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A FIXED POINT APPROACH TO STABILITY OF THE QUARTIC EQUATION IN 2-BANACH SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam-Rassias stability of the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

by using the direct method and the fixed point method in 2-Banach spaces.

Keywords: Hyers-Ulam stability; 2-Banach space; Quartic functional equation.

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1. Introduction and preliminaries

In 1940, S. M. Ulam [19] asked the first question on the stability problem for mappings. In 1941, D. H. Hyers [12] solved the problem of Ulam. This result was generalized by Aoki [4] for additive mappings and by Th. M. Rassias [18] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. In 1994, a further generalization was obtained by P. Găvruta

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[11]. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings.

In the middle of 1960s, S. Gähler [9,10] introduced the concept of linear 2-normed spaces.

We recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. *let X be a real linear space with $dimX > 1$ and $\|.,.\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

(1) $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\lambda x, y\| = |\lambda| \|x, y\|$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a linear 2-normed space. Sometimes the condition (4) called the triangle inequality.

Example 1.2. *For $x = (x_1, x_2)$, $y = (y_1, y_2) \in E = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_E$ is defined by*

$$\|x, y\|_E = |x_1 y_2 - x_2 y_1| .$$

Definition 1.3. *A sequence $\{x_k\}$ in a 2-normed space X is called a convergent sequence if there is an $x \in X$ such that*

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ as $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4. *A sequence $\{x_k\}$ in a 2-normed space X is said to be a Cauchy sequence with respect to the 2-norm if*

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space.

Now, we state the following results as lemma (See [16] for the details).

Lemma 1.5. *Let X be a 2-normed space. Then,*

(1) $|\|x, z\| - \|y, z\|| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,

(2) *if* $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$,

(3) *for a convergent sequence* x_n *in* X,

$$\lim_{n \longrightarrow \infty} \|x_n, z\| = \left\| \lim_{n \longrightarrow \infty} x_n, z \right\|$$

for all $z \in X$.

In [16], Won-Gil Park has investigated approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. In [3], A. Alotaibi and S.A. Mohiuddine have investigated stability of the cubic functional equation in random 2-normed spaces.

In [15], S.H. Lee, S.M. Im and I.S. Hwang considered the following functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \tag{1}$$

and they established the general solution and the stability problem for the functional equation (1) (see also [17]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

In this paper, we prove the Hyers-Ulam-Rassias stability of the quartic functional equation (1) in 2-Banach spaces by using the direct method and fixed point method.

2. Stability of the functional equation (1): Direct method

In this section, we investigate the generalized Hyers-Ulam-Rassias stability of the quartic functional equation (1) in 2-Banach spaces. Let X be a linear space and Y be a

2-Banach space with $dimY > 1$. For convenience, we use the following abbreviation for a given mapping $f : X \to Y$

$$Df(x,y) := f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) \tag{2}$$

for all $x, y \in X$.

Theorem 2.1. *Let $\varphi : X \times X \longrightarrow [0,+\infty)$ be a function such that*

$$\widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^{4k}} \varphi(2^k x, 2^k y) < \infty \tag{3}$$

$$\lim_{n\longrightarrow\infty} \frac{1}{2^{4n}} \varphi(2^n x, 2^n y) = 0 \tag{4}$$

for all $x, y \in X$. Suppose that $f : X \longrightarrow Y$ be a mapping with

$$\|Df(x,y), z\| \le \varphi(x,y) \tag{5}$$

for all $x, y \in X$ and all $z \in Y$. Then, there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x), z\| \le \frac{1}{32}\widetilde{\varphi}(x,0) \tag{6}$$

for all $x \in X$ and all $z \in Y$.

Proof. *Putting $x = y = 0$ in (5), we obtain $f(0) = 0$. Putting $y = 0$ in (5), we get*

$$\left\|\frac{1}{16}f(2x) - f(x), z\right\| \le \frac{1}{32}\varphi(x,0) \tag{7}$$

for all $x \in X$ and all $z \in Y$. If we replace x by $2^n x$ in (7) and divide both sides of (7) by 2^{4n}, we infer that

$$\left\|\frac{1}{2^{4(n+1)}}f(2^{n+1}x) - \frac{1}{2^{4n}}f(2^n x), z\right\| \le \frac{1}{2^{4n+5}}\varphi(2^n x, 0)$$

for all $x \in X$, all $z \in Y$ and integers $n \ge 1$. Hence, we have

$$\left\|\frac{1}{2^{4(n+1)}}f(2^{(n+1)}x) - \frac{1}{2^{4m}}f(2^m x), z\right\| \le \sum_{i=m}^{n} \left\|\frac{1}{2^{4(i+1)}}f(2^{(i+1)}x) - \frac{1}{2^{4i}}f(2^i x), z\right\|$$

$$\le \frac{1}{32}\sum_{i=m}^{n} \frac{1}{2^{4i}}\varphi(2^i x, 0) \tag{8}$$

for all $x \in X$*, all* $z \in Y$ *and all non-negative integers* m *and* n *with* $n \geq m$*. Therefore, we conclude from* (3)*,* (4) *and* (8) *that the sequence* $\left\{\frac{1}{2^{4n}} f(2^n x)\right\}$ *is a Cauchy sequence in* Y *for all* $x \in X$*. Since* Y *is complete, the sequence* $\left\{\frac{1}{2^{4n}} f(2^n x)\right\}$ *converges in* Y *for all* $x \in X$*. So, we can define the mapping* $Q : X \longrightarrow Y$ *by*

$$Q(x) := \lim_{n \longrightarrow \infty} \frac{1}{2^{4n}} f(2^n x) \tag{9}$$

for all $x \in X$*. That is*

$$\lim_{n \longrightarrow \infty} \left\| \frac{1}{2^{4n}} f(2^n x) - Q(x), z \right\| = 0$$

for all $x \in X$ *and all* $z \in Y$*. Letting* $m = 0$ *and passing the limit* $n \longrightarrow \infty$ *in* (8)*, we get the inequality* (6)*. Now, we show that* $Q : X \longrightarrow Y$ *is a quartic mapping. It follows from* (3)*,* (5)*,* (9) *and Lemma* 1.5 *that*

$$\|DQ(x, y), z\| = \lim_{n \longrightarrow \infty} \frac{1}{2^{4n}} \|Df(2^n x, 2^n y), z\| \leq \lim_{n \longrightarrow \infty} \frac{1}{2^{4n}} \varphi(2^n x, 2^n y) = 0$$

for all $x, y \in X$ *and all* $z \in Y$*. By Lemma* 1.6*, we obtain that* $DQ(x, y) = 0$ *for all* $x, y \in X$*. So, the mapping* $Q : X \longrightarrow Y$ *is quartic. To prove the uniqueness of* Q*, let* $A : X \longrightarrow Y$ *be another quartic mapping satisfying* (6)*. Since the mapping* $A : X \longrightarrow Y$ *satisfies* (1)*, then by letting* $y = 0$ *in* (1) *we get* $A(2x) = 2^4 f(x)$ *for all* $x \in X$*. Therefore, we have*

$$\|Q(x) - A(x), z\| = \lim_{n \longrightarrow \infty} \frac{1}{2^{4n}} \|Q(2^n x) - A(2^n x), z\| \leq \frac{1}{32} \lim_{n \longrightarrow \infty} \widetilde{\varphi}(2^n x, 0) = 0$$

for all $x \in X$ *and all* $z \in Y$*. By Lemma* 1.6*,* $\|Q(x) - A(x)\| = 0$ *for all* $x \in X$*. So* $Q = A$*. This proves the uniqueness of* Q*.*

Corollary 2.2. *Let* $(X, \|.\|_X)$ *be a normed space and* $(Y, \|., .\|_Y)$ *be a 2-Banach space. Let* ϵ *and* p *be nonnegative real numbers with* $p < 4$ *and let* $f : X \longrightarrow Y$ *be a mapping fulfilling*

$$\|Df(x, y), z\|_Y \leq \epsilon(\|x\|_X^p + \|y\|_X^p) \tag{10}$$

for all $x, y \in X$ and all $z \in Y$. Then there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x), z\|_Y \leq \frac{\epsilon}{32 - 2^{p+1}} \|x\|_X^p \tag{11}$$

for all $x \in X$ and all $z \in Y$.

Proof. *In Theorem 2.1, let $\varphi(x, y) = \epsilon\,(\|x\|_X^p + \|y\|_X^p)$ for all $x, y \in X$. Then (10) implies that $f(0) = 0$. So we obtain (11) from (6).*

Theorem 2.3. *Let $\varphi : X \times X \longrightarrow [0, +\infty)$ be a function such that*

$$\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} \frac{1}{3^{4k}} \varphi(3^k x, 3^k y) < \infty \tag{12}$$

$$\lim_{n \longrightarrow \infty} \frac{1}{3^{4n}} \varphi(3^n x, 3^n y) = 0 \tag{13}$$

for all $x, y \in X$. Suppose that $f : X \longrightarrow Y$ be a mapping with

$$\|Df(x, y), z\| \leq \varphi(x, y) \tag{14}$$

for all $x, y \in X$ and all $z \in Y$. Then, there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x), z\| \leq \frac{1}{81} \left(\widetilde{\varphi}(x, x) + 2\widetilde{\varphi}(x, 0)\right) \tag{15}$$

for all $x \in X$ and all $z \in Y$.

Proof. *Putting $x = y = 0$ in (14), we get $f(0) = 0$. Replacing y by x in (14), we get*

$$\|f(3x) - 4f(2x) - 17f(x), z\| \leq \varphi(x, x) \tag{16}$$

for all $x, y \in X$ and all $z \in Y$. Letting $y = 0$ in (14), we obtain

$$\|2f(2x) - 32f(x), z\| \leq \varphi(x, 0) \tag{17}$$

for all $x, y \in X$ and all $z \in Y$. From (16) and (17), we get

$$\|\frac{1}{81} f(3x) - f(x), z\| \leq \frac{1}{81} \left(\varphi(x, x) + 2\varphi(x, 0)\right) \tag{18}$$

for all $x, y \in X$ and all $z \in Y$. We replace x by $3^n x$ in (18) and divide both sides of (18) by 3^{4n}, we infer that

$$\left\|\frac{1}{3^{4(n+1)}}f(3^{n+1}x)-\frac{1}{3^{4n}}f(3^n x),z\right\|\leq\frac{1}{3^{4n+4}}\left(\varphi(x,x)+2\varphi(x,0)\right)$$

for all $x \in X$, all $z \in Y$ and integers $n \geq 1$. Hence, we have

$$\left\|\frac{1}{3^{4(n+1)}}f(3^{(n+1)}x)-\frac{1}{3^{4m}}f(3^m x),z\right\|\leq\sum_{i=m}^{n}\left\|\frac{1}{3^{4(i+1)}}f(3^{(i+1)}x)-\frac{1}{3^{4i}}f(3^i x),z\right\|$$

$$\leq\frac{1}{81}\sum_{i=m}^{n}\frac{1}{3^{4i}}\varphi(3^i x,3^i x)+\frac{1}{81}\sum_{i=m}^{n}\frac{1}{3^{4i}}\varphi(3^i x,0) \tag{19}$$

for all $x \in X$, all $z \in Y$ and all non-negative integers m and n with $n \geq m$. Therefore, we conclude from (12), (13) and (19) that the sequence $\left\{\frac{1}{3^{4n}}f(3^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, there exists a mapping $Q : X \longrightarrow Y$ defined by

$$Q(x) := \lim_{n\longrightarrow\infty}\frac{1}{3^{4n}}f(3^n x) \tag{20}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \longrightarrow \infty$ in (19), we get the inequality (15). The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.4. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|., .\|_Y)$ be a 2-Banach space. Let ϵ and p be nonnegative real numbers with $p < 4$ and lat $f : X \longrightarrow Y$ be a mapping fulfilling*

$$\|Df(x,y),z\|_Y\leq\epsilon\left(\|x\|_X^p+\|y\|_X^p\right) \tag{21}$$

for all $x, y \in X$ and all $z \in Y$. Then there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x)-Q(x),z\|_Y\leq\frac{4\epsilon}{81-3^p}\|x\|_X^p \tag{22}$$

for all $x \in X$ and all $z \in Y$.

Proof. *In Theorem 2.3, let $\varphi(x, y) = \epsilon\left(\|x\|_X^p + \|y\|_X^p\right)$ for all $x, y \in X$. Then (21) implies that $f(0) = 0$. So we obtain (22) from (15).*

Corollary 2.5. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|_Y)$ be a 2-Banach space. Let ϵ, p and q be nonnegative real numbers with $p + q < 4$ and let $f : X \longrightarrow Y$ be a mapping fulfilling*

$$\|Df(x, y), z\|_Y \leq \epsilon \left(\|x\|_X^p . \|y\|_X^q\right) \tag{23}$$

for all $x, y \in X$ and all $z \in Y$. Then there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x), z\|_Y \leq \frac{\epsilon}{81 - 3^{p+q}} \|x\|_X^{p+q} \tag{24}$$

for all $x \in X$ and all $z \in Y$.

Proof. *In Theorem* 2.3*, let $\varphi(x, y) = \epsilon \left(\|x\|_X^p + \|y\|_X^q\right)$ for all $x, y \in X$. Then* (23) *implies that $f(0) = 0$. So we obtain* (24) *from* (15)*.*

3. Stability of the functional equation (1): Fixed point method

In this section, we investigate the generalized Hyers-Ulam-Rassias stability of the quartic functional equation (1) by using fixed point method in 2-Banach spaces. We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *generalized metric* on X if d satisfies :

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 3.1. [8] *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, with the Lipshitz constant $L < 1$. If there exists a nonnegative integer k such that*

$$d(J^k x, J^{k+1} x) < \infty$$

for some $x \in X$, then the following are true:

(I) the sequence $J^n x$ converges to a fixed point x^* of J;

(II) x^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^k x, y) < \infty\}$;

(III) $d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 3.2. *Let* $f : X \longrightarrow Y$ *be a mapping for which there exists a function* $\varphi : X^2 \longrightarrow [0, \infty)$ *satisfying*

$$\phi(x) := \varphi(x, x) + 2\varphi(x, 0),$$

$$\|Df(x, y), z\| \leq \varphi(x, y) \tag{25}$$

and

$$\lim_{n \longrightarrow \infty} \frac{1}{3^{4n}} \varphi(3^n x, 3^n y) = 0 \tag{26}$$

for all $x, y \in X$ *and all* $z \in Y$. *Let* $0 < L < 1$ *be a constant such that* $\varphi(x, y) \leq 81L\varphi(\frac{x}{3}, \frac{y}{3})$ *for all* $x, y \in X$. *Then, there exists a unique quartic mapping* $Q : X \longrightarrow Y$ *satisfying*

$$\|f(x) - Q(x), z\| \leq \frac{1}{81(1-L)} \phi(x) \tag{27}$$

for all $x \in X$ *and all* $z \in Y$.

Proof. *Let us consider the set* $S := \{g : X \longrightarrow Y\}$ *and introduce the generalized metric on* S *as follows:*

$$d(g, h) = inf\{\alpha \in [0, \infty) : \|g(x) - h(x), z\| \leq \alpha\phi(x), \forall x \in X \text{ and } \forall z \in Y\}$$

where, as usual, $inf\emptyset = +\infty$. *The proof of the fact that* (S, d) *is a complete generalized metric space can be found in* [6]. *Now, we consider the linear mapping* $J : S \longrightarrow S$ *defined by*

$$Jg(x) := \frac{1}{81} g(3x)$$

for all $g \in S$ *and all* $x \in X$. *First we assert that* J *is strictly contractive on* S. *For given* $g, h \in S$, *let* $\alpha \in [0, \infty)$ *be an arbitrary constant with* $d(g, h) \leq \alpha$, *that is*

$$\|g(x) - h(x), z\| \leq \alpha\phi(x).$$

So we have

$$\|Jg(x) - Jh(x), z\| = \frac{1}{81}\|g(3x) - h(3x), z\| \leq \frac{1}{81}\alpha\phi(3x) \leq \alpha L\phi(x)$$

for all $g, h \in S$*, all* $x \in X$ *and all* $z \in Y$*. Then,* $d(Jg, Jh) \leq Ld(g, h)$ *,* $\forall g, h \in S$*; that is,* J *is a strictly contractive self-mapping on* S *with the Lipschitz constant* L*. Replacing* y *by* x *in* (25)*, we have*

$$\|f(3x) - 4f(2x) - 17f(x), z\| \leq \varphi(x, x) \tag{28}$$

for all $x \in X$ *and all* $z \in Y$*. Letting* $y = 0$ *in* (25)*, we get*

$$\|2f(2x) - 32f(x), z\| \leq \varphi(x, 0) \tag{29}$$

for all $x \in X$ *and all* $z \in Y$*. From the inequalities* (29) *and* (30)*, it follows that*

$$\|f(3x) - 81f(x), z\| \leq (\varphi(x, x) + 2\varphi(x, 0))$$

Then,

$$\|\frac{1}{81}f(3x) - f(x), z\| \leq \frac{1}{81}\phi(x) \tag{30}$$

for all $x \in X$ *and all* $z \in Y$*. Hence,*

$$d(f, Jf) \leq \frac{1}{81}$$

for all $f \in S$*. By Theorem* 3.1*, there exists a unique mapping* $Q : X \longrightarrow Y$ *satisfying the following:*

- Q *is fixed point of* J*, that is,* $Q(3x) = 81Q(x)$ *for all* $x \in X$*. The mapping* Q *is a unique fixed point of* J *in the set* $M = \{g \in S : d(f, g) \leq \infty\}$*. This implies that* Q *is a unique mapping such that there exists* $\alpha \in (0, \infty)$ *satisfying* $\|f(x) - Q(x), z\| \leq \alpha\phi(x)$*, for all* $x \in X$ *and* $z \in Y$*.*
- $d(J^n, Q) \longrightarrow 0$ *as* $n \longrightarrow \infty$*, which implies the equality*

$$\lim_{n\to+\infty} J^n f(x) = \lim_{n\to+\infty} \frac{f(3^n x)}{3^{4n}} = Q(x) \tag{31}$$

for all $x \in X$*.*

•

$$d(f,Q) \leq \frac{1}{1-L} d(f,Jf) \leq \frac{1}{81(1-L)} \tag{32}$$

which implies the inequality (28)

It follows from (25), (26) *and* (32), *that*

$$\|DQ(x,y),z\| = \lim_{n\to+\infty} \frac{1}{3^{4n}} \|Df(3^n x, 3^n y), z\| \leq \lim_{n\to+\infty} \frac{1}{3^{4n}} \varphi(3^n x, 3^n y) = 0$$

for all $x, y \in X$ *and all* $z \in Y$. *Hence,* $Q : X \longrightarrow Y$ *is a quartic mapping, as desired.*

Corollary 3.3. *Let* $(X, \|.\|_X)$ *be a normed space and* $(Y, \|.,.\|_Y)$ *be a 2-Banach space. Let* ϵ *and* p *be nonnegative real numbers with* $p < 4$ *and let* $f : X \longrightarrow Y$ *be a mapping fulfilling*

$$\|Df(x,y),z\|_Y \leq \epsilon\left(\|x\|_X^p + \|y\|_X^p\right) \tag{33}$$

for all $x, y \in X$ *and all* $z \in Y$. *Then there exists a unique quartic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x), z\|_Y \leq \frac{4\epsilon}{81 - 3^p} \|x\|_X^p \tag{34}$$

for all $x \in X$ *and all* $z \in Y$.

Proof. *Taking* $\varphi(x,y) = \epsilon\left(\|x\|_X^p + \|y\|_X^p\right)$ *for all* $x, y \in X$ *and choosing* $L = 3^{p-4}$ *in Theorem* 3.2, *we get the desired result.*

Corollary 3.4. *Let* $(X, \|.\|_X)$ *be a normed space and* $(Y, \|.,.\|_Y)$ *be a 2-Banach space. Let* ϵ, p *and* q *be nonnegative real numbers with* $p + q < 4$ *and let* $f : X \longrightarrow Y$ *be a mapping fulfilling*

$$\|Df(x,y),z\|_Y \leq \epsilon\left(\|x\|_X^p . \|y\|_X^q\right) \tag{35}$$

for all $x, y \in X$ *and all* $z \in Y$. *Then there exists a unique quartic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x), z\|_Y \leq \frac{\epsilon}{81 - 3^{p+q}} \|x\|_X^{p+q} \tag{36}$$

for all $x \in X$ *and all* $z \in Y$.

Proof. *Taking* $\varphi(x, y) = \epsilon\left(\|x\|_X^p.\|y\|_X^q\right)$ *for all* $x, y \in X$ *and choosing* $L = 3^{p+q-4}$ *in Theorem* 3.2*, we get the desired result.*

References

[1] M. Almahalebi, A fixed point approach of quadratic functional equations, Int. Journal of Math. Analysis, Vol.7 (2013), no.30, 1471-1477.

[2] M. Almahalebi and S. Kabbaj, A fixed point approach to the orthogonal stability of an additive - quadratic functional equation, Advances in Fixed Point Theory, (2013) (submitted).

[3] A. Alotaibi, S.A. Mohiuddine, On the stability of a cubic functional equation in random 2-normed spaces, Advan. in Diff. Equat.,doi:10.1186 /1687-1847-201 2-39, 2012.

[4] T. Aoki, On the stability of the linear transformation in Banach spaces, Journal of the Mathematical Society of Japan, vol. 2, pp. 64?6, 1950.

[5] L. Cădariu and V. Radu, Fixed points and the stability of Jensen s functional equation, J. Inequal Pure Appl Math.4(1), Art.ID4 (2003).

[6] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, Grazer Math. Ber. 346 , 43-52 (2004).

[7] S. Czerwik, Functional equations and Inequalities in Several Variables, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[8] J. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull Amer Math Soc.74, 305-309 (1968).

[9] S. Gähler, 2-metrische Räume und ihre topologische Struktur, Math. Nachr. 26 (1963) 115-148.

[10] S. Gähler, Linear 2-normiete Räumen, Math. Nachr. 28 (1964) 1-43.

[11] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431-436.

[12] D. H.Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27, (1941), 222-224.

[13] D.H. Hyers, G. Isac,and Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, Basel 1998.

[14] G. Isac and Th.M. Rassias, Stability of ψ-additive mappings: applications to nonlinear analysis, Intern J Math Math Sci.19, 219-228 (1996).

[15] S.H. Lee, S.M. Im, I.S. Hwang, Quartic functional equation, J. Math. Anal. Appl. 307(2005) 787-394.

[16] W.G. Park, Approximate additive mappings in 2-Banach spaces and related topics, J. Math. Anal. Appl. 376 (2011) 193-202.

[17] J.M. Rassias, Solution of the Ulam stability problem for quartic mappings, Glas. Mat. 34(1999) 243-252.

[18] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proceedings of the American Mathematical Society, vol. 72, no. 2, pp. 297?00, 1978.

[19] S.M. Ulam, A collection of the Mathematical Problems, Interscience Publ. New York, (1960), 431-436.