Available online at http://scik.org
J. Math. Comput. Sci. 2025, 15:8
https://doi.org/10.28919/jmcs/9357
ISSN: 1927-5307

MATHEMATICAL MODELING AND ANALYSIS OF THE DYNAMICS OF AN ECOLOGICAL SYSTEM WITH PREY REFUGE AND TWO TIME DELAYS

SOUKAINA BOUZIANE[1,*], KHALID HATTAF[1,2], JAMAL MOULINE[1]

[1]Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Sick, Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco

[2]Equipe de Recherche en Modélisation et Enseignement des Mathématiques (ERMEM), Centre Régional des Métiers de l'Education et de la Formation (CRMEF), 20340 Derb Ghalef, Casablanca, Morocco

Abstract. This paper proposes a prey-predator model with the Hattaf-Yousfi functional response, incorporating a prey refuge and two time delays. Prey refuge is crucial for stabilizing dynamics and preventing the extinction of prey species, as it provides a way for prey to avoid predation. In addition, the paper presents the mathematical analysis of the proposed model and discusses the stability results. Finally, some numerical simulations are given to illustrate the theoretical analysis.

Keywords: ecology; prey-predator; delays; refuge; stability; Hopf-bifurcation; Hattaf-Yousfi functional response.

2020 AMS Subject Classification: 34D20, 37G10, 92D25.

1. INTRODUCTION

Ecological modeling is a fundamental tool for understanding and managing ecosystems. Furthermore, it plays an important role in explaining the dynamics of interactions between organisms and their environment. One of the first major contributions to ecological modeling dates back to 1798, when Malthus [1] introduced his work on human population dynamics, his essay

---

[*] Corresponding author

E-mail address: soukainabouziane4@gmail.com

Received May 16, 2025

proposed the concept of exponential population growth tends to outpace the growth of resources essential for human survival. However, it was not until the early $20th$ century that mathematical modeling of biological interactions took a decisive turn with the contributions of Lotka [2] and Volterra [3]. They developed the renowned prey-predator model, which mathematically describes the dynamics of species coexistence and population regulation within ecosystems. Prey-predator models are typically expressed using differential equations or delay differential equations, enabling researchers to capture the complexity of ecological interactions and predict population evolution under various ecological conditions. Their impact remains fundamental in modern studies of population dynamics and ecosystem management. For a more detailed exploration of the dynamics of such models, we refer to the work of Tripathi et al. [4], who proposed a delayed prey-predator model incorporating the Crowley-Martin functional response [5]. More recently, in 2022, Bouziane et al. [6] extended the model of [4] by introducing the Hattaf-Yousfi functional response [7], along with a delay in the second equation to represent the gestation period of predators.

In ecosystems, the refuge parameter is defined as any mechanism that reduces predation risk and helps maintain ecological stability. In systems where predators and prey interact, the prey refuge regulates prey-predator dynamics and prevents the extinction of prey species, as well as promotes biodiversity within ecosystems. The refuge can manifest in diverse forms, including the physical barriers like burrows, dense vegetation and rocky shelters, the behavioural adaptations like nocturnal activity to evade diurnal predators, as well as the population-level strategies like migration to regions with reduced predation risk. Mathematical models incorporating the refuge parameter have been studied and analyzed by many researchers in ecology to understand the influence of refuge on prey-predator models. In 2005, Kar [8] introduced a prey-predator model with Holling type II functional response [9] incorporating a prey refuge. Simultaneously, Tripathi et al. [10] studied a prey-predator model using the Beddington-DeAngelis functional response [11, 12]. Recently, Peng et al. [13] investigated a delayed prey-predator model with a prey refuge where the predator population eats the prey according to the Beddington-DeAngelis type functional response. However, it is known that the functional response is an essential factor in ecology which describes the interaction between prey and predator. Therefore, it interesting to study the dynamics of ecosystems by a general functional response like Hattaf-Yousfi that includes the Holling type II, the Beddington-DeAngelis and the Crowley-Martin functional responses. Additionally, this functional response it will be interesting, it used in various aeras of science such as virology [14, 15], epidemology [16], economics [17] and ecology [6, 18].

Motivated by the above ecological and mathematical considerations, we propose a mathematical model that describes the dynamics of prey-predator interaction taking into account refuge and two time delays. Into our proposed model, the first delay represents the feedback of preys, while the second ones describes the gestation period of predators. Furthermore, our proposed model extends and generalises numerous prey-predator models presented in the literature [2, 3, 4, 6, 18, 8, 10, 13] by considering other important ecological factors.

The present study is organized as follows. The next section is devoted to the formulation of our proposed model and the existence of equilibria. Section 3 establishes the local stability and the sufficient conditions of the existence of Hopf bifurcation. Section 4 provides the numerical simulations of our theoretical results. Finally, the study ends with a brief conclusion.

2. FORMULATION OF THE MODEL AND EQUILIBRIA

This section presents the formulation of our prey-predator model, discusses the existence of non-negative solutions and equilibria.

First, we formulate our model by delay differential equation as follows

$$(1)\quad \begin{cases} \frac{dX(t)}{dt} = rX(t)(1-\frac{X(t-\tau_1)}{K}) - \frac{a(1-m)X(t)Y(t)}{\alpha_0+\alpha_1(1-m)X(t)+\alpha_2 Y(t)+\alpha_3(1-m)X(t)Y(t)}, \\ \frac{dY(t)}{dt} = \frac{ab(1-m)X(t-\tau_2)Y(t-\tau_2)}{\alpha_0+\alpha_1(1-m)X(t-\tau_2)+\alpha_2 Y(t-\tau_2)+\alpha_3(1-m)X(t-\tau_2)Y(t-\tau_2)} - cY(t) - dY^2(t), \end{cases}$$

where $X(t)$ and $Y(t)$ denote the prey and predator densities at time t, respectively. r is the prey intrinsic growth rate; τ_1 is the feedback delay about the prey population; K is the environmental carrying capacity for prey; $m \in [0,1)$ is the refuge parameter; $(1-m)X$ represents the amount of preys available to the predators; b is the conversion rate of prey to predator; c is the death rate of predator; d is the rate of competition between predators and τ_2 is a time delay that represents the gestation period of predators. The interaction between prey and predator is modeled by Hattaf-Yousfi functional response of the form $\frac{aXY}{\alpha_0+\alpha_1 X+\alpha_2 Y+\alpha_3 XY}$, where a represents the rate of prey capture by the predator called also consumption rate, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \geq 0$ are the saturation factors measuring the inhibitory or psychological effect.

Next, we focus on non-negativity of solutions. Firstly, we introduce the necessary notations.

Let $\tau = \max\{\tau_1, \tau_2\}$ and $\mathscr{C} = C([-\tau, 0], \mathbb{R}_+^2)$ be the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbb{R}_+^2$ with the usual supremum norm. Based on the fundamental theory of functional differential equations [19], it is evident that there exists a unique solution $(X(t), Y(t))$ of system (1) with initial conditions $(X_0, Y_0) \in \mathscr{C}$. For ecological reasons, we suppose that the initial conditions for system (1) satisfy

$$X_0(s) = \phi_1(s), \quad Y_0(s) = \phi_2(s), \qquad \text{for all } s \in [-\tau, 0]. \tag{2}$$

Theorem 2.1. *All the solutions of system* (1) *starting in* $\mathbb{R}_+^2$ *stay non-negative.*

Proof. From (1), we have

$$\begin{aligned} X(t) &= \phi_1(0) e^{\int_0^t \left[r\left(1 - \frac{X(s-\tau_1)}{K}\right) - \frac{a(1-m)Y(s)}{\alpha_0 + \alpha_1(1-m)X(s) + \alpha_2 Y(s) + \alpha_3(1-m)X(s)Y(s)} \right] ds}, \\ Y(t) &= \phi_2(0) e^{\int_0^t \left[\frac{ab(1-m)X(s-\tau_2)Y(s-\tau_2)}{Y(s)(\alpha_0 + \alpha_1(1-m)X(s-\tau_2) + \alpha_2 Y(s-\tau_2) + \alpha_3(1-m)X(s-\tau_2)Y(s-\tau_2))} - c - dY(s) \right] ds}, \end{aligned}$$

Since $\phi_1(0) \geq 0$ and $\phi_2(0) \geq 0$, we have $X(t)$ and $Y(t)$ are non-negative. ■

Clearly, system (1) exhibits three equilibrium points that are :

(i) The trivial equilibrium $E^0(0,0)$ that denotes the absence of prey and predator.

(ii) The predator free axial equilibrium $E^1(K,0)$ called also the prey population, it reaches in the carrying capacity in the absence of predators.

(iii) The coexistence equilibrium point $E^*(X^*, Y^*)$ that denotes the steady state of coexistence between prey and predators. By a simple computation similar as in [6], we have $Y^* = \frac{r\left(1 - \frac{X^*}{K}\right)[\alpha_0 + \alpha_1(1-m)X^*]}{a(1-m) - r\left(1 - \frac{X^*}{K}\right)[\alpha_2 + \alpha_3(1-m)X^*]}$ and X^* is the unique positive solution of the following algebraic equation :

$$a_5 X^5 + a_4 X^4 + a_3 X^3 + a_2 X^2 + a_1 X + a_0 = 0, \tag{3}$$

where

$$a_5 = \frac{ab\alpha_3^2(1-m)^2 K^3}{r\alpha_0^3},$$

$$\begin{aligned}
a_4 &= \frac{2ab\alpha_3(1-m)K^2}{r^2\alpha_0^3}(r\alpha_2-a),\\
a_3 &= \frac{a\alpha_1(1-m)^2K}{r^2\alpha_0^3}(d\alpha_1K-c\alpha_3K)+\frac{abK}{r^2\alpha_0^3}(r\alpha_2^2+r\alpha_3^2(1-m)^2K^2+2a\alpha_3(1-m)K\\
&\quad -4r\alpha_2\alpha_3(1-m)K),\\
a_2 &= \frac{a}{r^2\alpha_0^3}(-c\alpha_0\alpha_3(1-m)K+c\alpha_1(1-m)K(\alpha_3(1-m)K-\alpha_2))\\
&\quad -\frac{ad\alpha_1(1-m)K}{r^2\alpha_0^3}(\alpha_1(1-m)K+2\alpha_0)+\frac{2abK}{r^2\alpha_0^3}(a\alpha_2-a\alpha_3(1-m)K-r\alpha_2^2+r\alpha_2\alpha_3(1-m)K),\\
a_1 &= \frac{ac}{r\alpha_0^3}(\alpha_3(1-m)K-\alpha_2)+\frac{ac\alpha_1(1-m)K}{r^3\alpha_0^3}(r\alpha_2-a)-\frac{ad}{r^2\alpha_0^2}(\alpha_0+2\alpha_1(1-m)K)\\
&\quad +\frac{abK}{r^3\alpha_0^3}(a^2-r^2\alpha_2^2-2ra\alpha_2),\\
a_0 &= \frac{ac}{r^3\alpha_0^2}(r\alpha_2-a)-\frac{ad}{r^2\alpha_0}.
\end{aligned}$$

3. STABILITY ANALYSIS AND HOPF-BIFURCATION

This section analyses the stability of the three equilibria and investigates the existence of Hopf bifurcation in different cases of delays.

Theorem 3.1. *The trivial equilibrium* $E^0(0,0)$ *is unstable.*

Proof. At $E^0(0,0)$, the characteristic equation of system (1) is given by

$$(\lambda-r)(c+\lambda)=0, \tag{4}$$

where the roots of Eq. (4) are $\lambda_1=r>0$ and $\lambda_2=-c<0$, which implies that $E^0(0,0)$ is unstable. ■

Theorem 3.2. *Let* $R_0=\frac{ab(1-m)K}{c(\alpha_0+\alpha_1(1-m)K)}$. *The predator free axial equilibrium* $E^1(K,0)$ *is locally asymptotically stable if* $R_0<1$ *and it becomes unstable if* $R_0>1$.

Proof. At $E^1(K,0)$, the characteristic equation of system (1) is given by

$$\left(\lambda+re^{-\lambda\tau_1}\right)\left(\lambda-c(R_0e^{-\lambda\tau_2}-1)\right)=0. \tag{5}$$

Done $\lambda_1=-re^{-\lambda_1\tau_1}$ and $\lambda_2=c(R_0e^{-\lambda_2\tau_2}-1)$. Obviously, $\lambda_1<0$ for all $\tau_1\geq 0$ and $\lambda_2<0$ if $R_0<1$ for all $\tau_2\geq 0$. Hence, $E^1(K,0)$ is locally asymptotically stable when $R_0<1$.

For the case $R_0 > 1$, we consider the following function

$$f(\lambda) = \lambda - c(R_0 e^{-\lambda\tau_2} - 1).$$

We have $f(0) = -c(R_0 - 1) < 0$ and $\lim\limits_{\lambda\to+\infty} f(\lambda) = +\infty$. Then there exists a $\bar{\lambda} \in (0, +\infty)$ such that $f(\bar{\lambda}) = 0$. Hence, $E^1(K, 0)$ is unstable when $R_0 > 1$. ■

Now, we discuss the stability of the coexistence equilibrium $E^*(X^*, Y^*)$. The corresponding characteristic equation of system (1) at E^* is given by

$$\lambda^2 + A\lambda + B + (C\lambda + D)e^{-\lambda\tau_1} + (E\lambda + F)e^{-\lambda\tau_2} + Ge^{-\lambda(\tau_1+\tau_2)} = 0, \tag{6}$$

where

$$\begin{aligned}
A &= -r\left(1 - \frac{X^*}{K}\right) + \frac{a(1-m)Y^*(\alpha_0 + \alpha_2 Y^*)}{(\alpha_0 + \alpha_1(1-m)X^* + \alpha_2 Y^* + \alpha_3(1-m)X^*Y^*)^2} + c + 2dY^*,\\
B &= \left(-r\left(1 - \frac{X^*}{K}\right) + \frac{a(1-m)Y^*(\alpha_0 + \alpha_2 Y^*)}{(\alpha_0 + \alpha_1(1-m)X^* + \alpha_2 Y^* + \alpha_3(1-m)X^*Y^*)^2}\right)(c + 2dY^*),\\
C &= \frac{r}{K}X^*,\\
D &= \frac{r}{K}X^*(c + 2dY^*),\\
E &= \frac{-ab(1-m)X^*(\alpha_0 + \alpha_1(1-m)X^*)}{(\alpha_0 + \alpha_1(1-m)X^* + \alpha_2 Y^* + \alpha_3(1-m)X^*Y^*)^2},\\
F &= r\left(1 - \frac{X^*}{K}\right)\left(\frac{ab(1-m)X^*(\alpha_0 + \alpha_1(1-m)X^*)}{(\alpha_0 + \alpha_1(1-m)X^* + \alpha_2 Y^* + \alpha_3(1-m)X^*Y^*)^2}\right),\\
G &= \frac{-rX^*}{K}\left(\frac{ab(1-m)X^*(\alpha_0 + \alpha_1(1-m)X^*)}{(\alpha_0 + \alpha_1(1-m)X^* + \alpha_2 Y^* + \alpha_3(1-m)X^*Y^*)^2}\right).
\end{aligned}$$

So, we distinguish three cases.

The case $\tau_1 = \tau_2 = 0$. In this case, the corresponding characteristic equation (6) becomes

$$\lambda^2 + (A + C + E)\lambda + B + D + F + G = 0. \tag{7}$$

Based on Routh-Hurwitz criteron, all the roots of Eq. (7) have negative real parts if and only if

$$A + C + E > 0 \quad , \qquad B + D + F + G > 0. \qquad (C_0)$$

Then the equilibrium E^* is locally asymptotically stable if the condition (C_0) is satisfied.

The case $\tau_1 = 0$ **and** $\tau_2 \neq 0$**.** For $\tau_1 = 0$ and $\tau_2 \neq 0$, then Eq. (6) becomes

$$\lambda^2 + (A + C)\lambda + B + D + (E\lambda + F + G)e^{-\tau_2} = 0. \tag{8}$$

Let $\lambda = i\omega(\omega > 0)$ be a root of (8) and separating real and imaginary parts, we have

$$\begin{cases} (A+C)\omega & = (F+G)\sin(\omega\tau_2) - E\omega\cos(\omega\tau_2), \\ \omega^2 - (B+D) & = (F+G)\cos(\omega\tau_2) + E\omega\sin(\omega\tau_2), \end{cases} \tag{9}$$

which implies that

$$\omega^4 + ((A+C)^2 - E^2 - 2(B+D))\omega^2 + (B+D)^2 - (F+G)^2 = 0. \tag{10}$$

Let $z = \omega^2$, Eq. (10) becomes

$$g(z) := z^2 + b_1 z + b_0 = 0, \tag{11}$$

where $b_1 = (A+C)^2 - E^2 - 2(B+D)$ and $b_0 = (B+D)^2 - (F+G)^2$.

It is easy to show the following result.

Lemma 3.3.

(i) If $b_0 < 0$, then Eq. (11) has at least one positive root.

(ii) If $b_0 \geq 0, \quad \Delta = 0$ and $b_1 < 0$, then Eq. (11) has one positive root.

(iii) If $b_0 \geq 0, \quad \Delta > 0$ and $b_1 < 0$, then Eq. (11) has two positive roots.

(iv) If $b_0 \geq 0, \quad \Delta < 0$ or $b_1 \geq 0$, then Eq. (11) has no positive roots.

So, we consider the following conditions :

(C_1) $b_0 < 0$;

(C_2) $b_0 \geq 0, \quad \Delta = 0$ and $b_1 < 0$;

(C_3) $b_0 \geq 0, \quad \Delta > 0$ and $b_1 < 0$.

(C_4) $b_0 \geq 0, \quad \Delta < 0$ or $b_1 \geq 0$.

Theorem 3.4. *For $\tau_1 = 0$, assume that the condition (C_0) is satisfied. If the condition (C_4) holds, then the equilibrium E^* is locally asymptotically stable for all $\tau_2 \geq 0$.*

Next, we show the transversality condition for the Hopf bifurcation by considering the delay τ_2 as a parameter of bifurcation. Without loss of generality, we assume that Eq. (11) has two positive roots, denoted by z_1 and z_2 with $z_1 < z_2$. Then Eq. (10) has two positive roots $\omega_1 = \sqrt{z_1}$ and $\omega_2 = \sqrt{z_2}$.

From (9), we obtain

$$\tau_{2,n}^k = \frac{1}{\omega_k} \arccos\left\{ \frac{((F+G)-(A+C)E)\omega_k^2 - (B+D)(F+G)}{(F+G)^2 + E^2\omega_k^2} \right\} + \frac{2n\pi}{\omega_k}, \tag{12}$$

where $k = 1,2$ and $n \in \mathbb{N}$. Then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (8) with $\tau_2 = \tau_{2,n}^k$. Define

$$\tau_{2,0} = \tau_{2,0}^{k_0} = \min_{1 \le k \le 2}\{\tau_{2,0}^k\} \quad \text{and} \quad \omega_0 = \omega_{k_0}. \tag{13}$$

Let $\lambda(\tau_2) = \mu(\tau_2) + i\omega(\tau_2)$ be a root of equation (8) satisfying $\mu(\tau_{2,n}^k) = 0$ and $\omega(\tau_{2,n}^k) = \omega_k$. Differentiating both sides of equation (8) with respect τ_2, we have

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{-2\lambda + (A+C)}{\lambda(\lambda^2 + (A+C)\lambda + (B+D))} + \frac{E}{\lambda(E\lambda + (F+G))} - \frac{\tau_2}{\lambda},$$

By a simple computation, we get

$$\begin{aligned} Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}|_{\tau_2 = \tau_{2,n}^k} &= \frac{2\omega_k^2 + (A+C)^2 - 2(B+D) - E^2}{E^2\omega_k^2 + (F+G)^2} \\ &= \frac{g'(\omega_k^2)}{E^2\omega_k^2 + (F+G)^2}. \end{aligned}$$

Since, sign $\left\{\frac{d(Re\lambda)}{d\tau}|_{\tau_2=\tau_{2,n}^k}\right\}$ =sign $\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\tau_2=\tau_{2,n}^k}\right\} = sign\{g'(\omega_k^2)\}$. It is simple to find out that $g'(\omega_k) \neq 0$ for all $k = 1,2$. Hence, the transversality condition holds and we get the following result.

Theorem 3.5. *For $\tau_1 = 0$, assume that the condition (C_0) is satisfied, if one of the conditions $(C_1) - (C_3)$ holds, then the equilibrium E^* is locally asymptotically stable for $\tau_2 < \tau_{2,0}$ and becomes unstable for $\tau_2 > \tau_{2,0}$. Further, the system (1) undergoes a Hopf bifurcation at E^* when $\tau_2 = \tau_{2,n}^k$.*

The case $\tau_1 \neq 0$ and $\tau_2 \neq 0$. In this case, we consider Eq. (6) with τ_2 in the stable regions. Regarding τ_1 as a parameter of bifurcation. According to Ruan and Wei [20], we have the following lemma.

Lemma 3.6. *If all roots of equation (8) have negative real parts for $\tau_2 > 0$, then there exists a $\tau_1^*(\tau_2) > 0$, such that when $0 \le \tau_1 < \tau_1^*(\tau_2)$ all roots of equation (6) have negative real parts.*

Proof. The expression on the left hand side of Eq. (6) is analytic in λ and τ_1. We conclude from [20] that when τ_1 varies, the sum of the multiplicities of zeros of the left hand side of Eq. (6) in the open right half-plane can change only if a zero on or cross the imaginary axis. ■

Theorem 3.7. *For τ_2 in the stable regions and $\tau_1 > 0$, assume that the condition (C_0) is satisfied.*

(i) If the condition (C_4) holds, then for any delay $\tau_2 \geq 0$, there exists a $\tau_1^*(\tau_2)$ such that the equilibrium E^* is locally asymptotically stable, when $\tau_1 \in [0, \tau_1^*(\tau_2))$,

(ii) If one of the conditions $(C_1) - (C_3)$ holds, then for any delay $\tau_2 \in [0, \tau_{2,0})$, there exists a $\tau_1^*(\tau_2)$ such that the equilibrium E^* is locally asymptotically stable, when $\tau_1 \in [0, \tau_1^*(\tau_2))$.

Proof. It is clear that (i) is a direct consequence of the Lemma 3.3 (iv), Lemma 3.6, and Theorem 3.5.

Now, it remains to prove (ii). Based on Theorem 3.5, we conclude that the equilibrium E^* is locally asymptotically stable for $\tau_2 < \tau_{2,0}$. Moreover, all roots of Eq. (8) have negative real parts. According to Lemma 3.6, there exists a $\tau_1^*(\tau_2) > 0$, such that when $0 \leq \tau_1 \leq \tau_1^*(\tau_2)$ all roots of Eq. (6) have negative real parts. Then the equilibrium E^* is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*(\tau_2))$. ■

4. NUMERICAL SIMULATIONS

In this section, we give some numerical simulations to illustrate our analytical results. For these simulations, we choose $r = 1, K = 1, a = 5, m = 0.1, \alpha_0 = 1, \alpha_1 = 2.1, \alpha_2 = 1.1, \alpha_3 = 0.001, b = 1, c = 0.2, d = 0.01$. First, we investigate the impact of the two delays τ_1 and τ_2 on the dynamical behaviors of our model.

4.1. Impact of delays. For $\tau_1 = \tau_2 = 0$, we have $A + C + E = 0.0355 > 0$ and $B + D + F + G = 0.1332 > 0$. Hence, the condition (C_0) is satisfied. Then the positive equilibrium $E^*(0.0657, 0.3025)$ is locally asymptotically stable (see, Figure 1).

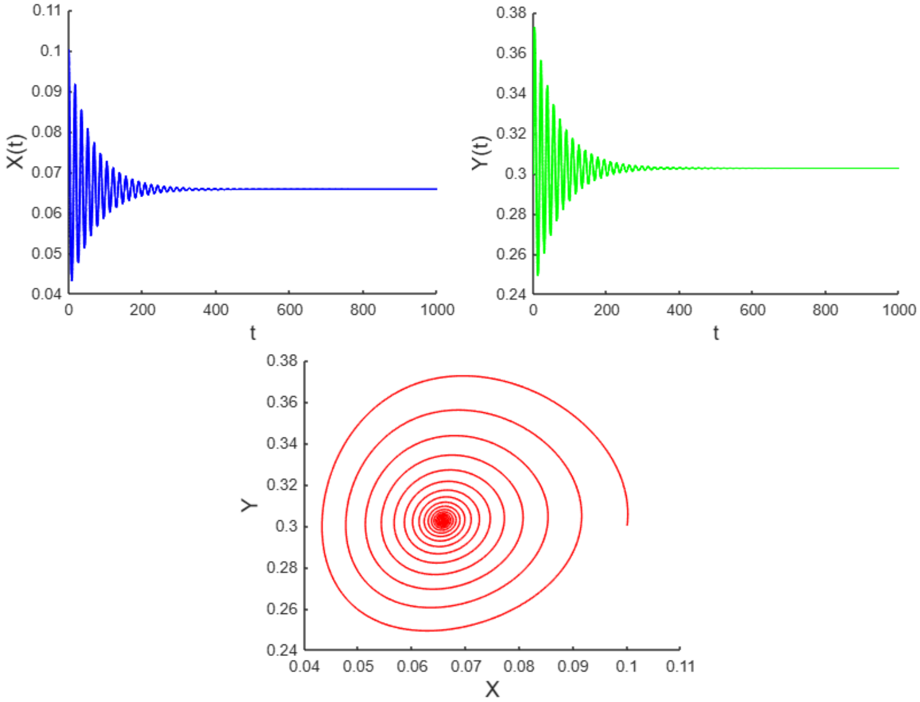

FIGURE 1. Dynamical behavior of system (1) at the coexistence equilibrium E^* when $\tau_1 = \tau_2 = 0$.

For $\tau_1 = 0$ and $\tau_2 \neq 0$, the condition $b_0 = -0.0185 < 0$ is satisfied. Then the positive equilibrium $E^*(0.0657, 0.3025)$ is locally asymptotically stable if $\tau_2 = 0.05 < \tau_{2,0}$, where $\tau_{2,0} = 4.24$. Figure 2 illustrates this result. However, when $\tau_2 = 4.3 > \tau_{2,0} = 4.24$ the positive equilibrium $E^*(0.0657, 0.3025)$ becomes unstable. Figure 3 demonstrates this finding. In addition, Figure 4 shows the dynamical behavior of our model when $\tau_1 = 1$ and $\tau_2 = 1.5 \in [0, \tau_{2,0})$.

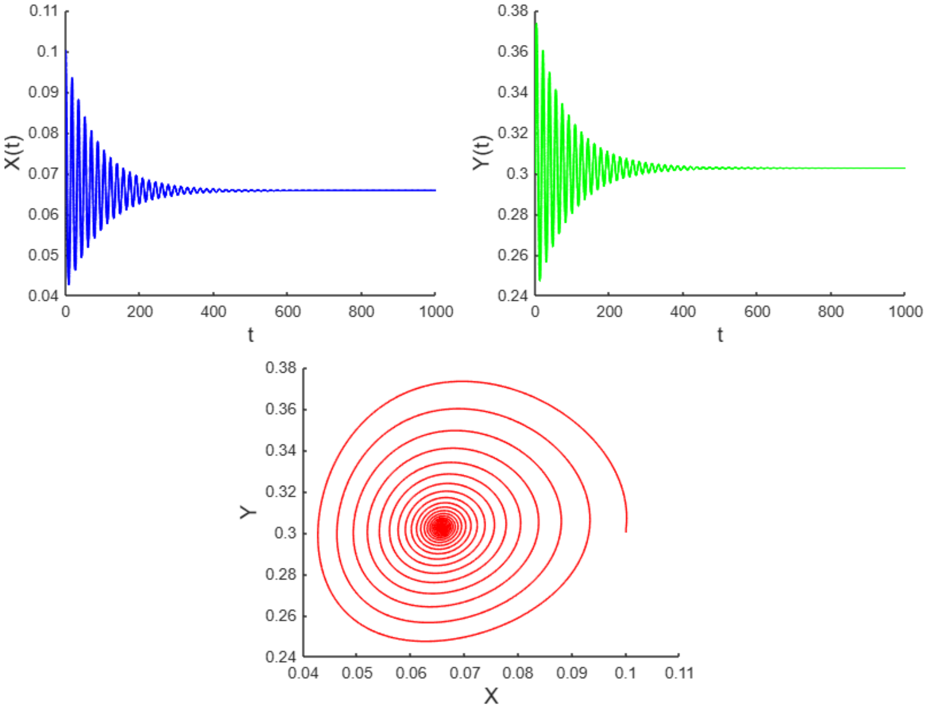

FIGURE 2. Dynamical behavior of system (1) at the coexistence equilibrium E^* when $\tau_2 = 0.05 < \tau_{2,0} = 4.24$.

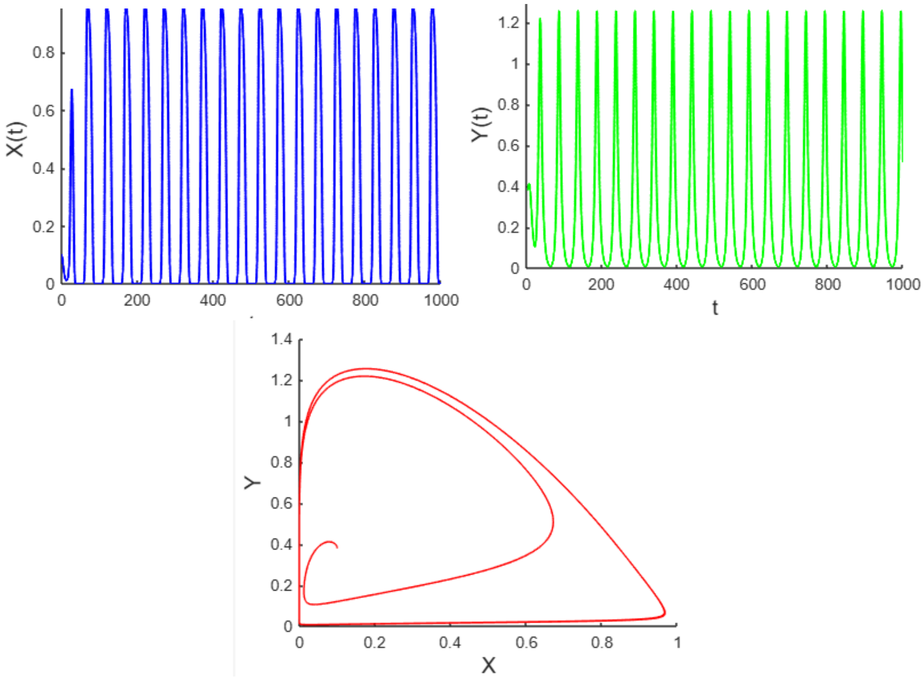

FIGURE 3. Dynamical behavior of system (1) at the coexistence equilibrium E^* when $\tau_2 = 4.3 > \tau_{2,0} = 4.24$

.

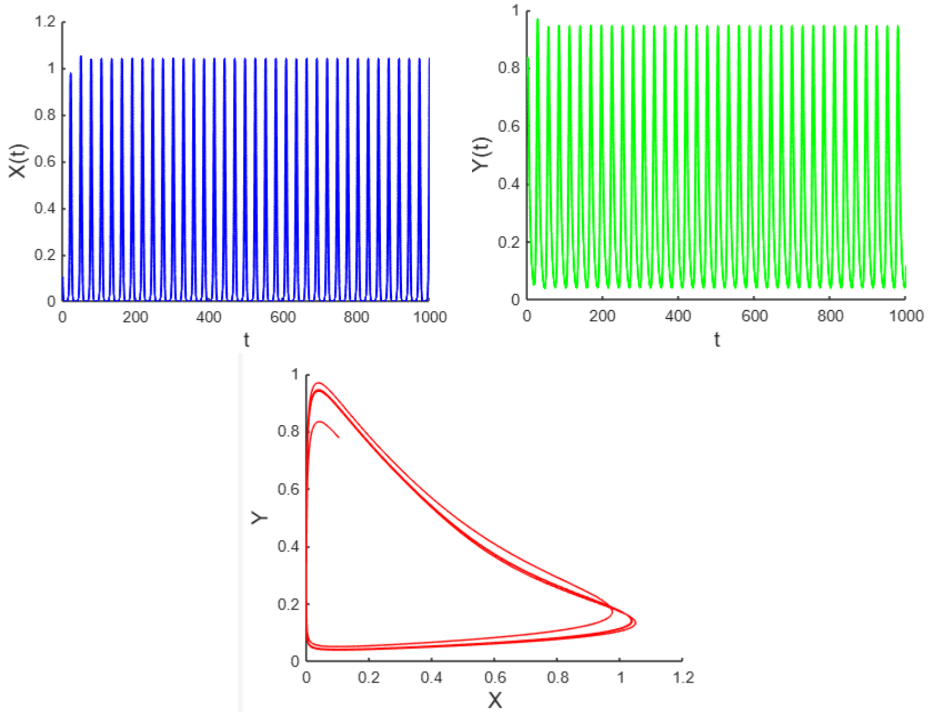

FIGURE 4. Dynamical behavior of system (1) at the coexistence equilibrium E^* when $\tau_1 = 1$ and $\tau_2 = 1.5$.

4.2. Impact of prey refuge. It follows from Theorem 3.2 that the dynamics of our model depends on the threshold parameter R_0. According to the expression of R_0 given in this Theorem, we can deduce that $R_0 = R_0(m)$ is a decreasing function of prey refuge m with $R_0(1) = 0$. So, there exists a critical value of prey refuge m^* (see, Figure 5). More precisely, if $m < m^* < 1$, then R_0 exceeds one. In this case, the predator is able to survive. However, when $m > m^*$, threshold parameter R_0 becomes less than one and the predator population will disappear. On the other hand, a bifurcation diagram is shown in Figure 6 treating m as a bifurcation parameter.

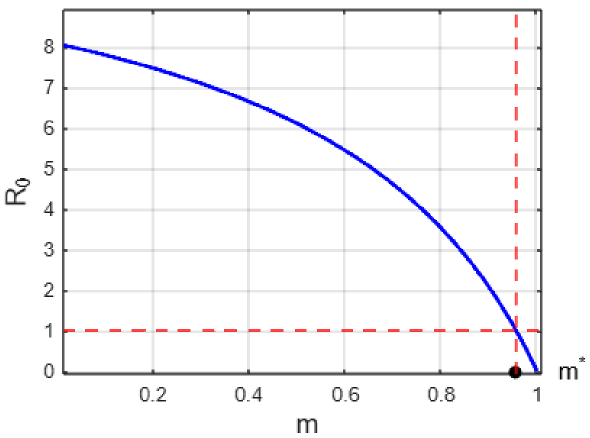

FIGURE 5. Basic reproduction rate R_0 as a function of the prey refuge m

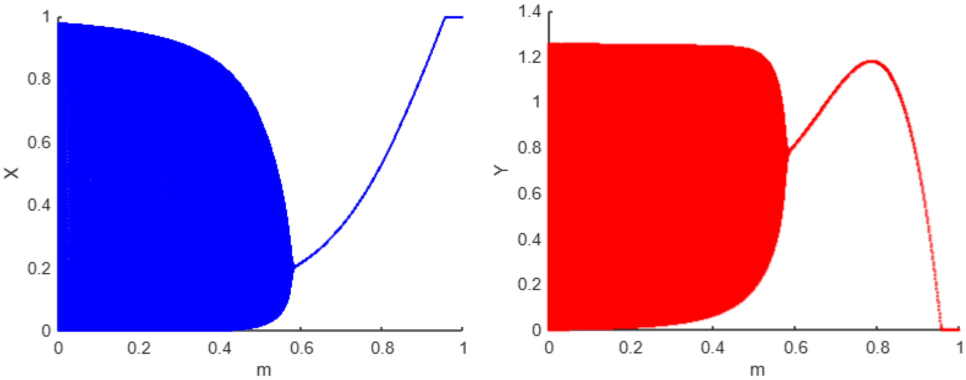

FIGURE 6. A bifurcation diagram for system (1) with m as the bifurcation parameter when $\tau_1 = 0$ and $\tau_2 = 4.3 > \tau_{2,0}$.

5. CONCLUSION

In this work, we have proposed a prey-predator model with the Hattaf-Yousfi functional response, prey refuge and two time delays. First, we focused on analyzing the local stability of the three equilibrium points. We demonstrated that the trivial equilibrium E^0 is always unstable, the predator free equilibrium E^1 is locally asymptotically stable when $R_0 < 1$ and it becomes unstable when $R_0 > 1$. For the interior equilibrium E^*, the stability was analyzed under three different cases of delays. Moreover, sufficient conditions for the existence of the Hopf bifurcation have been established. Finally, the numerical simulations illustrated our theoretical results.

Based on our analytical and numerical results, we can conclude that the time delays and prey refuge have a significant impact on the dynamical behavior of our prey-predator model. The memory effect on dynamics of our model will be done in our future work by using the new generalized Hattaf mixed fractional derivative [21, 22] that includes many types of fractional operators with singular and non-singular kernels.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] T. R. Malthus, An essay on the principle of population, London, 1798.

[2] A. J. Lotka, Elements of physical biology, Williams and Wilkins, 1925.

[3] V. Volterra, Fluctuations in the abundance of a species considered mathematically, Nature. 118 (1926), 558-560.

[4] J.P. Tripathi, S. Tyagi and S. Abbas, Global analysis of a delayed density dependent predator-prey model with Crowley-Martin functional response, Commun. Nonlinear Sci. Numer. Simul. 30 (2016), 45–69.

[5] P.H. Crowley and E.K. Martin, Functional responses and interference within and between year classes of a dragonfly population, J. North Am. Benthol. Soc. 8 (1989), 211–221.

[6] S. Bouziane, E. Lotfi, K. Hattaf and N. Yousfi, Dynamics of a delayed prey-predator model with Hattaf-Yousfi functional response, Commun. Math. Biol. Neurosci. (2022), 1–13.

[7] K. Hattaf and N. Yousfi, A class of delayed viral infection models with general incidence rate and adaptive immune response, Int. J. Dyn. Control (2016), 254–265.

[8] T.K. Kar, Stability analysis of a prey-predator model incorporating a prey refuge, Commun. Nonlinear Sci. Numer. Simul. 10 (2005), 681–691.

[9] C.S. Holling, The components of predation as revealed by a study of small-mammal predation of the European pine sawfly, Can. Entomol. 91 (1959), 293–320.

[10] J.P. Tripathi, S. Abbas and M. Thakur, Dynamical analysis of a prey-predator model with Beddington-DeAngelis type function response incorporating a prey refuge, Nonlinear Dyn. 80 (2015), 177–196.

[11] J.R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, J. Anim. Ecol. 44 (1975), 331–340.

[12] D.L. DeAngelis, R.A. Goldstein and R.V. O'Neill, A model for tropic interaction, Ecology 56 (1975), 881–892.

[13] M. Peng, R. Lin, Y. Chen, Z. Zhang and M.M.A. Khater, Qualitative analysis in a Beddington-DeAngelis type predator-prey model with two time delays, Symmetry 14 (2022), 2535.

[14] A. Nangue and B.N. Tchiffo, Global analysis of a spatiotemporal cellular model for the transmission of hepatitis C virus with Hattaf-Yousfi functional response, arXiv preprint arXiv:2111.03312 (2021).

[15] M. Bachraoui, M.A. Ichou, K. Hattaf and N. Yousfi, Spatiotemporal dynamics of a fractional model for hepatitis B virus infection with cellular immunity, Math. Model. Nat. Phenom. 16 (2021), 5.

[16] H. El Mamouni, K. Hattaf and N. Yousfi, Existence of traveling waves by means of fixed point theory for an epidemic model with Hattaf-Yousfi incidence rate and temporary immunity acquired by vaccination, Adv. Fixed Point Theory 13 (2023).

[17] D. Riad, K. Hattaf and N. Yousfi, Dynamics of capital-labour model with Hattaf-Yousfi functional response, Br. J. Math. Comput. Sci. 18 (5) (2016), 1–7.

[18] S. Bouziane, E. Lotfi, K. Hattaf and N. Yousfi, Stability analysis and Hopf bifurcation of a delayed prey-predator model with Hattaf-Yousfi functional response and Allee effect, Math. Model. Comput. 10 (2023), 668–673.

[19] J. Hale and S.M. Verduyn Lunel, Introduction to functional differential equations, Springer, New York, 1993.

[20] S. Ruan and J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equation with two delays, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 10 (2003), 863–874.

[21] K. Hattaf, Useful results for the qualitative analysis of generalized Hattaf mixed fractional differential equations with applications to medicine, Computation 13 (2025), 167.

[22] K. Hattaf, A new generalized class of fractional operators with weight and respect to another function, J. Fract. Calc. Nonlinear Syst. 5 (2024), 53–68.