_{L^X}, M_2, L_1, e_{L^Y})$ *is a dual Galois connection with isotone maps* f *and* g *iff* $(e_{L^X}, I_2, J_1, e_{L^Y})$ *is a residuated connection with isotone maps* f *and* g*.*

(5) (e_X, f, g, e_Y) *is a dual Galois connection iff* $(e_{L^X}, F_2, G_2, e_{L^Y})$ *is a Galois connection with antitone maps* f *and* g *iff* $(e_{L^X}, K_2, H_2, e_{L^Y})$ *is a dual residuated connection with antitone maps* f *and* g *iff* $(e_{L^X}, M_2, L_2, e_{L^Y})$ *is a dual Galois connection with antitone maps* f *and* g *iff* $(e_{L^X}, I_2, J_2, e_{L^Y})$ *is a residuated connection with antitone maps* f *and* g*.*

(6) (e_X, f, g, e_Y) *is a dual residuated connection iff* $(e_{L^X}, F_1, G_2, e_{L^Y})$ *is a Galois connection with isotone maps* f *and* g *iff* $(e_{L^X}, K_1, H_2, e_{L^Y})$ *is a dual residuated connection with isotone maps* f *and* g *iff* $(e_{L^X}, M_1, L_2, e_{L^Y})$ *is a dual Galois connection with isotone maps* f *and* g *iff* $(e_{L^X}, I_1, J_2, e_{L^Y})$ *is a residuated connection with isotone maps* f *and* g*.*

(7) *If* $e_X(x,y) \le e_Y(f(x),f(y))$, *then*

$$\begin{array}{ll} F_1((e_X)_z) = (e_Y)^{-1}_{f(z)}, & F_2((e_X)^{-1}_z) = (e_Y)_{f(z)}, \\ K_1(((e_X)^{-1}_z)^*) = ((e_Y)^{-1}_{f(z)})^*, & K_2((e_X)^*_z) = (e_Y)^*_{f(z)} \\ I_1((e_X^{-1})_z) = (e_Y)^{-1}_{f(z)}, & I_2((e_X)_z) = (e_Y)_{f(z)}, \\ M_1((e_X^{-1})^*_z) = (e_Y)^{-1}_{f(z)}, & M_2((e_X)^*_z) = (e_Y)_{f(z)}. \end{array}$$

(8) *If* $e_X(x,y) \le e_Y(f(y),f(x))$, *then*

$$\begin{array}{ll} F_1((e_X)^{-1}_z) = (e_Y)^{-1}_{f(z)}, & F_2((e_X)_z) = (e_Y)_{f(z)}, \\ K_1(((e_X)_z)^*) = ((e_Y)^{-1}_{f(z)})^*, & K_2((e_X^{-1})^*_z) = (e_Y)^*_{f(z)}, \\ I_1((e_X)_z) = (e_Y)^{-1}_{f(z)}, & I_2((e_X)^{-1}_z) = (e_Y)_{f(z)}, \\ M_1((e_X)^*_z) = (e_Y)^{-1}_{f(z)}, & M_2((e_X^{-1})^*_z) = (e_Y)_{f(z)}. \end{array}$$

(9) *If* $e_Y(x,y) \le e_X(g(x),g(y))$, *then*

$$\begin{array}{ll} G_1((e_Y)_y) = (e_X)^{-1}_{g(y)}, & G_2((e_Y)^{-1}_y) = (e_X)_{g(y)}, \\ H_1((e_Y)^{-1}_y) = (e_X)^{-1}_{g(y)} & H_2((e_Y)_y) = (e_X)_{g(y)}, \\ J_1(((e_Y)^{-1}_y)^*) = ((e_X)^{-1}_{g(y)})^*, & J_2((e_Y)^*_y) = (e_X)^*_{g(y)}, \\ L_1((e_Y^{-1})^*_y) = (e_X)^{-1}_{g(y)}, & L_2((e_Y)^*_y) = (e_X)_{g(y)}. \end{array}$$

(10) *If* $e_Y(x,y) \le e_X(g(y),g(x))$, *then*

$$\begin{array}{ll} G_1((e_Y)^{-1}_y) = (e_X)^{-1}_{g(y)}, & G_2((e_Y)_y) = (e_X)_{g(y)}, \\ H_1((e_Y)_y) = (e_X)^{-1}_{g(y)}, & H_2((e_Y)^{-1}_y) = (e_X)_{g(y)}, \\ J_1(((e_Y)_y)^*) = ((e_X)^{-1}_{g(y)})^*, & J_2((e_Y^{-1})^*_y) = (e_X)^*_{g(y)}, \\ L_1((e_Y)^*_y) = (e_X)^{-1}_{g(y)}, & L_2((e_Y^{-1})^*_y) = (e_X)_{g(y)}. \end{array}$$

Proof. (1) and (2) follow from their definitions.

(3) Let $e_X(x,g(y)) = e_Y(y,f(x))$ be given. Since $e_X(g(y),g(y)) = e_Y(y,f(g(y)) = 1$, then g is an antitone map from:

$$\begin{aligned} e_Y(y_1,y_2) &= e_Y(y_1,y_2) \odot e_Y(y_2,f(g(y_2))) \\ &\le e_Y(y_1,f(g(y_2))) = e_X(g(y_2),g(y_1)). \end{aligned}$$

Similarly, f is an antitone map.

First, we will show that $e_X(x,g(y)) = e_Y(y,f(x))$ iff $e_{L^X}(A,G_1(B)) = e_{L^Y}(B,F_1(A))$.

Let $e_X(x,g(y)) = e_Y(y,f(x))$ be given. By Lemma 1.2 (2,5), we have

$$\begin{aligned}
e_{L^Y}(B,F_1(A)) &= \bigwedge_{y\in Y}(B(y)\to F_1(A)(y))\\
&= \bigwedge_{y\in Y}\Big(B(y)\to \bigwedge_{x\in X}(A(x)\to e_Y(y,f(x)))\Big)\\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}\Big(A(x)\to (B(y)\to e_X(x,g(y)))\Big)\\
&= \bigwedge_{x\in X}\Big(A(x)\to G_1(B)(x)\Big)\\
&= e_{L^X}(A,G_1(B)).
\end{aligned}$$

Conversely, put $A = 1_x$ and $B = 1_y$. By (1) and (2), we have

$$\begin{aligned}
e_Y(y,f(x)) &= F_1(1_x)(y) = e_{L^Y}(1_y,F_1(1_x))\\
&= e_{L^X}(1_x,G_1(1_y)) = G_1(1_y)(x) = e_X(x,g(y)).
\end{aligned}$$

Second, we will show that $e_X(x,g(y)) = e_Y(y,f(x))$ iff $e_{L^X}(H_1(B),A) = e_{L^Y}(B,K_1(A))$. Let $e_X(x,g(y)) = e_Y(y,f(x))$ be given. By Lemma 1.2 (3,5), we have

$$\begin{aligned}
e_{L^X}(H_1(B),A) &= \bigwedge_{x\in X}(H_1(B)(x)\to A(x))\\
&= \bigwedge_{x\in X}\Big(\bigvee_{y\in Y}(e_X(x,g(y))\odot B(y))\to A(x)\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y)\to (e_X(x,g(y))\to A(x))\Big)\\
&= \bigwedge_{y\in Y}\Big(B(y)\to \bigwedge_{x\in X}(e_Y(y,f(x))\to A(x))\Big)\\
&= \bigwedge_{y\in Y}\Big(B(y)\to K_1(A)(y)\Big)\\
&= e_{L^Y}(B,K_1(A))
\end{aligned}$$

Conversely, put $A = 1^*_x$ and $B = 1_y$. By (1) and (2), we have

$$\begin{aligned}
e^*_X(x,g(y)) &= H_1(1_y)^*(x) = e_{L^X}(H_1(1_y),1^*_x)\\
&= e_{L^Y}(1_y,K_1(1^*_x)) = K_1(1^*_x)(y) = e^*_Y(y,f(x)).
\end{aligned}$$

Third, we will show that $e_X(x,g(y)) = e_Y(y,f(x))$ iff $e_{L^X}(L_1(B),A) = e_{L^Y}(M_1(A),B)$.

Let $e_X(x,g(y)) = e_Y(y,f(x))$ be given. By Lemma 1.2 (3,5,6), we have

$$\begin{aligned} e_{L^Y}(M_1(A),B) &= \bigwedge_{y\in Y}(M_1(A)(y) \to B(y)) \\ &= \bigwedge_{y\in Y}((\bigvee_{z\in X}(A^*(z)\odot e_Y(y,f(z))) \to B(y)) \\ &= \bigwedge_{y\in Y}\bigwedge_{z\in X}(A^*(z) \to (e_Y(y,f(z)) \to B(y))) \\ &= \bigwedge_{z\in X}(A^*(z) \to \bigwedge_{y\in Y}(e_Y(y,f(z)) \to B(y)) \\ &= \bigwedge_{z\in X}(A^*(z) \to (\bigvee_{y\in Y}(e_Y(y,f(z))\odot B^*(y)))^*) \\ &= \bigwedge_{z\in X}(\bigvee_{y\in Y}(e_Y(y,f(z))\odot B^*(y))) \to A(z)) \\ &= e_{L^X}(L_1(B),A). \end{aligned}$$

Conversely, put $A = 1_x^*$ and $B = 1_y^*$. Since $M_1(1_x^*)(y) = e_Y(y,f(x))$ and $L_1(1_y^*)(x) = e_X(x,g(y))$ from (1) and (2). Hence we have

$$\begin{aligned} & e_Y^*(y,f(x)) = M_1(1_x^*)^*(y) = e_{L^Y}(M_1(1_x^*),1_y^*) \\ & = e_{L^X}(L_1(1_y^*),1_x^*) = L_1(1_y^*)^*(x) = e_X^*(x,g(y)). \end{aligned}$$

Finally, we will show that $e_X(x,g(y)) = e_Y(y,f(x))$ iff $e_{L^X}(A,J_1(B)) = e_{L^Y}(I_1(A),B)$. Let $e_X(x,g(y)) = e_Y(y,f(x))$. Then

$$\begin{aligned} e_{L^Y}(I_1(A),B) &= \bigwedge_{y\in Y}(I_1(A)(y) \to B(y)) \\ &= \bigwedge_{y\in Y}((\bigvee_{x\in X}(A(x)\odot e_Y(y,f(x))) \to B(y)) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}(A(x) \to (e_Y(y,f(x)) \to B(y))) \\ &= \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in Y}(e_Y(y,f(x)) \to B(y))) \\ &= \bigwedge_{x\in X}(A(x) \to J_1(B)(x)) \\ &= e_{L^X}(A,J_1(B)). \end{aligned}$$

Conversely, put $A = 1_x$ and $B = 1_y^*$. Since $I_1((e_X)_x)(y) = e_Y(y,f(x))$ and $J_1((e_Y)_y^*)(x) = e_X(x,g(y))^*$ from (1) and (2),

$$\begin{aligned} & e_Y^*(y,f(x)) = I_1(1_x)^*(y) = e_{L^Y}(I_1(1_x),1_y^*) \\ & = e_{L^X}(1_x,J_1(1_y^*)) = J_1(1_y^*)(x) = e_Y^*(x,g(y)). \end{aligned}$$

(4) Let $e_X(x,g(y)) = e_Y(f(x),y)$ be given. Since $e_X(g(y),g(y)) = e_Y(f(g(y),y) = 1$, then g is an isotone map from:

$$\begin{aligned} e_Y(y_1,y_2) &= e_Y(y_1,y_2)\odot e_Y(f(g(y_1)),y_1) \\ &\le e_Y(f(g(y_1)),y_2) = e_X(g(y_1),g(y_2)). \end{aligned}$$

Similarly, f is an isotone map.

First, we will show that $e_X(x,g(y)) = e_Y(f(x),y)$ iff $e_{L^X}(A,G_1(B)) = e_{L^Y}(B,F_2(A))$.

Let $e_X(x,g(y)) = e_Y(f(x),y)$ be given. By Lemma 1.2(2,5), we have

$$\begin{aligned}
e_{L^Y}(B,F_2(A)) &= \bigwedge_{y\in Y}(B(y)\to F_2(A)(y))\\
&= \bigwedge_{y\in Y}\Big(B(y)\to \bigwedge_{x\in X}(A(x)\to e_Y(f(x),y))\Big)\\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}\Big(A(x)\to (B(y)\to e_X(x,g(y)))\Big)\\
&= \bigwedge_{x\in X}\Big(A(x)\to \bigwedge_{y\in Y}(B(y)\to e_X(x,g(y)))\Big)\\
&= \bigwedge_{x\in X}\Big(A(x)\to G_1(B)(x)\Big)\\
&= e_{L^X}(A,G_1(B)).
\end{aligned}$$

Conversely, put $A = 1_x$ and $B = 1_y$. By (1) and (2), $F_2(1_x) = (e_Y)_{f(x)}$ and $G_1(1_y) = (e_X)^{-1}_{g(y)}$.

$$\begin{aligned}
&e_Y(f(x),y) = F_2(1_x)(y) = e_{L^Y}(1_y,F_2(1_x))\\
&= e_{L^X}(1_x,G_1(1_y)) = G_1(1_y)(x) = e_X(x,g(y)).
\end{aligned}$$

Second, we will show that $e_X(x,g(y)) = e_Y(y,f(x))$ iff $e_{L^X}(H_1(B),A) = e_{L^Y}(B,K_2(A))$.

If $e_X(x,g(y)) = e_Y(f(x),y)$, then

$$\begin{aligned}
e_{L^X}(H_1(B),A) &= \bigwedge_{x\in X}(H_1(B)(x)\to A(x))\\
&= \bigwedge_{x\in X}\Big((\bigvee_{y\in Y}(e_X(x,g(y))\odot B(y)))\to A(x)\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y)\to (e_X(x,g(y))\to A(x))\Big)\\
&= \bigwedge_{y\in Y}\Big(B(y)\to \bigwedge_{x\in X}(e_Y(f(x),y)\to A(x))\Big)\\
&= \bigwedge_{y\in Y}\Big(B(y)\to K_2(A)(y)\Big)\\
&= e_{L^Y}(B,K_2(A)).
\end{aligned}$$

Put $A = 1^*_x$ and $B = 1_y$. By (1) and (2), $K_2(1^*_x) = (e_Y)^*_{f(x)}$ and $H_1(1_w) = (e_X)^{-1}_{g(w)}$. Hence

$$\begin{aligned}
&e^*_X(x,g(y)) = K_2(1^*_x)(y) = e_{L^X}(H_1(1_y,1^*_x)\\
&= e_{L^Y}(1_y,K_2(1^*_x) = H_1(1_y)^*(x) = e^*_X(x,g(y)).
\end{aligned}$$

Other cases, (5) and (6) are similarly proved in (3).

(7) We have $F_2((e_X)^{-1}_z) = (e_Y)_{f(z)}$ from:

$$\begin{aligned} F_2((e_X)_z^{-1})(y) &= \bigwedge_{x\in X}((e_X)_z^{-1}(x) \to e_Y(f(x),y)) \\ &\le (e_X)_z^{-1}(z) \to e_Y(f(z),y) = e_Y(f(z),y). \end{aligned}$$

Since f is an isotone map,

$$e_Y(f(z),y) \odot e_X(x,z) \le e_Y((f(z),y) \odot e_Y(f(x),f(z)) \le e_Y(f(x),y),$$

$$e_Y(f(z),y) \le \bigwedge_{z\in X}((e_X)_z^{-1}(x) \to e_Y(f(x),y)) = F_2((e_X)_z^{-1})(y).$$

$$\begin{aligned} K_2((e_X)_x^*)(y) &= \bigwedge_{z\in X}(e_Y(f(z),y) \to (e_X)_x^*(z)) \\ &\le (e_Y(f(x),y) \to \bot) = e_Y(f(x),y)^*. \end{aligned}$$

Thus, $K_2((e_X)_x^*) \le (e_Y)_{f(x)}^*$. Furthermore, $K_2((e_X)_x^*) \ge (e_Y)_{f(x)}^*$ from:

$$\begin{aligned} &e_Y(f(z),y) \odot e_X(x,z) \le e_Y(f(z),y) \odot e_Y(f(x),f(z)) \le e_Y(f(x),y) \\ &\text{iff } (e_Y(f(x),y))^* \le e_Y(f(z),y) \to (e_X(x,z))^*. \end{aligned}$$

(9) We have $G_1((e_Y)_y) \le (e_X)_{g(y)}^{-1}$ from:

$$G_1((e_Y)_y)(x) = \bigwedge_{w\in Y}((e_Y)_y(w) \to e_X(x,g(w))) \le e_X(x,g(y)).$$

Moreover, $G_1((e_Y)_y) \ge (e_X)_{g(y)}^{-1}$ from:

$$e_X(x,g(y)) \odot e_Y(y,w) \le e_X(x,g(y)) \odot e_X(g(y),g(w)) \le e_X(x,g(w))$$

$$e_X(x,g(y)) \le e_Y(y,w) \to e_X(x,g(w)).$$

We have $H_1((e_Y)_w^{-1}) = (e_X)_{g(w)}^{-1}$ from:

$$H_1((e_Y)_w^{-1})(x) = \bigwedge_{y\in Y}((e_Y)_w^{-1}(y) \odot e_X(x,g(y))) \ge (e_X)_{g(w)}(x).$$

$$e_X(x,g(y)) \odot e_Y(y,w) \le e_X(x,g(y)) \odot e_X(g(y),g(w)) \le e_X(x,g(w).$$

Since $J_1(((e_Y)_y^{-1})^*)(x) = \bigwedge_{w\in Y}(e_X(x,g(w)) \to ((e_Y)_y^{-1})^*(w) \le (e_X(x,g(y)))^*$, then $J_1(((e_Y)_y^{-1})^*) \le ((e_X)_{g(y)}^{-1})^*$.

Since $e_X(x,g(w)) \odot e_Y(w,y) \le e_X(x,g(w)) \odot e_X(g(w),g(y)) \le e_X(x,g(y))$, then

$$e_X(x,g(w)) \to e_Y(w,y)^* \ge (e_X(x,g(y)))^*.$$

Thus, $J_1(((e_Y)_y^{-1})^*) \ge ((e_X)_{g(y)}^{-1})^*$. Hence $J_1(((e_Y)_y^{-1})^*) = ((e_X)_{g(y)}^{-1})^*$.

Other cases in (7) and (9), (8) and (10) are similarly proved.

Example 2.2.Define a binary operation $\odot$ (called Łukasiewicz conjection) on $L = [0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\},\ x \to y = \min\{1 - x + y, 1\}.$$

Let $(X = \{a, b, c\}, e_X)$ and $(Y = \{x, y, z\}, e_Y)$ be a fuzzy poset with $e_X = (e_X(a, b))$, $e_Y = (e_Y(x, y))$ and $e_Y^0 = (e_Y^0(x, y))$ as follows:

$$e_X = \begin{pmatrix} 1.0 & 0.7 & 0.4 \\ 0.3 & 1.0 & 0.6 \\ 0.5 & 0.5 & 1.0 \end{pmatrix} \quad e_Y = \begin{pmatrix} 1.0 & 0.8 & 0.6 \\ 0.6 & 1.0 & 0.5 \\ 0.7 & 0.6 & 1.0 \end{pmatrix}$$

$$e_Y^0 = \begin{pmatrix} 0.4 & 0.6 & 1.0 \\ 1.0 & 0.3 & 0.5 \\ 0.7 & 1.0 & 0.5 \end{pmatrix}$$

(1) We define $f : X \to Y$ with $f(a) = x, f(b) = f(c) = y$. Then f is an isotone map. It satisfies Theorem 2.1(7). For examples,

$$\begin{aligned} F_2((e_X)_a^{-1}) &= F_2(1, 0.3, 0.5) = (1, 0.8, 0.6) = (e_Y)_{f(a)} = (e_Y)_x, \\ F_2((e_X)_b^{-1}) &= F_2(0.7, 1, 0.5) = (0.6, 1, 0.5) = (e_Y)_{f(b)} = (e_Y)_y, \\ F_2((e_X)_c^{-1}) &= F_2(0.4, 0.3, 1) = (0.6, 1, 0.5) = (e_Y)_{f(c)} = (e_Y)_y. \end{aligned}$$

(2) We define $h : X \to Y$ with $h(a) = x, h(b) = h(c) = z$. Then f is an antitone map. It satisfies Theorem 2.1(8). For examples,

$$\begin{aligned} K_2((e_X^{-1})_a^*) &= K_2(0, 0.7, 0.5) = (0, 0.2, 0.4) = (e_Y)_{h(a)}^*, \\ K_2((e_X^{-1})_b^*) &= K_2(0.3, 0, 0.5) = (0.3, 0.4, 0) = (e_Y)_{h(b)}^*, \\ K_2((e_X^{-1})_c^*) &= K_2(0.6, 0.4, 0) = (0.3, 0.4, 0) = (e_Y)_{h(c)}^*. \end{aligned}$$

(3) We define f and g as $f(a) = x, f(b) = y, f(c) = z$ and $g(x) = c, g(y) = a, f(z) = b$. Then $e_Y^0(x, f(a)) = e_X(a, g(x))$ for all $a \in X, x \in Y$. By Theorem 2.1, (e_X, f, g, e_Y^0) is a Galois connection, $(e_{L^X}, F_1, G_1, e_{L^Y})$ is a Galois connection with antitone maps f and g, $(e_{L^X}, K_1, H_1, e_{L^Y})$ is a dual residuated connection with antitone maps f and g, $(e_{L^X}, M_1, L_1, e_{L^Y})$ is a dual Galois connection with antitone maps f and g and $(e_{L^X}, I_1, J_1, e_{L^Y})$

is a residuated connection with antitone maps f and g. It satisfies Theorem 2.1(8)and (10). For examples,

$$\begin{aligned}
F_1((e_X)_a^{-1})(z) &= F_1(1,0.3,0.5)(z) = 0.7 = e_Y^0(z,x)\\
F_2((e_X)_b^{-1}) &= F_2(0.7,1,0.5) = (0.7,0.6,1) = (e_Y)_{f(b)}\\
F_2((e_X)_c^{-1}) &= F_2(0.4,0.3,1) = (0.7,0.6,1) = (e_Y)_{f(c)}
\end{aligned}$$

Example 2.3.Let $X = \{a,b,c\}$ be a set and $f : X \to X$ a function as $f(a) = b, f(b) = a, f(c) = c$. Define a binary operation $\odot$ (called Łukasiewicz conjection) on $L = [0,1]$ as Example 2.2.

(1) Let $(X = \{a,b,c\}, e_1 = (e_X(a,b)))$ be a fuzzy poset as follows:

$$e_1 = \begin{pmatrix} 1.0 & 0.6 & 0.5\\ 0.6 & 1.0 & 0.5\\ 0.7 & 0.7 & 1.0 \end{pmatrix}$$

Since $e_1(f(x),y) = e_1(x,f(y))$, then (e_1,f,f,e_1) are both residuated and dual residuated connections. It satisfies Theorem 2.1 (4) and (6). Since f is an isotone map, it satisfies Theorem 2.1 (7) and (9). For examples,

$$\begin{aligned}
&e_1(f(a),c) = 0.5 = F_2((e_1)_a^{-1})(c) = (1\to 0.5)\wedge(0.6\to 0.5)\wedge(0.7\to 1)\\
&= e_{L^X}((e_1)_c, F_2((e_1)_a^{-1})) = (0.7\to 0.6)\wedge(0.7\to 1)\wedge(1\to 0.5)\\
&= e_{L^X}((e_1)_a^{-1}, G_1((e_1)_c)) = (1\to 0.5)\wedge(0.6\to 0.5)\wedge(0.7\to 0.8)\\
&= G_1((e_1)_c)(a) = (0.7\to 0.6)\wedge(0.7\to 1)\wedge(1\to 0.5)\\
&= e_1(a,f(c)) = (e_1)_{f(c)}^{-1}(a).
\end{aligned}$$

$$\begin{aligned}
&e_1^*(f(c),a) = 0.3 = K_2((e_1)_c^*)(a) = (0.6\to 0.3)\wedge(1\to 0.3)\wedge(0.7\to 0)\\
&= e_{L^Y}((e_1)_a^{-1}, K_2((e_1)_c^*) = (1\to 0.3)\wedge(0.6\to 0.3)\wedge(0.7\to 0)\\
&= e_{L^X}(H_1((e_1)_a^{-1}), (e_1)_c^*) = (0.6\to 0.3)\wedge(1\to 0.3)\wedge(0.7\to 0)\\
&= H_1((e_1)_a^{-1})^*(c) = e_1^*(c,f(a)).
\end{aligned}$$

(2) Let $(X = \{a, b, c\}, e_2 = (e_2(a, b)))$ be a fuzzy poset as follows:

$$e_2 = \begin{pmatrix} 1.0 & 0.6 & 0.5 \\ 0.6 & 1.0 & 0.7 \\ 0.7 & 0.5 & 1.0 \end{pmatrix}$$

Since $e_1(y, f(x)) = e_1(x, f(y))$, then (e_1, f, f, e_1) are both both Galois and dual Galois connections. It satisfies Theorem 2.1 (3) and (5). Since f is an antitone map, it satisfies Theorem 2.1 (8) and (10). For examples,

$$\begin{aligned} e_2(b, f(c)) &= 0.7 = F_1((e_2)_c^{-1})(b) = (0.5 \to 1) \wedge (0.7 \to 0.6) \wedge (1 \to 0.7) \\ &= e_{L^Y}((e_Y)_y^{-1}, F_1((e_X)_x^{-1})) = (0.6 \to 0.5) \wedge (1 \to 0.7) \wedge (0.5 \to 1) \\ &= e_{L^X}((e_X)_x^{-1}, G_1((e_Y)_y^{-1})) = (0.5 \to 1) \wedge (0.7 \to 0.6) \wedge (1 \to 0.7) \\ &= G_1((e_2)_b^{-1})(c) = e_2(c, f(b)). \end{aligned}$$

$$\begin{aligned} e_2^*(a, f(a)) &= 0.4 = H_1((e_2)_a^*(a) = \Big((0.6 \odot 1) \vee (1 \odot 0.6) \vee (0.5 \odot 0.5)\Big)^* \\ &= e_{L^X}(H_1((e_2)_a, (e_2)_a^*) = (0.6 \to 0) \wedge (1 \to 0.4) \wedge (0.5 \to 0.5) \\ &= e_{L^Y}((e_2)_2, K_1((e_2)_a^*)) = (1 \to 0.4) \wedge (0.6 \to 0) \wedge (0.5 \to 0.5) \\ &= K_1((e_2)_a^*)(a) = e_2(a, f(a)). \end{aligned}$$

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