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SOME PROPERTIES OF UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY BESSEL FUNCTIONS

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Abstract. The aim of the present paper is to investigate some characterization for generalized Bessel functions of the first kind to be a subclass of analytic functions. Furthermore, we study coefficient estimates, radius of starlikeness, convexity, close - to - convexity, convex linear combinations for the class $TS(\lambda,\sigma,\rho)$. Finally we prove integral means inequalities for this class.

Keywords: analytic function; Bessel function; starlike and Hadamard product.

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1. Introduction

Let A be the class of functions r normalized by

$$r(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and T denote the class of functions in the form of

$$r(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0), \tag{2}$$

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which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. This subclass was given in [18].

For $r \in A$ given by (1) and $g(z)$ as provided by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution indicated by $(r * g)$, is defined as

$$(r * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * r)(z),\ z \in \Delta.$$

Let $T^*(\alpha)$ and $C(\alpha)$ be indicate starlike and convex functions of order $\alpha, (0 \leq \alpha < 1)$, respectively.

The classes $UCV(\alpha, \sigma)$ consists of uniform $\sigma-$convex functions of order α and $SP(\alpha, \sigma)$ consists of parabolic $\sigma-$ starlike functions of order $\alpha, -1 < \alpha \leq 1, \sigma \geq 0$, generalizes the class UCV and SP respectively, were given in [9] such that

$$UCV(\alpha, \sigma) = \left\{ r \in A : \Re\left\{ 1 + \frac{zr''(z)}{r'(z)} - \alpha \right\} > \sigma \left| \frac{zr''(z)}{r'(z)} \right|,\ z \in \Delta \right\} \tag{3}$$

and

$$SP(\alpha, \sigma) = \left\{ r \in A : \Re\left\{ \frac{zr'(z)}{r(z)} - \alpha \right\} > \sigma \left| \frac{zr'(z)}{r(z)} - 1 \right|,\ z \in \Delta \right\}. \tag{4}$$

It is obvious from (3) and (4) that $r \in UCV(\alpha, \sigma)$ if and only if $zr'(z) \in SP(\alpha, \sigma)$.

One of the most important special functions is the Bessel function. As a result, it is critical for addressing many issues in engineering, physics, and mathematics . For example, it is used to calculate velocity and tension in the rotating flow of Burge's fluid via an unbounded circular tube. Many academics have recently focused on determining various circumstances under which a Bessel function has specific geometric qualities such as close-to-convexity (univalency), starlikeness, and convexity in the frame of a unit disc (see [1, 3]).

There has been a continues interest shown on the geometric and other related properties of Bessel functions (like hypergeometric function) after many papers have been published by Baricz [1] in recent times. One such problem of Baricz [2] was to find condition on the triple p, b and c such that the function $\vartheta_{p,b,c}$ is starlike and convex of order α. In earlier investigations, finding conditions on the parameters for which the Gaussian hypergeometric functions belong to the various classes of functions have been discuss in details by Shanmugam [17], Naeem et al. [14] and Sivasubramanina et al. [21] .

Let us take into consideration second order linear homogenous differential equation (see [3])

$$z^2\omega''(z)+bz\omega'(z)+[cz^2-p^2+(1-b)p]\omega(z)=0,\ (p,b,c\in\mathbb{C}). \tag{5}$$

As a particular solution of (5) generalized Bessel function of the first kind of order p, is defined in [3] as follows:

$$\omega(z)=\omega_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p},\ z\in\Delta, \tag{6}$$

where Γ stands for the Euler gamma function and $k=p+\frac{b+1}{2}\notin\mathbb{Z}_0^-=\{0,-1,-2,\cdots\}$.

Though the series given in (6) is convergent everywhere, the function $\omega_{p,b,c}$ is not univalent in Δ.

Specially, choosing $b=c=1$ in (6), we get Bessel function of the first kind of order p given in [3] as

$$J_p(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(p+n+1)}\left(\frac{z}{2}\right)^{2n+p},z\in\Delta. \tag{7}$$

Choosing $b=1$ and $c=-1$ in (6), we get the modified Bessel function of the first kind order of p given in [3] as

$$I_p(z)=\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(p+n+1)}\left(\frac{z}{2}\right)^{2n+p},z\in\Delta. \tag{8}$$

Further choosing $b=2$ and $c=1$ in (6), the functions $\omega_{p,b,c}$ reduces to $\sqrt{\frac{2}{\pi}}\ j_p(z)$, where j_p is the spherical Bessel function of the first kind of order p, given in [3] as

$$j_p(z)=\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(p+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p},z\in\Delta. \tag{9}$$

The function $\vartheta_{p,b,c}$ is defined in [7] as

$$\vartheta_{p,b,c}(z)=2^p\Gamma\left(p+\frac{b+1}{2}\right)z^{1-\frac{p}{2}}\omega_{p,b,c}(\sqrt{z}) \tag{10}$$

in terms of generalized Bessel function $\omega_{p,b,c}$.

By the help of Pochhammer symbol, Gamma function is defined as

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mu = 0, \lambda \in \mathbb{C}\setminus\{0\}; \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } \mu = n \in \mathbb{N}, \lambda \in \mathbb{C}; \end{cases}$$

$$(\lambda)_0 = 1$$

and we get $\vartheta_{p,b,c}$ given in (10) as

$$\vartheta_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (k)_n n!} z^{n+1}, \tag{11}$$

where $k = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$ and $\mathbb{N} = \{1,2,3,\cdots\}$.

We will write $\vartheta_{k,c}(z) = \vartheta_{p,b,c}(z)$ for convenience.

Now, we consider the operator $B_k^c : A \to A$, which is define by the Hadamard product as

$$\begin{aligned} B_k^c r(z) &= \vartheta_{k,c}(z) * r(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n (k)_n n!} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1} a_n}{4^{n-1} (k)_{n-1} (n-1)!} z^n = z + \sum_{n=2}^{\infty} E(c,k,n) a_n z^n \end{aligned}$$

$$\text{where } E(c,k,n) = \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!};$$

$$\mathrm{E}(c,k,2) = \frac{-c}{4k},\ k = \left(p + \frac{b+1}{2}\right) \notin \mathbb{Z}_0^-, c \in \mathbb{C}. \tag{12}$$

By the help of (12), we get

$$z\left[B_{k+1}^c r(z)\right]' = k B_k^c r(z) - (k-1) B_{k+1}^c r(z), \tag{13}$$

where $k = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$.

The function $B_k^c r(z)$ is an elementary transformation of the generalized hypergeometric function, so that $B_k^c r(z) = z\, {}_0F_1(k; \frac{-c}{4} z) * r(z)$ and $\vartheta_{k,c}\left(\frac{-c}{4} z\right) = z\, {}_0F_1(k; z)$.

The univalence of some integral operators involving the normalization form of the both ordinary and generalized Bessel function of the first kind also characterizations for certain subclasses of starlike and convex functions associated with Bessel functions was studied in [4, 6, 7, 13, 15]. Motivated by the new technique given in [11, 16, 22, 23], we introduce the following new subclass of uniformly convex functions given by Bessel functions.

Definition 1.1. The following relationship is satisfied if $r \in S(\lambda, \sigma, \rho)$:

$$\Re\left\{\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-\rho\right\} > \sigma\left|\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right|, \tag{14}$$

for $0 \leq \lambda \leq 1,\ 0 \leq \rho < 1$ and r is given by (1).

Further we denote $TS(\lambda, \sigma, \rho) = S(\lambda, \sigma, \rho) \cap T$.

The aim of this paper is to study some characterization for generalized Bessel functions of first kind is to be subclass of analytic functions. Further, we studied coefficient estimates, radius of starlikeness, convexity, close-to- convexity, convex linear combinations for the class $TS(\lambda, \sigma, \rho)$. Finally, we proved integral means inequalities for the class.

2. Main Theorems

Theorem 2.1. A sufficient condition for a function $r \in S(\lambda, \sigma, \rho)$ is that the following inequality holds.

$$\sum_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]\mathrm{E}(c,k,n)|a_n| \leq 1-\rho, \tag{15}$$

where $0 \leq \lambda \leq 1, 0 \leq \rho < 1, \sigma \geq 0$.

Proof. It is enough to show that

$$\sigma\left|\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right|-\Re\left\{\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right\} \leq 1-\rho.$$

We have

$$\begin{aligned}
&\sigma\left|\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right|-\Re\left\{\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right\}\\
\leq&(1+\sigma)\left|\frac{z\left(B_k^c r(z)\right)'}{(1-\lambda)z+\lambda B_k^c r(z)}-1\right|\\
\leq&\frac{(1+\sigma)\sum\limits_{n=2}^{\infty}(n-\lambda)\mathrm{E}(c,k,n)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda\mathrm{E}(c,k,n)|a_n||z|^{n-1}}\\
\leq&\frac{(1+\sigma)\sum\limits_{n=2}^{\infty}(n-\lambda)\mathrm{E}(c,k,n)|a_n|}{1-\sum\limits_{n=2}^{\infty}\lambda\mathrm{E}(c,k,n)|a_n|}.
\end{aligned}$$

The right hand of the last inequality is bounded above by $(1-\rho)$ if

$$\sum_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]\mathrm{E}(c,k,n)|a_n| \leq (1-\rho)$$

and this completes the proof. □

Theorem 2.2. $r \in TS(\lambda,\sigma,\rho)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)||a_n| \leq (1-\rho), \tag{16}$$

where $0 \leq \lambda \leq 1,\ 0 \leq \rho < 1, \sigma \geq 0$, r is in the form (2).

Proof. According to the Theorem 2.1, we will prove only the necessary. If $r \in TS(\lambda,\sigma,\rho)$ and z is real then

$$\Re\left\{\frac{1-\sum\limits_{n=2}^{\infty} n\mathrm{E}(c,k,n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda\mathrm{E}(c,k,n)a_n z^{n-1}}\right\} > \sigma\left|\frac{\sum\limits_{n=2}^{\infty}(n-\lambda)\mathrm{E}(c,k,n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda\mathrm{E}(c,k,n)a_n z^{n-1}}\right|.$$

Taking $z \to 1$ along the real axis, we get

$$\sum_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)| \leq (1-\rho).$$

□

Corollary 2.1. If $r \in TS(\lambda,\sigma,\rho)$ then

$$|a_n| \leq \frac{(1-\rho)}{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}z^n, \tag{17}$$

where $0 \leq \lambda \leq 1,\ 0 \leq \rho < 1, \sigma \geq 0$. Equality is satisfied for the function

$$r(z) = z - \frac{(1-\rho)}{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}z^n. \tag{18}$$

Remark 2.1.

(i) For the choice of $\sigma = 1$ in Theorems 2.1, 2.2 and Corollary 2.1, we obtain the result of Thirupathi Reddy and Venkateswarlu [23], Theorems 2.1, 2.2 and Corollary 2.1 .

(ii) For the choice of $\sigma = 1$ and $\lambda = 1$ in Theorems 2.1, 2.2 and Corollary 2.1, we obtain the results of Thirupathi Reddy and Venkateswarlu [22], Theorems 1,2 and Corollary 1.

Theorem 2.3. Let $r_1(z) = z$ and

$$r_n(z) = z - \frac{(1-\rho)}{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|} z^n, \ n \geq 2. \tag{19}$$

Then $r \in TS(\lambda,\sigma,\rho)$ if and only if it has the following form

$$r(z) = \sum_{n=1}^{\infty} \omega_n r_n(z), \ \sum_{n=1}^{\infty} \omega_n = 1. \tag{20}$$

Proof. Assume that one can write r as in (20). Then

$$r(z) = z - \sum_{n=2}^{\infty} \frac{(1-\rho)}{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|} z^n.$$

Now $\displaystyle\sum_{n=2}^{\infty} \omega_n \frac{(1-\rho)[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(1-\rho)[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|} = \sum_{n=2}^{\infty} \omega_n = (1-\omega_1) \leq 1.$

Thus $r \in TS(\lambda,\sigma,\rho)$.

Conversely suppose that $r \in TS(\lambda,\sigma,\rho)$. Then by using (17), we get,

$$\omega_n = \frac{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{1-\rho} a_n, \ n \geq 2 \text{ and } \omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n.$$

Thus we obtain $r(z) = \sum_{n=1}^{\infty} \omega_n r_n(z)$. □

Theorem 2.4. The class $TS(\lambda,\sigma,\rho)$ is a convex set.

Proof. Let the functions

$$r_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \ a_{n,j} \geq 0, \ j = 1,2 \tag{21}$$

be in $TS(\lambda,\sigma,\rho)$. It is enough to prove that the function h given by $h(z) = \xi r_1(z) + (1-\xi) r_2(z), 0 \leq \xi < 1$ is in $TS(\lambda,\sigma,\rho)$.

Due to the fact that

$$h(z) = z - \sum_{n=2}^{\infty} [\xi a_{n,1} + (1-\xi) a_{n,2}] z^n$$

by help of Theorem 2.2 and by an easy calculation, we get

$$\sum_{n=2}^{\infty} [n(1+\sigma)-\lambda(\sigma+\rho)] \xi |\mathrm{E}(c,k,n)| a_{n,1} + \sum_{n=2}^{\infty} [n(1+\sigma)-\lambda(\sigma+\rho)](1-\xi)|\mathrm{E}(c,k,n)| a_{n,2}$$

$$\leq \xi(1-\rho) + (1-\xi)(1-\rho) \leq (1-\rho),$$

which means that $h \in TS(\lambda,\sigma,\rho)$. Hence the $TS(\lambda,\sigma,\rho)$ is convex.

□

In the next theorem the radii of close-to-convexity, starlikeness and for the class $TS(\lambda,\sigma,\rho)$ will be obtained.

Theorem 2.5. Let r is given by (2) is in $TS(\lambda,\sigma,\rho)$. Then r is close-to-convex of order $\delta, (0 \leq \delta < 1)$ in the disc $|z| < t_1$, where

$$t_1 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum\limits_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{n(1-\rho)}\right]^{\frac{1}{n-1}}, \; n \geq 2. \tag{22}$$

The result is sharp with the extremal function $r(z)$ given by (18).

Proof. If $r \in T$ and r is close-to-convex of order δ, then we get

$$|r'(z)-1| < (1-\delta). \tag{23}$$

For the left hand side of (23), we obtain

$$|r'(z)-1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1},$$

$(1-\delta)$ is greater than the right hand side of this inequality. Thus

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \leq 1.$$

We know that $r(z) \in TS(\lambda,\sigma,\rho)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(1-\rho)} a_n \leq 1.$$

Then (23) holds true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(1-\rho)}$$

or equivalently

$$|z| \leq \left[\frac{(1-\delta)[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{n(1-\rho)}\right]^{\frac{1}{n-1}}$$

and hence the proof is completed. ◻

Theorem 2.6. Let $r \in TS(\lambda,\sigma,\rho)$. Then r is starlike of order $\delta, (0 \leq \delta < 1)$ in the disc $|z| < t_2$, where

$$t_2 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(n-\delta)(1-\rho)}\right]^{\frac{1}{n-1}}, \; n \geq 2. \tag{24}$$

The result is sharp with the extremal function given by (18).

Proof. Since $r \in T$ and r is starlike of order δ, we get

$$\left|\frac{zr'(z)}{r(z)} - 1\right| < (1-\delta). \tag{25}$$

For the left hand side of (25), we have

$$\left|\frac{zr'(z)}{r(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}},$$

$(1-\delta)$ is greater than the right hand side of the last relation if

$$\sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)}a_n|z|^{n-1} < 1.$$

We know that $r \in TS(\lambda,\sigma,\rho)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(1-\rho)}a_n \leq 1.$$

Thus (25) is true if

$$\frac{(n-\delta)}{(1-\delta)}|z|^{n-1} \leq \frac{[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(1-\rho)}$$

or equivalently

$$|z|^{n-1} \leq \frac{(1-\delta)[n(1+\sigma)-\lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|}{(n-\delta)(1-\rho)}.$$

It yield starlikeness of the family. □

Remark 2.2.

(i) For the choice of $\sigma = 1$ in Theorems 2.3, 2.4, 2.5 and 2.6, we obtained the results of Thirupathi Reddy and Venkateswarlu [23] , Theorems 2.3, 2.4, 2.5 and 2.6.

(ii) For the choice of $\sigma = 1$ and $\lambda = 1$ in Theorems 2.3, 2.4, 2.5 and 2.6. we obtained the results of Thirupathi Reddy and Venkateswarlu [22], Theorems 3,4,5and 6.

Now, we will prove integral means inequality for the function $r \in TS(\lambda,\sigma,\rho)$.

In [19], the function $r_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T was found by Silverman. Also this function was applied to resolve his integral means inequality, conjectured [19] and settled in [20], that

$$\int_0^{2\pi} \left| r(te^{i\theta}) \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| r_2(te^{i\theta})^{\eta} \right| d\theta,$$

for all $r \in T, \eta > 0$ and $0 < t < 1$. Also his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T was proved in [19].

To prove Silverman's conjecture for the class of functions $TS(\lambda,\sigma,\rho)$, let us remember the concept of subordination between analytic functions given in [10]. Two functions r and s, which are analytic in Δ, the function ω is said to be subordinate to v in Δ if there exists a function ω analytic in Δ with $\omega(0) = 0,\ |\omega(z)| < 1, (z \in \Delta)$ such that $r(z) = s(\omega(z)),\ (z \in \Delta)$. This subordination is denoted by $r(z) \prec s(z)$.

Lemma 2.1. [10] If the functions r and s are analytic in Δ with $r(z) \prec s(z)$, then for $\eta > 0$ and $0 < t < 1$,

$$\int_0^{2\pi} \left| s(te^{i\theta}) \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| r(te^{i\theta}) \right|^{\eta} d\theta.$$

Now, we argue that the integral means inequalities for functions $r \in TS(\lambda,\sigma,\rho)$ and $\int_0^{2\pi} \left| s(te^{i\theta}) \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| r(te^{i\theta}) \right|^{\eta} d\theta$.

Theorem 2.7. Suppose $r \in TS(\lambda,\sigma,\rho), \eta > 0, 0 \leq \lambda < 1, 0 \leq \rho < 1, \sigma \geq 0$ and r_2 be defined by

$$r_2(z) = z - \frac{1-\rho}{\phi_2(\lambda,\sigma,\rho)} z^2, \tag{26}$$

where $\phi_2(\lambda,\sigma,\rho) = [2(1+\sigma) - \lambda(\sigma+\rho)]|\mathrm{E}(c,k,2)|$ and $|\mathrm{E}(c,k,2)|$ is given by (12). Then for $0 < t < 1$, has

$$\int_0^{2\pi} \left| r(te^{i\theta}) \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| r_2(te^{i\theta}) \right|^{\eta} d\theta. \tag{27}$$

Proof. Since $r(z) = z\left(1 - \sum\limits_{n=2}^{\infty} a_n z^{n-1}\right)$ and $r_2(z) = z\left(1 - \frac{1-\rho}{\phi_2(\lambda,\sigma,\rho)} z\right)$,
(27) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n t^{n-1} e^{i(n-1)\theta} \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1-\rho}{\phi_2(\lambda,\sigma,\rho)} t e^{i\theta} \right|^{\eta} d\theta.$$

By Lemma 2.1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\rho}{\phi_2(\lambda,\sigma,\rho)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\rho}{\phi_2(\lambda,\sigma,\rho)} \omega(z) \tag{28}$$

and using (16), we can say that $\omega(z)$ is analytic in $U, \omega(z) = 0$ and if

$$|\omega(z)| = \left| \frac{\phi_2(\lambda,\sigma,\rho)}{1-\rho} \sum_{n=2}^{\infty} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{\phi_n(\lambda,\sigma,\rho)}{1-\rho} |a_n| \leq |z|,$$

where $\phi_n(\lambda,\sigma,\rho) = [n(1+\sigma) - \lambda(\sigma+\rho)]|\mathrm{E}(c,k,n)|$. This completes the proof. □

Remark 2.3.

(i) For the choice of $\sigma = 1$ in Theorem 2.7, we obtained the results of Thirupathi Reddy and Venkateswarlu [23], Theorems 2.7.

(ii) For the choice of $\sigma = 1$ and $\lambda = 1$ in Theorem 2.7, we obtained the results of Thirupathi Reddy and Venkateswarlu [22], Theorems 7.

Conclusion: The work has successfully extended the theory of generalized Bessel functions and shown how these functions fit within the broader framework of analytic function theory. We have established key geometric properties, provided coefficient estimates, and derived important integral inequalities that will pave the way for future research. The class represents a valuable contribution to the study of special functions and their geometric properties, opening up new avenues for exploring their applications in mathematical physics, complex analysis, and other areas.

Conflict of Interests

The authors declare that there is no conflict of interests.

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