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VARIATIONS ON DOMINATION IN GRAPHS CONTAINING EDGE-DISJOINT CYCLES

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Abstract. The concept of domination in graphs, with its many variations, is now well studied in graph theory. In this paper, independent sets and two variations on domination theme are discussed. These two variations on domination theme that are well studied in graph theory, called total domination and restrained domination. A sub set $S \subseteq V(G)$ is an independent set, if no two vertices of S are adjacent. A subset $S \subseteq V(G)$ is called a dominating set, if every vertices of $V - S$ is adjacent to a member of S. The results are focused on independent sets and variant dominating sets in the graphs containing edge-disjoint cycles.

Keywords: Domination set, independent set, total dominating set, restrained dominating set, Graph.

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1. INTRODUCTION:

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics and its results have applications in many areas of the computing, social, and natural sciences. The fastest growing area within graph theory is the study of domination, the reason being its many and varied applications in such fields as social sciences, communications networks, algorithmic designs etc. Dominating and independent sets are among the most well-

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studied graph sets. Domination can be a useful tool for determining business network and making decisions.

The topic of domination was given formal Mathematical definition by C. Berge in 1958 and O. Ore [14] in 1962. Berge called the domination as external stability and domination number of coefficient of external stability. Ore introduced the world domination in his famous book [14]. This concept lived in hibernation until 1975 when a paper [8] published in 1977. This paper brought to light new ideas and potentiality of being applied in variety of areas. The research in domination theory has been broadly classified in [16], [17]. In this paper we study the dominating sets and its variation in graphs contain edge- disjoint cycles. Also we focus on independent sets and two variations on domination sets that are called total domination and restrained domination.

2. PRELIMINARIES AND NOTATIONS:

Let $G=(V,E)$ be a simple graph (i.e., undirected, without loops and multi edges). The number of vertices namely the cardinality of V is called the order of G and is denoted by $|G|$. The number of edges of a graph namely the cardinality of E is called the size of G and is denoted by $|E|$. We write $e=v_iv_j \in E(G)$ to mean the pair $v_i, v_j \in E(G)$ and if $e=v_iv_j \in E(G)$ we say that v_i and v_j are adjacent and e and v_i, e and v_j are incident.

The open neighbourhood $N(v)$ of the vertex v consists of the set of vertices adjacent to v. That is $N(v)=\{w \in V : vw \in E\}$. The closed neighbourhood of v is $N[v]=N(v)\cup\{v\}$. For a set $S \subset V$, the open neighbourhood N(S) is defined by $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighbourhood $N[S]$ by $N[S]=N(S)\cup S$. A vertex $v \in S$ is called "an enclave of S", if $N[S] \subset S$. A vertex $v \in S$ is called "an isolate of of S", if $N(S) \subset V-S$.

The degree of a point v is denoted by $\deg(v)$ is defined as the number of edges incident with v. That is $\deg(v)=|N(v)|$. The maximum and minimum of the degree of vertices of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. If $\Delta(G)=\delta(G)=r$, then G is said to be a regular graph of degree r or simply r-regular.

Now, we describe independent set, dominating set and variant dominating sets of graphs.

3. INDEPENDENT SET, DOMONATING SET AND ITS VARIATIONS:

Independent sets were introduced into the communication theory on noisy channels [9]. From application point of view, domination problems appear in numerous practical setting, ranging from strategic decision such as locating radar stations or emergency services through computational biology to voting system. Variations of dominations such as multiple domination, even/odd domination, distance domination, directed domination, independent domination and connected domination have found numerous application and significant theoretical interest recently. The concept of domination in graphs, with its many variations, is now well studied in graph theory.

A sub set $S \subseteq V(G)$ is an independent set, if no two vertices of S are adjacent. Moreover, the subset containing only one vertex and the empty set also are independent. The number of all independent sets in G is denoted by $NI(G)$. For a graph G on $V(G) = \phi$, we put $NI(G) = 1$. Independence number $\beta(G)$ of graph G is the maximal cardinality of an independent set of vertices.

A subset $S \subseteq V(G)$ is called a dominating set, if every vertices of $V - S$ is adjacent to a member of S. A dominating set of G with minimum cardinality is called a minimum dominating set and the cardinality of a minimum dominating set is called the domination number and denoted by $\gamma(G)$. The upper domination number of a graph G denoted by $\Gamma(G)$ is defined as the maximum cardinality of a minimum dominating set of G.

A set S of vertices in a graph G is called an independent dominating set of G if S is both an independent and a dominating set of G. This set is also called a Stable set or a Kernel of the graph. Independent dominating sets were introduced into the theory of games by Neumann and Margenstern in 1944 [13]. The independent domination number $i(G)$ is the cardinality of the smallest independent domination set.

A set $S \subseteq V(G)$ is a total dominating set of G, if every vertex is adjacent to a vertex in S (other than itself). The total domination number of G is the minimum cardinality of a total dominating set and denoted by $\gamma_t(G)$.

A dominating set S is a Restrained Dominating Set (RDS) if every vertex in V-S is adjacent to another vertex of V-S. The restrained domination number of G is the minimum cardinality of a restrained Dominating set (RDS) and denoted by $\gamma_r(G)$. A RDS of G of cardinality $\gamma_r(G)$. is called a γ_r-set of G.

Domination can be a useful tool for determining business network and making decisions. Business would benefit from the use of the concept of domination to strategically plan the location of their stores in order to reach the maximum amount of areas with minimal stores locations. Suppose a company wants to open new showrooms in a heavily populated region. The company has purchased land in the centre of each neighborhood. However, the company would like to minimize the number of showrooms that it must build and yet still be accessible (within 10 miles of a showroom) to as many people in the region as possible. Let vertices of the graph are the 7 land locations purchased by the company. Two vertices will be joined by an edge if the two locations are no more than 5 miles apart. What is the minimum number of showrooms that the company must build so that its showrooms are accessible to everyone in the seven neighborhoods?

Let S be the set of seven locations, $S = \{a, b, c, d, e, f, g\}$. Here (i) d is no more than 5 miles from b, c and e, (ii) e is no more than 5 miles from d, f and g, (iii) a is no more than 5 miles from b. Since d and e are accessible to a maximum number of neighborhoods, the company would benefit from putting showrooms at these locations. Because location a is accessible to itself and b, then a showroom can be located at either location. A domination set is $S = \{d, e, a\}$ and the domination number $\gamma(G) = 3$. Therefore the company would need a minimum of 3 showrooms to cover all seven neighborhoods.

4. THE MAIN RESULTS:

In this section, we present some results on domination in Graphs containing Edge-disjoint cycles.

Case (1): If Graph having two edge-disjoint cycles.

Theorem 4.1 Let G be a graph of order n and having two edge-disjoint cycles. A dominating set $S \subseteq V$ is a total dominating set if and only if (i) if $N(u) \not\subset V-S, u \in S$ (ii) $\exists$ a vertex $v \in V-S$ for which $N(v) \cap S = \{u, x\}$.

Proof: Let S be a total dominating set of graph G having two edge-disjoint cycles. First we are to prove that $N(u) \not\subset V-S.$

Let $u \in S$ and $w \in N(u)$ then $\exists$ a vertex $w \in S$ such that $uw \in E(G)$ [4.1]

since S is a total dominating set. Thus at least one element of $N(u)$ and V-S will be different. Hence $N(u) \not\subset V-S.$

Suppose for every $v \in V-S$, $N(v) \bigcap S \neq \{u, x\}$, then $\exists$ a vertex $t_v \in S$ such that $t_v \neq u, t_v \neq x$ and $t_v \in N(v) \cap S.$ [4.2]

Since S is a total dominating set, we get $N(v) \bigcap S \neq \phi$

Consider $S - \{u, x\},$ then $V - (S - \{u, x\}) = V - S \cup \{u, x\}.$

Let $v \in V - S \cup \{u, x\}.$ if $v \neq u, v \neq x, then$ $N(v) \bigcap (S - \{u, x\}) \neq \phi.$

Thus $S - \{u, x\}$ is a total dominating set which is contradiction..

Hence for a vertex $v \in V-S$, $N(v) \bigcap S = \{u, x\}.,$

Conversely, Let S be a dominating set and let $u \in S.$

Suppose (i) holds, i.e., $N(u) \not\subset V-S,$ then at least one element of $N(u)$ and V-S will be different.

Suppose (ii) holds, then $N(v) \bigcap (S - \{u, x\}) = \phi.$ for some $v \in V-S.$

Hence $S-\{u, x\}$ is not a total dominating set. Thus S is a total dominating set.

Theorem 4.2 Let G be a graph of order n and having two edge-disjoint cycles. A dominating set $S \subseteq V$ is a restrained dominating set if and only if (i) if $N(u) \subset V-S, u \in S$ (ii) $\exists$ a vertex $v \in V-S$ for which $N(v) \cap S = \{u, y\}$.

Proof: Let S be a restrained dominating set of graph G having two edge-disjoint cycles. First we are to prove that $N(u) \subset V-S$.

Suppose $N(u) \not\subset V-S$, $\exists$ a vertex $w \in S$ such that $uw \in E(G)$ [4.3]

Every vertex of V-S is not adjacent to another vertex of V-S, which is contradiction.

Hence $N(u) \subset V-S$.

Suppose for every $v \in V-S$, $N(v) \bigcap S \neq \{u, y\}$, then $\exists$ a vertex $t_v \in S$ such that $t_v \neq u, t_v \neq x$ and $t_v \in N(v) \cap S$. [4.4]

Since S is a restrained dominating set, we get $N(v) \bigcap S \neq \phi$

Consider $S-\{u, y\}$, then $V-(S-\{u, y\}) = V-S \cup \{u, y\}$.

Let $v \in V-S \cup \{u, y\}$. if $v \neq u, v \neq y,$ *then* $N(v) \bigcap (S-\{u, y\}) \neq \phi$.

Thus $S-\{u, y\}$ is a restrained dominating set which is contradiction.

Hence for a vertex $v \in V-S$, $N(v) \bigcap S = \{u, y\}$.,

Conversely, Let S be a dominating set and let $u \in S$.

Suppose (i) holds, i.e., $N(u) \subset V-S$, then u is adjacent to at least one element of V-S.

Suppose (ii) holds, i.e., for every $v \in V-S$, $N(v) \bigcap S = \{u, y\}$, then every element of V-S is adjacent to another element of V-S. Hence S is a restrained dominating set.

Theorem 4.3: Let G be a graph of order n having two edge-disjoint cycles. Then

(i) $\gamma_r(G) \geq \left(n+1-\frac{2m}{3} \right)$,

(ii) $\gamma_r(G) \geq \gamma_t(G) \geq \gamma(G)$, where $\gamma_r(G), \gamma_t(G)$ and $\gamma(G)$ are restrained domination number, total domination number and domination number respectively.

Proof (i): Let G be a graph of order n having two edge-disjoint cycles Let S be a restrained dominating set of graph G. Let m_1 be the size of V-S.

Thus $m_1 = \sum_{v \in V-S} \deg(v) \geq \frac{1}{2}\left(n+1-\gamma_r(G)\right)$. [4.5]

Let m_2 be the number of edges between S and V-S.

Since S is restrained dominating set, so every vertex in V-S is adjacent to at least one vertex in S as well as in V-S.

Thus $m_2 \geq \left(n+1-\gamma_r(G)\right)$. [4.6]

Combine these results, we get

$$m = m_1 + m_2 \geq \frac{1}{2}\left(n+1-\gamma_r(G)\right) + n+1-\gamma_r(G),$$
$$m \geq \frac{3(n+1)}{2} - \frac{3}{2}\gamma_r(G), \text{ which implies that}$$

$$\gamma_r(G) \geq \left(n+1-\frac{2m}{3} \right).$$

Proof (ii): Let S, S_t, S_r be a dominating set, total dominating set and restrained dominating set of G contains two edge-disjoint cycles.

Since S_t is a total dominating set and $S_t = V_i \cup V_j$, where V_i and V_j are vertex subsets of minimum cardinality.

Thus $\gamma(G) \leq \left|V_i \cup V_j\right| = \left|V_i\right| + \left|V_j\right| = \gamma_t(G)$. [4.7]

Since S_r is a restrained dominating set and $S_r = V_k \cup V_l$, where V_k and V_l are vertex subsets of minimum cardinality and may be contain some vertex of V_i and V_j.

Thus $\gamma_r(G) = |V_k \cup V_l| = |V_k| + |V_l| \geq |V_i| + |V_j| = \gamma_t(G)$, i.e., $\gamma_r(G) \geq \gamma_t(G)$, [4.8]

Combine both the inequalities, we get

$\gamma_r(G) \geq \gamma_t(G) \geq \gamma(G)$

Case (2): If Graph having three edge-disjoint cycles.

Theorem 4.4: Let G be a graph of order n and having three edge-disjoint cycles. A dominating set $S \subseteq V$ is a total dominating set if and only if (i) if $N(u) \not\subset V - S, u \in S$ (ii) $\exists$ a vertex $v \in V - S$ for which $N(v) \cap S = \{u, x\}$.

The proof can be given same as theorem 4.1.

Theorem 4.5: Let G be a graph of order n and having three edge-disjoint cycles. A dominating set $S \subseteq V$ is a restrained dominating set if and only if (i) if $N(u) \subset V - S, u \in S$ (ii) $\exists$ a vertex $v \in V - S$ for which $N(v) \cap S = \{u, y\}$.

The proof can be given same as theorem 4.2.

Theorem 4.6: Let G be a graph of order n having three edge-disjoint cycles. Then

(i) $\gamma_r(G) \geq \left(n + 1 - \frac{2m}{3}\right)$,

(ii) $\gamma_r(G) \leq \gamma_t(G) \geq \gamma(G)$, where $\gamma_r(G), \gamma_t(G)$ and $\gamma(G)$ are restrained domination number, total domination number and domination number respectively.

Proof (i): Let G be a graph of order n having three edge-disjoint cycles Let S be a restrained dominating set of graph G. Let m_1 be the size of V-S.

Thus $m_1 = \sum_{v \in V-S} \deg(v) \geq \frac{1}{2}\left(n + 1 - \gamma_r(G)\right)$. [4.9]

Let m_2 be the number of edges between S and V-S.

Since S is restrained dominating set, so every vertex in V-S is adjacent to at least one vertex in S as well as in V-S.

Thus $m_2 \geq \left(n+1-\gamma_r(G)\right)$. [4.10]

Combine both the inequalities, we get

$$m = m_1 + m_2 \geq \frac{1}{2}\left(n+1-\gamma_r(G)\right) + n + 1 - \gamma_r(G),$$

$$m \geq \frac{3(n+1)}{2} - \frac{3}{2}\gamma_r(G), \text{ which implies that}$$

$$\gamma_r(G) \geq \left(n+1-\frac{2m}{3}\right).$$

Proof (ii): This can be given same as theorem 4.3.

Next we state two theorems related to total and restrained dominating sets of graphs containing three edge-disjoint cycles.

Theorem 4.7: Let G be a graph of order n and having three edge-disjoint cycles. A dominating set $S \subseteq V$ is a total dominating set if and only if (i) for vertex $v \in V - S \; \exists u, w \in S$ such that $N(u) \bigcup N(w) = V$ and (ii) either $N(u) \bigcap N(w) = \phi$ or $N(u) \bigcap N(w) = \{v\}$.

Theorem 4.8: Let G be a graph of order n and having three edge-disjoint cycles. A dominating set $S \subseteq V$ is a restrained dominating set if and only if (i) for vertex $v \in V - S \; \exists u, w \in S$ such that $N(u) \bigcup N(w) \subset V$ and (ii) either $N(u) \bigcap N(w) = \phi$ or $N(u) \bigcap N(w) = \{v\}$.

CONCLUSION:

Independent set, dominating set and its variant sets in graphs containing edge-disjoint cycles are discussed in this paper. Also presented and derived some identities on total and restrained domination numbers of graphs containing edge-disjoint cycles.

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