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ON THE CORDIAL LABELING OF $T_l(P_n)$

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Abstract. Let G be a simple graph that has n vertices and m edges. Let $V(G)$ be the set of all vertices of G and $E(G)$ the set of all edges of G. Let $f: V(G) \to \{1,2,\ldots,k\}$ be a function that assigns to each vertex $v \in V(G)$ a positive integer $f(v) \in \{1,2,\ldots,k\}$. This integer is called the label of v. For each edge $uv \in E(G)$ we assign a label which is the gcd $(f(u), f(v))$. Let $v_f(i)$ be the number of vertices labeled with i for $i \geqslant 1$. The function f is called k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$ for all $i, j \in \{1,2,\ldots,k\}$ and $|e_f(1) - e_f(0)| \leqslant 1$ where $e_f(1)$ denotes the number of edges that are labeled with 1 and $e_f(0)$ denotes the number of edges that are not labeled with 1. In this paper we introduce the concept of the l-fold of a graph G, $T_l(G)$, and we show that the l-fold of a path P_n, $T_l(P_n)$, is a 4-prime cordial graph.

Keywords: cordial labeling; 4-prime cordial graphs; path.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph, where $V(G)$ be the set of all vertices of G, and $E(G)$ is the edge set of G. Let n be the number of vertices of G and m be the number of edges in $E(G)$. We call such a graph an (n,m)-graph. The number of edges incident to a vertex $v \in V(G)$ is called the *degree* of v, and is denoted by $d_G(v)$. We say that two vertices

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$u, v \in V(G)$ are adjacent or neighbors if uv is an edge of G, that is, $uv \in E(G)$. The concepts of labeling of graphs has been popular in the last 65 years due to its wide range of applications. In 2020, J.A. Gallian in [2] gave a survey and a classification of graph labeling. The concept of Cordial labeling of graphs was introduced in [1] by Cahit. In [6], M. Sandoran, R. Ponrag and S. Somasudaran, introduced the concept of prime cordial labeling and discussed the prime labeling of certain graphs. In [4], M. Hailat introduced the concept of trigraph of a graph G, $T_3(G)$, and showed that the trigraph of a path P_n, $T_3(P_n)$, and the trigraph of a cycle C_n, $T_3(C_n)$ are 4-prime cordial graphs.

In this paper we introduce the concept of the l-fold of a graph G, $T_l(G)$, and we show that the l-fold of a path P_n, $T_l(P_n)$, is a 4-prime cordial graph.

Definition 1. Let G be an (n,m)-graph and let $f\colon V(G) \to \{1,2,\ldots,k\}$ be a function. For each edge $uv \in E(G)$ we assign the label gcd $(f(u), f(v))$. We say f is a k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$ for all $i, j \in \{1,2,\ldots,k\}$ and $|e_f(1) - e_f(0)| \leqslant 1$ where $e_f(1)$ denotes the number of edges that are labeled with 1 and $e_f(0)$ denotes the number of edges that are not labeled with 1, respectively.

Definition 2. A graph G is called k-prime cordial graph if it admits a k-prime cordial labeling [5].

Definition 3. An l-fold of a connected graph G, $T_l(G)$, is constructed by taking l copies of $G = G_1, G_2, \ldots, G_l$ and joining each vertex $v_{is} \in G_i$ to the neighbors of the corresponding $v_{js} \in G_j$ and vice-versa [5].

Example 1. The graph below is $T_3(P_4)$

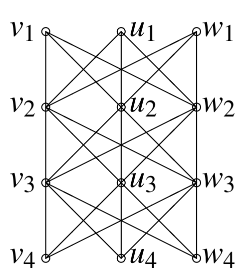

Figure 1

It follows that

$$V(T_l(P_n)) = \{v_{11},\ldots,v_{in}v_{21}\ldots,v_{2n},\ldots,v_{l1}\ldots,v_{ln}\}$$

and

$$E(T_l(P_n)) = \{v_{ij}v_{i(j+1)} | 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n-1\} \cup \{v_{ij}v_{s(j+1)}, v_{ij}v_{s(j-1)} | 1 \leqslant i,s \leqslant l, 2 \leqslant j \leqslant n-1\}$$

Proposition 1. *: $|V(T_l(P_n))| = ln$ and $|E(T_l(P_n))| = l^2(n-1)$.*

Proof. : Since we have l copies of P_n and each P_n has n vertices, therefore, $|V(T_l(P_n))| = ln$. Since $\deg(v_{i1}) = \deg(v_{in}) = l$ for all $i = 1,\ldots,s$ and $\deg(v_{ij}) = 2l$ for $2 \leqslant j \leqslant n-1$ then

$$\begin{aligned} |E(T_l(P_n))| &= \frac{1}{2}\left(\sum_{i=1}^{l}(\deg(v_{i1}) + \deg(v_{in})) + \sum_{j=2}^{n-2}\sum_{i=1}^{l}\deg(v_{ij})\right) \\ &= \frac{1}{2}(2l^2 + 2l^2(n-2)) = l^2(n-1). \end{aligned}$$

2. 4-Prime Cordial Labeling of $T_l(P_n)$

In this section we will discus the structure of the l-fold path graph, $T_l(P_n)$, and show that $T_l(P_n)$ is a 4-prime cordial graph.

Theorem 1. *The l-fold path graph, $T_l(P_n)$, is a 4-prime graph.*

Proof. : Let us define a function $f\colon V(T_l(P_n)) \to \{1,2,3,4\}$. To do that, we will divide the discussion into four cases depending on the value of n.

Case 1: Suppose that $n \equiv 0 \pmod 4$. That is, $n = 4t$. In this case we define f as follows:

$$\begin{aligned} f(v_{ij}) &= 2 && \text{for } 1 \leqslant i \leqslant \left\lfloor \tfrac{l}{2} \right\rfloor, 1 \leqslant j \leqslant \tfrac{n}{2} \\ f(v_{ij}) &= 4 && \text{for } \left\lfloor \tfrac{l}{2} \right\rfloor + 1 \leqslant i \leqslant 2\left\lfloor \tfrac{l}{2} \right\rfloor, 1 \leqslant j \leqslant \tfrac{n}{2} \\ f(v_{ij}) &= 1 && \text{for } 1 \leqslant i \leqslant l, j = \tfrac{n}{2}+1, \tfrac{n}{2}+3, \ldots, n-1 \\ f(v_{ij}) &= 3 && \text{for } 1 \leqslant i \leqslant l, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4, \ldots, n \end{aligned}$$

If l is even then $\lfloor \frac{l}{2} \rfloor = \frac{l}{2}$ and $2\lfloor \frac{l}{2} \rfloor = l$, so we assign an integer 2 or 4 to any vertex v_{ij} such that $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant \frac{n}{2}$, as defined above. If l is odd then we assign integers to the remaining vertices of the form v_{ij}, $1 \leqslant j \leqslant \frac{n}{2}$ as follows:

$$f(v_{l1})=2, f(v_{l2})=4, f(v_{l3})=2, f(v_{l4})=4\ldots$$

It follows from the above assignment that,

$$v_f(1)=v_f(2)=v_f(3)=v_f(4)=\frac{ln}{4}.$$

For $1\leqslant j\leqslant \frac{n}{2}$ all edges are either labeled with 2, since $2=\gcd(2,2)=\gcd(2,4)$ or $4=\gcd(4,4)$. That is the number of edges labeled with 1 or with not 1 are given as

$$\begin{aligned} e_f(1) &= \frac{2(l)(l)(\frac{n}{2}-2)+l^2+l^2+l^2}{2} \\ &= \frac{l^2(n-4)+3l^2}{2}=\frac{l^2(n-1)}{2}. \end{aligned}$$

This implies that

$$e_f(0)=l^2(n-1)-\frac{l^2(n-1)}{2}=\frac{l^2(n-1)}{2}.$$

We summarize the previous information in the table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 1

From Table 1, f satisfies the definition of 4-prime cordial labeling for $T_l(P_n)$ since $|v_f(i)-v_f(1)|=0\leqslant 1$ and $|e_f(0)-e_f(1)|=0\leqslant 1$.

Case 2: Suppose that $n\equiv 1 \pmod 4$. That is $n=1+4t$ for a positive integer t. We define $f\colon V(T_l(P_n))\to\{1,2,3,4\}$ as follows:

$$\begin{aligned} f(v_{ij})&=2 && \text{for } 1\leqslant i\leqslant \lfloor \tfrac{l}{2}\rfloor, 1\leqslant j\leqslant 2t \\ f(v_{ij})&=4 && \text{for } \lfloor \tfrac{l}{2}\rfloor+1\leqslant i\leqslant 2\lfloor \tfrac{l}{2}\rfloor, 1\leqslant j\leqslant 2t \\ f(v_{ij})&=1 && \text{for } 1\leqslant i\leqslant l, j=2t+2,,2t+4,\ldots,4t=n-1 \\ f(v_{ij})&=3 && \text{for } 1\leqslant i\leqslant l, j=2t+3,2t+5,\ldots,4t+1=n \end{aligned}$$

This cordial labeling f is defined for all v_{ij} except v_{il} if l is an odd integer and if $i=2t+1$. To define f on these exceptions we divide the discussion into four subcases:

Subcase 2.1: Suppose $l \equiv 0 \pmod 4$. Then l is an even integer and in this subcase $2\lfloor \frac{l}{2} \rfloor = l$. Therefore, f is defined for all v_{ij} except when $i = 2t+1$. For the latter case we define

$$f(v_{1(2t+1)}) = 1, f(v_{2(2t+1)}) = 2, f(v_{3(2t+1)}) = 3,$$

$$f(v_{4(2t+1)}) = 4, f(v_{5(2t+1)}) = 1, \ldots, f(v_{l(2t+1)}) = 4.$$

It follows that

$$\begin{aligned}
v_f(1) &= (2t)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{4tl+l}{4} = \frac{l(4t+1)}{4} = \frac{ln}{4} \\
v_f(2) &= (2t)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{l(4t+1)}{4} = \frac{ln}{4} \\
v_f(3) &= (2t)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{ln}{4} \\
v_f(4) &= (2t)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{l(4t+1)}{4} = \frac{ln}{4} \\
e_f(1) &= \frac{2l^2(2l0 + l^2 - l^2}{2} = \frac{l^2(4t+1) - l^2}{2} = \frac{l^2(n-1)}{2} \\
e_f(0) &= l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}.
\end{aligned}$$

We summarize the previous information in the table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 2

From Table 2, f satisfies the definition of 4-prime cordial labeling for $T_l(P_n)$ since $|v_f(i) - v_f(1)| = 0 \leqslant 1$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$.

Subcase 2.2 Suppose $l \equiv 1 \pmod 4$. Then l is an odd integer and in this subcase $2\lfloor \frac{l}{2} \rfloor = l-1$. Therefore we define f for the vertices of the form v_{il} as:

$$\begin{aligned}
f(v_{il}) &= 2 && \text{if } i \text{ is odd and } i < 2t+1 \\
f(v_{il}) &= 4 && \text{if } i \text{ is even and } i < 2t+1 \\
f(v_{il}) &= 1 && \text{if } i \text{ is odd and } i > 2t+3 \\
f(v_{il}) &= 3 && \text{if } i \text{ is even and } i > 2t+2
\end{aligned}$$

$$f(v_{1(2t+1)}) = 1 \qquad f(v_{2(2t+1)}) = 2, \quad f(v_{3(2t+1)}) = 3$$

$$f(v_{4(2t+1)}) = 4 \qquad f(v_{5(2t+1)}) = 1, \ldots, f(v_{(l-1)(2t+1)}) = 4, f(v_{l(2t+1)} = 1.$$

It follows that

$$\begin{aligned}
v_f(1) &= (2t)\Big(\frac{l-1}{2}\Big) + \frac{l-1}{4} + t + 1 = \frac{n(l-1)}{4} + t + 1 = \frac{ln+3}{4} \\
v_f(2) &= (2t)\Big(\frac{l-1}{2}\Big) + \frac{l-1}{4} + t = \frac{n(l-1)}{4} + t = \frac{ln-1}{4} \\
v_f(3) &= (2t)\Big(\frac{l-1}{2}\Big) + \frac{l-1}{4} + t = \frac{n(l-1)}{4} + t = \frac{ln-1}{4} \\
v_f(4) &= (2t)\Big(\frac{l-1}{2}\Big) + \frac{l-1}{4} + t = \frac{n(l-1)}{4} + t = \frac{ln-1}{4}
\end{aligned}$$

Note that $v_f(1) + v_f(2) + v_f(3) + v_f(4) = n(l-1) + 4t + 1 = n(l-1) + n = nl$. Also,

$$\begin{aligned}
e_f(1) &= \frac{2l^2(2t-1) + 2l^2 - 2l + 2l}{2} = \frac{2l^2 2t}{2} = \frac{l^2(n-1)}{2} \\
e_f(0) &= l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}
\end{aligned}$$

We summarize the above result as:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln+3}{4}$	$\frac{ln-1}{4}$	$\frac{ln-1}{4}$	$\frac{ln-1}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 3

From the above table, we have $|v_f(i) - v_f(1)| = 0$ 0r $1 \leqslant 1$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

<u>Subcase 2.3:</u> Suppose that $l \equiv 2 \pmod 4$. Then l is an even integer and in this subcase $2\lfloor \frac{l}{2} \rfloor = l$. This subcase is similar in calculation to Subcase 2.1.

<u>Subcase 2.4:</u> Suppose that $l \equiv 3 \pmod 4$. Then l is an odd integer and in this subcase $2\lfloor \frac{l}{2} \rfloor = l - 1$. This subcase is the same as subcase 2.2.

<u>Case 3:</u> Suppose that $n \equiv 2 \pmod 4$. That is $n = 2 + 4t$ for a positive integer t. We define $f\colon V(T_l(P_n)) \to \{1,2,3,4\}$ as follows:

$$f(v_{ij}) = 2 \qquad \text{for } 1 \leqslant i \leqslant \lfloor \tfrac{l}{2} \rfloor, 1 \leqslant j \leqslant 2t+1$$

$$f(v_{ij}) = 4 \qquad \text{for } \lfloor \tfrac{l}{2} \rfloor + 1 \leqslant i \leqslant 2\lfloor \tfrac{l}{2} \rfloor, 1 \leqslant j \leqslant 2t+1$$

$$f(v_{ij}) = 1 \qquad \text{for } 1 \leqslant i \leqslant l, j = 2t+2, 2t+4, \ldots, 4t+1 = n-1$$

$$f(v_{ij}) = 3 \qquad \text{for } 1 \leqslant i \leqslant l, j = j = 2t+3, 2t+5, \ldots, 4t+2 = n$$

This cordial labeling f is defined for all v_{ij} except v_{il} if l is an odd integer. We divide the discussion to the following subcases depending on the values of l.

Subcase 3.1: Suppose $l \equiv 0 \pmod 4$ or $l \equiv 2 \pmod 4$. Then l is an even integer, and therefore $2\lfloor \frac{l}{2} \rfloor = l$, so that f is defined on all vertices of $T_l(P_n)$. It follows that

$$v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{ln}{4}$$
$$e_f(1) = \frac{2l^2(2t+1)l - l^2}{2} = \frac{l^2(n-1)}{2}.$$
$$e_f(0) = l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}.$$

We summarize the above results as

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 4

From the above table, we have $|v_f(i) - v_f(1)| = 0 \leqslant 1$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

Subcase 3.2: Suppose that $l \equiv 1 \pmod 4$ or $l \equiv 3 \pmod 4$. That is l is an odd integer. In this subcase we have to define f on v_{il} since $2\lfloor \frac{l}{2} \rfloor = l-1$. We define $f(v_{il})$ as follows:

$$f(v_{ij}) = \begin{cases} 2 & \text{if } i \text{ is odd and } i \leqslant 2t+1 \\ 4 & \text{if } i \text{ is even and } i \leqslant 2t+1 \\ 1 & \text{if } i \text{ is odd and } i \geqslant 2t+2 \\ 3 & \text{if } i \text{ is even and } i \geqslant 2t+2 \end{cases}$$

It follows that

$$v_f(2) = \frac{(\frac{l-1}{2})(2t+1) + \frac{2t+2}{2}}{2} = \frac{(\frac{l-1}{2})\frac{n}{2} + \frac{\frac{n}{2}+1}{2}}{2} = \frac{nl+2}{4}$$
$$v_f(4) = \frac{(\frac{l-1}{2})(2t+1) + \frac{2t}{2}}{2} = \frac{(\frac{n(l-1)}{4}) + \frac{n-2}{4}}{2} = \frac{nl-2}{4}$$

$$v_f(1) = \frac{nl+2}{4} \quad \text{as in } v_f(2)$$
$$v_f(3) = \frac{nl-2}{4} \quad \text{as in } v_f(4)$$
$$e_f(1) = \frac{2l^2(2t+1)-l^2}{2} = \frac{2l^2(\frac{n}{2})-l^2}{2} = \frac{l^2(n-1)}{2}$$
$$e_f(0) = l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}$$

We summarize the above result as:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(1)$	$e_f(0)$
$\frac{nl+2}{4}$	$\frac{nl+2}{4}$	$\frac{nl-2}{4}$	$\frac{nl-2}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 5

From Table 5, we have $|v_f(i) - v_f(j)| = 0$ 0r $1 \leqslant 1$ for all $i, j = 1,2,3,4$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

Case 4: Suppose that $n \equiv 3 \pmod 4$. That is $n = 3+4t$ for positive integer t. We define $f\colon V(T_l(P_n)) \to \{1,2,3,4\}$ as follows:

$$\begin{aligned}
f(v_{ij}) &= 2 && \text{for } 1 \leqslant i \leqslant \lfloor \tfrac{l}{2} \rfloor, 1 \leqslant j \leqslant 2t+1\\
f(v_{ij}) &= 4 && \text{for } \lfloor \tfrac{l}{2} \rfloor + 1 \leqslant i \leqslant 2\lfloor \tfrac{l}{2} \rfloor, 1 \leqslant j \leqslant 2t+1\\
f(v_{ij}) &= 1 && \text{for } 1 \leqslant i \leqslant l, j = 2t+2,,2t+4,\ldots,4t+1 = n-3\\
f(v_{ij}) &= 3 && \text{for } 1 \leqslant i \leqslant l, j = 2t+3,2t+5,\ldots,4t+2 = n-2
\end{aligned}$$

For the remaining vertices in the last two rows $v_{(n-1)j}$ and v_{nj}we divide the discussion into four cases depending on l.

Subcase 4.1: Suppose $l \equiv 0 \pmod 4$. Then $l = 4s$ where s is a positive integer. Then we define

$$\begin{aligned}
f(v_{(n-1)j}) &= 1 && \text{for } j = 1,3,5,\ldots,l-1\\
f(v_{(n-1)j}) &= 3 && \text{for } j = 2,4,\ldots,l\\
f(v_{nj}) &= 3 && \text{for } j = 1,3\ldots,\frac{l}{2}-1\\
f(v_{nj}) &= 2 && \text{for } j = \frac{l}{2}+1, \frac{l}{2}+3,\ldots,l-1
\end{aligned}$$

$$f(v_{nj}) = 4 \qquad\qquad \text{for } j = \frac{l}{2}+2, \frac{l}{2}+4, \ldots, l$$

It follows that

$$v_f(1) = (2t)\left(\frac{l}{2}\right) + \frac{3l}{4} = \frac{4tl+3l}{4} = \frac{(n-3)l+3l)}{4} = \frac{ln}{4}$$
$$v_f(2) = (2t+1)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{(n-1)l}{4} + \frac{l}{4} = \frac{ln}{4}$$
$$v_f(3) = (2t)\left(\frac{l}{2}\right) + \frac{3l}{4} = \frac{4tl+3l}{4} = \frac{(n-3)l+3l)}{4} = \frac{ln}{4}$$
$$v_f(4) = (2t+1)\left(\frac{l}{2}\right) + \frac{l}{4} = \frac{(n-1)l}{4} + \frac{l}{4} = \frac{ln}{4}$$
$$e_f(1) = \frac{2(2t)l^2}{2} + \frac{2l(\frac{l}{2})}{2} + \frac{l(\frac{l}{4})}{2} + + \frac{l(\frac{l}{2})}{2} = \frac{(n-3)l^2}{2} + l^2 = \frac{l^2(n-1)}{2}$$
$$e_f(0) = l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}.$$

We summarize the previous information in the following table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{ln}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 6

From Table 6 we have $|v_f(i) - v_f(j)| = 0 \leqslant 1$ for all $i, j = 1, 2, 3, 4$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph. Subcase 4.2: Suppose $l \equiv 1 \pmod 4$. Then $l = 1+4s$ for some positive integer s. Then we define f as we did in Subcase 4.1 by replacing l with $l-1 \equiv 0 \pmod 4$. That is f is defined for all v_{ij} as in the previous subcase, except v_{lj} for $1 \leqslant j \leqslant n$. We define f on v_{lj} as follows

$$f(v_{lj}) = \begin{cases} 2 & \text{if } j = 1, 3, \ldots, 2t+1, j = n-1 \\ 4 & \text{if } j = 2, 4, \ldots, 2t, j = n \\ 1 & \text{if } j = 2t+2, 2t+4, \ldots, n-3 \\ 3 & \text{if } j = 2t+3, 2t+5, \ldots, n-2 \end{cases}$$

This implies that

$$v_f(1) = \frac{2t(l-1)}{2} + \frac{3(l-1)}{4} + t$$

$$
\begin{aligned}
&= \frac{(\frac{n-3}{2})(l-1)}{2} + \frac{3(l-1)}{4} + \frac{n-3}{4} \\
&= \frac{nl-3}{4} \\
v_f(2) &= \frac{(2t+1)(l-1)}{2} + \frac{(l-1)}{4} + \frac{2t+2}{2} \\
&= (\frac{n-1}{4})(l-1) + \frac{(l-1)}{4} + \frac{\frac{n-3}{2}+2}{2} \\
&= \frac{ln+1}{4} \\
v_f(3) &= \Big(\frac{2t(l-1)}{2}\Big) + \frac{3(l-1)}{4} + t + 1 = \frac{nl-3}{4} + 1 = \frac{ln+1}{4} \\
v_f(4) &= \frac{ln+1}{4}
\end{aligned}
$$

<u>Note</u>: $v_f(1)+v_f(2)+v_f(3)+v_f(4) = \frac{ln}{4}$. It follows that, using the same idea as in Subcase 4.1, for $l-1 \equiv 0 \pmod 4$

$$
\begin{aligned}
e_f(1) &= \frac{2(2t)l^2}{2} + \frac{2l(\frac{l}{2})}{2} + \frac{l(\frac{l}{4})}{2} + \frac{l(\frac{l}{4})}{2} = \frac{(n-3)l^2}{2} + \frac{l^2}{2} + \frac{l^2}{4} + \frac{l^2}{4} = \frac{l^2(n-1)}{2} \\
e_f(0) &= l^2(n-1) - \frac{l^2(n-1)}{2} = \frac{l^2(n-1)}{2}.
\end{aligned}
$$

We summarize the previous information in the following table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{ln-3}{4}$	$\frac{ln+1}{4}$	$\frac{ln+1}{4}$	$\frac{ln}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 6

From Table 6 above, we have $|v_f(i)-v_f(j)| = 0$ or $1 \leqslant 1$ for all $i,j = 1,2,3,4$ and $|e_f(0)-e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

<u>Subcase 4.3</u>: Suppose that $l \equiv 2 \pmod 4$. That is $l = 2+4s$ for some positive integer s. We define $f\colon V(T_l(P_n)) \to \{1,2,3,4\}$ as follows:

$$
\begin{aligned}
f(v_{ij}) &= 2 && \text{for } 1 \leqslant i \leqslant \frac{l}{2}, 1 \leqslant j \leqslant 2t+1 \\
f(v_{ij}) &= 4 && \text{for } \frac{l}{2}+1 \leqslant i \leqslant l, 1 \leqslant j \leqslant 2t+1
\end{aligned}
$$

$$
\begin{aligned}
f(v_{i(2t+2)}) &= 1 && \text{for } 1 \leqslant i \leqslant \lfloor \tfrac{l}{4} \rfloor = \frac{l-2}{4} \\
f(v_{i(2t+2)}) &= 3 && \text{for } \frac{l+2}{4} \leqslant i \leqslant \frac{2l-4}{4} \\
f(v_{i(2t+2)}) &= 2 && \text{for } \frac{l}{2} \leqslant i \leqslant \frac{3(l-2)}{4} \\
f(v_{i(2t+2)}) &= 4 && \text{for } \frac{3l-2}{4} \leqslant i \leqslant l-2 \\
f(v_{ij}) &= 1 && \text{for } 1 \leqslant i \leqslant \frac{l}{2}, 2t+3 \leqslant n = 4t+3 \\
f(v_{ij}) &= 3 && \text{for } \frac{l}{2}+1 \leqslant i \leqslant l, 2t+3 \leqslant j \leqslant n = 4t+3 \\
f(v_{(l-1)i(2t+2)}) &= 3 \\
f(v_{l(2t+2)}) &= 4
\end{aligned}
$$

This implies that

$$
\begin{aligned}
v_f(1) &= (2t+1)\frac{l}{2} + \lfloor \tfrac{l}{4} \rfloor = \frac{n-1}{2}\frac{l}{2} + \frac{l-2}{4} = \frac{nl-2}{4} \\
v_f(2) &= (2t+1)\frac{l}{2} + \lfloor \tfrac{l}{4} \rfloor = \frac{nl-2}{4} \\
v_f(3) &= (2t+1)\frac{l}{2} + \lfloor \tfrac{l}{4} \rfloor + 1 = \frac{nl+2}{4} \\
v_f(4) &= (2t+1)\frac{l}{2} + \lfloor \tfrac{l}{4} \rfloor + 1 = \frac{nl+2}{4} \\
e_f(1) &= \frac{2(2t+1)l}{2} - \frac{l^2}{2} - \frac{l}{2}(\frac{l-2}{4}) + 3(\frac{l-2}{4})\frac{l}{2} + \frac{l}{2} \\
&= l^2(\frac{n-1}{2}) - \frac{4l^2 + l^2 - 3l^2 + 6l - 4l - 2l^2}{8} = l^2(\frac{n-1}{2}) \\
e_f(0) &= l^2(n-1) - l^2(\frac{n-1}{2}) = l^2(\frac{n-1}{2}).
\end{aligned}
$$

We summarize the previous results in the following table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(1)$	$e_f(0)$
$\frac{nl-2}{4}$	$\frac{nl-2}{4}$	$\frac{nl+2}{4}$	$\frac{nl+2}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Table 7

From Table 7 we conclude that $|v_f(i) - v_f(j)| = 0$ or $1 \leqslant 1$ for all $i, j = 1,2,3,4$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

Subcase 4.4: Suppose that $l \equiv 3 \pmod 4$ That is $l = 3+4s$ for some positive integer s. We define $f\colon V(T_l(P_n)) \to \{1,2,3,4\}$ as follows:

$$
\begin{aligned}
f(v_{ij}) &= 2 && \text{for } 1 \leqslant i \leqslant \frac{l-1}{2}, 1 \leqslant j \leqslant 2t+1\\
f(v_{ij}) &= 4 && \text{for } \frac{l+1}{2}+1 \leqslant i \leqslant l-1, 1 \leqslant j \leqslant 2t+1\\
f(v_{lj}) &= 2,4,2,\dots,4,2 && \text{for } j = 1,2,3,\dots,2t+1\\
f(v_{i(2t+2)}) &= 1 && \text{for } 1 \leqslant i \leqslant \lfloor \tfrac{l}{4} \rfloor = \frac{l-3}{2}\\
f(v_{i(2t+2}) &= 3 && \text{for } \frac{l+1}{4} \leqslant i \leqslant \frac{2l-6}{4}\\
f(v_{i(2t+2}) &= 2 && \text{for } \frac{2l-2}{4} \leqslant i \leqslant \frac{2l-9}{4}\\
f(v_{i(2t+2}) &= 4 && \text{for } \frac{3l-5}{4} \leqslant i \leqslant \frac{4l-12}{4} = l-3\\
f(v_{ij}) &= 1 && \text{for } 1 \leqslant i \leqslant \frac{l-1}{2}, 2t+3 \leqslant j \leqslant n = 4t+3\\
f(v_{ij}) &= 3 && \text{for } 1 \leqslant i \leqslant \frac{l+1}{2}, 2t+3 \leqslant j \leqslant n = 4t+3\\
f(v_{ij}) &= 1,3,1,3,\dots,3, && \text{for } j = 2t+3, 2t+4, 2t+5, \dots, 4t+3\\
f(v_{(l-2)(2t+2)} &= 2\\
f(v_{(l-1)(2t+2)} &= 3\\
f(v_{l(2t+2)} &= 4
\end{aligned}
$$

It follows that

$$
\begin{aligned}
v_f(1) &= (2t+1)\frac{l-1}{2}+t+\frac{l-3}{4} = \frac{n-1}{2}\cdot\frac{l-1}{2}+\frac{n+1}{4}+\frac{l-3}{4} = \frac{nl-1}{4}\\
v_f(2) &= (2t+1)\frac{l-1}{2}+t+1+\frac{l-3}{4} = \frac{(n-1)(l-1)+n+1+l-3}{4} = \frac{nl+1}{4}\\
v_f(3) &= (2t+1)\frac{l-1}{2}+t+\frac{l-3}{4}+1 = \frac{nl-1}{4}\\
v_f(4) &= (2t+1)\frac{l-1}{2}+t+\frac{l-3}{4}+1 = \frac{nl-1}{4}\\
e_f(1) &= \frac{2l(2t+1)}{2}+2l(\tfrac{l-2}{2})+\frac{3l-1}{2}-l^2+\frac{l-1}{2} = \frac{l^2(n-1)}{2}\\
e_f(0) &= l^2(n-1)-l^2(\frac{n-1}{2}) = l^2(\frac{n-1}{2}).
\end{aligned}
$$

We summarize the previous results in the following table

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(1)$	$e_f(0)$
$\frac{nl-1}{4}$	$\frac{nl+2}{4}$	$\frac{nl-1}{4}$	$\frac{nl-1}{4}$	$\frac{l^2(n-1)}{2}$	$\frac{l^2(n-1)}{2}$

Note that $v_f(1)+v_f(2)+v_f(3)+v_f(4)=nl=|V(G)|$. Also note that $|v_f(i)-v_f(j)|=0$ or $1 \leqslant 1$ for all $i,j=1,2,3,4$ and $|e_f(0)-e_f(1)|=0 \leqslant 1$, so that the graph is 4-prime cordial graph.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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