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ORE EXTENSIONS OVER NOETHERIAN δ-RINGS

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Abstract. Let R be a ring. Let σ be an automorphism of R such that $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and δ a σ-derivation of R such that $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R. In this paper we show that if R is Noetherian ring such that $\sigma(P) = P$ for all minimal prime ideal P of R, then $R[x; \sigma, \delta]$ is 2-primal.

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1. Introduction

Throughout this paper R will denote an associative ring with identity $1 \neq 0$. The field of complex numbers is denoted by $\mathbb{C}$, the set of real numbers is denoted by $\mathbb{R}$ and the field of rational numbers is denoted by $\mathbb{Q}$, the ring of integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The set of prime ideals of R is denoted by $Spec(R)$. The set of minimal prime ideals of R is denoted by $MinSpec(R)$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

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Let R be a ring and σ an endomorphism of R. Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all a, $b \in R$. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R.

Let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a map defined by $\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then δ is a σ-derivation of R if and only if ϕ is a ring homomorphism. Also let $R = K[x]$, K a field. Then the formal derivative d/dx is a derivation of R.

Recall that $R[x; \sigma, \delta]$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \Sigma_{i=0}^{n} x^i a_i$. We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. An ideal I of a ring R is called σ-invariant if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $I[x; \sigma, \delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$. In the case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$. In the case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$.

2-primal rings

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $O(R)$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [10], it has been shown that for a local ring R with a nilpotent maximal ideal, the Ore extension $O(R)$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $O(R)$ is 2-primal, it will satisfy an even stronger condition; in the case where $O(R)$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [8]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e., if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [[1]- [3], [8], [10]].

Example 1.1

(1) Let $R = F[x]$ be the polynomial ring over the field F. Then R is 2-primal with $P(R) = \{0\}$.

(2) Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so cannot be 2-primal.

Now let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$, σ be an automorphism of R such that $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and δ be a σ-derivation of R such that $a\delta(a) \in P(R)$ implies $a \in P(R)$. Let $P \in MinSpecR$. Then $P(O(R) = O(P(R))$ if and only if $O(R)$ is 2-primal. This is proved in Theorem (3.3).

2. Ore extensions

Definition 2.1 $\sigma(*)$**-Ring:** In Kwak [9], a ring R is said to be $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 2.2

(1) Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a + ib) = a - ib; a, b \in \mathbb{R}$. Then R is $\sigma(*)$-ring.

(2) Let $R = F[x]$ be the polynomial ring over the field F. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$.
Then R is not a $\sigma(*)$-ring.

Definition 2.3 δ**-Ring:** Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$.

Proposition 2.4 *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies R is 2-primal.*

Proof.

Let $a \in R$ be such that $a^2 \in P(R)$. Then
$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a)$
$\in \sigma(P(R)) = P(R)$.
Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.
i.e., every $\sigma(*)$-ring is a 2-primal ring but converse need not be true.

Example 2.5 Let $R = F[x]$ be the polynomial ring over the field F. Then R is 2-primal with $P(R) = \{0\}$.
Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$.
Then R is not a $\sigma(*)$-ring.

Proposition 2.6 *Let R be a 2-primal ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof.

Let $P \in MinSpec(R)$. Now P is a completely prime ideal, therefore, for any $a \in P$, there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of Shin [11]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \subseteq P(R)$; i.e., $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Now $\sigma(P) = P$ implies that $\sigma(b) \notin P$ and since P is completely prime in R, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$.

Theorem 2.7 *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof.

Define a map $\rho : R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that τ is an automorphism of $R/P(R)$ and ρ is a τ-derivation of $R/P(R)$. Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in $R/P(R)$. Thus as in Proposition (5) of Hong, Kim and Kwak [7], R is a reduced ring and, therefore as mentioned in introduction, R is 2-primal.

Proposition 2.8 *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then:*

(1) *For any completely prime ideal P of R with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.*

(2) *For any completely prime ideal U of $O(R)$, $U \cap R$ is completely prime ideal of R.*

Proof.

See Proposition (4) of [4].

Corollary 2.9 *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is moreover a δ-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Proof.

R is 2-primal by Theorem (2.7), and so by Proposition (2.6) $\delta(P) \subseteq P$. Further more as mentioned in Proposition (2.6) above, P is a completely prime ideal of R. Now use Proposition (2.8), and the proof is complete.

3. Main results

Proposition 3.1 *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ be a σ-derivation of R such that R is a δ-ring. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$ then $\delta(P) \subseteq P$.*

Proof.

Let $P \in MinSpec(R)$. Then by Proposition (1.1) of [3] $\delta(P(R)) \subseteq P(R)$ and by Theorem (2.7) R is 2-primal. Since $\sigma(P) = P$ by Proposition (2.6) $\delta(P) \subseteq P$.

Corollary 3.2 *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that R is a δ-ring. Let $P \in MinSpec(R)$. Then $O(P)$ is completely prime ideal of $O(R)$.*

Proof.

Let $P \in MinSpec(R)$. Then $\sigma(P) = P$ by Theorem (2) of [6] and $\delta(P) \subseteq P$ by Proposition (3.1). Also P is completely prime ideal of R by Theorem (2) of [6]. Now use Proposition (2.8), and the proof is complete.

We now prove the following Theorem, which is crucial in proving Theorem (3.5).

Theorem 3.3 *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that R is a δ-ring. Let $P \in MinSpec(R)$. Then $P(O(R)) = O(P(R))$ if and only if $O(R)$ is 2-primal.*

Proof.

Suppose $P(O(R)) = O(P(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \Sigma_{i=0}^{n} x^i b_i \in O(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in MinSpec(R)$. Now $\sigma(P) = P$ and P is completely prime by Theorem (2) of [6]. Also, R is 2-primal by Proposition (2.4). Therefore we have $b_n \in P$, for all $P \in MinSpec(R)$; i.e., $b_n \in P(R)$. Now since $\delta(P) \subseteq P$ for all $P \in MinSpec(R)$ by Proposition (3.1), we get $(\Sigma_{i=o}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in O(P(R))$, i.e., $g(x) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal.

Conversely, let $O(R)$ be 2-primal. Now by Corollary (3.2) $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \Sigma_{j=0}^{n} x^j a_j \in O(P(R))$. Now R is a 2-primal subring of $O(R)$ by Proposition (2.4) which implies that a_j is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $P(O(R)) = O(P(R))$.

Theorem 3.4 *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

1. *$P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*
2. *$P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$.*

Proof.

See Theorem (2.3) of [5].

Theorem 3.5 *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that R is a δ-ring. If $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal.*

Proof.

Let $P_1 \in MinSpec(R)$. Since R is 2-primal, $\sigma(P_1) = P_1$, and therefore Theorem (3.4) implies $O(P_1) \in MinSpec(O(R))$. Similarly for any $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ Theorem (3.4) implies that $P \cap R \in MinSpec(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using Theorem (3.3).

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