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ON SRIVASTAVA - ATTIYA INTEGRAL OPERATORS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{S}_\alpha^*$ denote the class of functions f analytic in the open unit disc $\mathcal{U}$ with normalizations $f(0) = 0 = f'(0) - 1$ satisfying

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \alpha, \quad z \in \mathcal{U}.$$

We determine β so that whenever $\mathrm{J}_{s,b}(f) \in \mathcal{S}_\beta^*$, then $\mathrm{J}_{s+1,b}(f) \in \mathcal{S}_\alpha^*$, for all $s \in \mathbb{C},\ b \neq 0, -1, -2, ...$ where $\mathrm{J}_{s,b}(f)$ is the Srivastava - Attiya integral operator.

Keywords: Alexander operator, Libera operator and Srivastava - Attiya operator.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2z^2 + ...$, analytic in the unit disc $\mathcal{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{P}_\alpha$ denote the class of functions p , analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\left| \frac{p(z) - 1}{p(z) + 1} \right| < \alpha, \quad 0 < \alpha \leq 1,\ z \in \mathcal{U}.$$

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Obviously $\mathcal{P}_\alpha \subset \mathcal{P}$, the class of functions with positive real part.

Let $\mathcal{S}^*_\alpha$ denote the class of functions in $\mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_\alpha, \quad z \in \mathcal{U}.$$

$\mathcal{S}^*_1$ is the well known class $\mathcal{S}^*$ of starlike functions with respect to the origin. Srivastava and Attiya [7] defined the operaor $\mathrm{J}_{s,b}(f)$ as

$$\mathrm{J}_{s,b}(f)(z) = G_{s,b}(z) * f(z), \quad (z \in \mathcal{U},\ f \in \mathcal{A}),$$

where $*$ denotes the Hadamard product or convolution and

$$G_{s,b}(z) = (1+b)^s \left[\phi(z,s,b) - b^{-s}\right], \quad (z \in \mathcal{U},\ s \in \mathbb{C},\ b \neq 0,-1,-2,....).$$

Here $\phi(z,s,b)$ is the general Hurwitz - Lerch Zeta function defined by [8]

$$\phi(z,s,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

where $s \in \mathbb{C},\ b \neq 0,-1,-2,....$, when $z \in \mathcal{U}$, $\Re\{s\} > 1$ when $|z| = 1$.

$$\begin{aligned}
\mathrm{J}_{0,b}(f)(z) &= f(z)\\
\mathrm{J}_{1,0}(f)(z) &= \int_0^z \frac{f(t)}{t}\,dt = \Lambda(f)(z)\\
\mathrm{J}_{1,1}(f)(z) &= \frac{2}{z}\int_0^z \frac{f(t)}{t}\,dt = \mathrm{L}(f)(z)\\
\mathrm{J}_{1,\gamma}(f)(z) &= \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}\,dt = \mathrm{I}_\gamma(f)(z)\\
&\quad (\gamma,\ \text{is real},\ \gamma > -1)\\
\mathrm{J}_{\sigma,1}(f)(z) &= \frac{2^\sigma}{z\,\Gamma(\sigma)}\int_0^z \left(\log\left(\frac{z}{t}\right)^{\sigma-1}\right) f(t)\,dt = \mathrm{I}^\sigma(f)(z)\\
&\quad (\sigma,\ \text{is real},\ \sigma > 0)
\end{aligned}$$

where $\Lambda(f)$, $\mathrm{L}(f)$, $\mathrm{I}_\gamma(f)$, $\mathrm{I}^\sigma(f)$ are Alexander [1], Libera [4], Bernardi [2] and Jund [3] operators respectively.

In this paper we determine β so that whenever $\mathrm{J}_{s,b}(f) \in \mathcal{S}^*_\beta$, then $\mathrm{J}_{s+1,b}(f) \in \mathcal{S}^*_\alpha$. We also consider a similar problem for

$$f \in \mathcal{R}_\alpha = \left\{ f \in \mathcal{A} : \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha \right\}.$$

$\mathcal{R}_1$ is the class of $f \in \mathcal{A}$ such that f' belong to the Caratheodry class of $\mathcal{P}$ of functions. We need the following Lemmas which we will be using in the sequel.

Lemma 1.1. [7] *If the function f belongs to $\mathcal{A}$, then*

$$z\mathrm{J}'_{s+1,b}(f)(z) = (1+b)\mathrm{J}_{s,b}(f)(z) - b\mathrm{J}_{s+1,b}(f)(z) \tag{1.1}$$

for $z \in \mathbb{C},\ s \in \mathbb{C},\ b \neq 0, -1, -2, \ldots.$

Lemma 1.2. [5] *Suppose that the function $\omega(z)$ is regular in $\mathcal{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, we have,*

(1) $z_0\omega'(z_0) = k\omega(z_0)$ *and*
(2) $\Re\left\{1 + \dfrac{z\omega''(z_0)}{\omega'(z_0)}\right\} \geq k$ *where k is real and $k \geq 1$.*

2. MAIN RESULTS

Theorem 2.1. *Let $\beta = \alpha\left(\dfrac{2+\alpha+b(1-\alpha)}{1+2\alpha+b(1-\alpha)}\right)$ and $\mathrm{J}_{s,b}$ be the Srivastava - Attiya operator. If $\mathrm{J}_{s,b}(f) \in \mathcal{S}^*_\beta$, then $\mathrm{J}_{s+1,b}(f) \in \mathcal{S}^*_\alpha$ for $0 < \alpha \leq 1,\ s \in \mathbb{C},\ b \neq 0, -1, -2, \ldots.$*

Proof. Let us define a function $\omega(z)$ by

$$\omega(z) = \frac{1}{\alpha}\left\{\frac{\dfrac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)} - 1}{\dfrac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)} + 1}\right\}, \quad \text{for, } 0 < \alpha \leq 1 \tag{2.1}$$

and $\omega(z) \neq 1$ for $z \in \mathcal{U}$. Then, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$. It is sufficient to show that $|\omega(z)| < 1$ in $\mathcal{U}$. From (1.1) we have

$$\frac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)} = \frac{1+\alpha\omega(z)}{1-\alpha\omega(z)}.$$

Logarithmic differentiation yields

$$1 + \frac{z\mathrm{J}''_{s+1,b}(f)(z)}{\mathrm{J}'_{s+1,b}(f)(z)} - \frac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)} - 1 = \frac{2\alpha z\omega'(z)}{1-\alpha^2\omega^2(z)}.$$

Taking logarithmic derivative of (1.1) we have

$$\frac{z\mathrm{J}'_{s,b}(f)(z)}{\mathrm{J}_{s,b}(f)(z)} = \frac{1+\alpha\omega(z)}{1-\alpha\omega(z)}\left\{\frac{2\alpha z\omega'(z)}{(1+\alpha\omega(z))(1+b+(1-b)\alpha\omega(z))}+1\right\}$$

Thus,

$$\frac{z\mathrm{J}'_{s,b}(f)(z)}{\mathrm{J}_{s,b}(f)(z)} = \frac{2\alpha z\omega'(z)}{(1-\alpha\omega(z))(1+b+(1-b)\alpha\omega(z))}+\frac{1+\alpha\omega(z)}{1-\alpha\omega(z)}.$$

Let there exist a point $z_0 \in \mathcal{U}$ such that $\max|\omega(z)| = |\omega(z_0)| = 1$, then by Lemma (1.2), $|z| < |z_0|$.

We have $z_0\omega'(z_0) = k\omega(z_0),\ k \geq 1$.

Then we obtain

$$\left\{\frac{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}-1}{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}+1}\right\} = \frac{\alpha\omega(z_0)\left(k+1+b+(1-b)\alpha\omega(z_0)\right)}{(1+b)+\alpha\omega(z_0)(1-b+k)}$$

and

$$\left|\frac{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}-1}{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}+1}\right| = \frac{\alpha\left|(k+1+b)+(1-b)\alpha e^{i\theta}\right|}{|1+b+\alpha e^{i\theta}(1-b+k)|} = \phi(\cos\theta), \tag{2.2}$$

where $\phi(t)$ is a decreasing function of $t = \cos\theta$ in $[-1, 1]$.

Hence from (2.2) we get

$$\left|\frac{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}-1}{\frac{z_0\mathrm{J}'_{s,b}(f)(z_0)}{\mathrm{J}_{s,b}(f)(z_0)}+1}\right| \geq \alpha\left\{\frac{(b+2)+\alpha(1-b)}{(2-b)\alpha+1+b}\right\} = \beta,$$

a contradiction to the hypothesis that $\mathrm{J}_{s,b}(f)(z) \in \mathcal{S}^*(\beta)$. Hence, we have

$$|\omega(z)| = \frac{1}{\alpha}\left|\frac{\frac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)}-1}{\frac{z\mathrm{J}'_{s+1,b}(f)(z)}{\mathrm{J}_{s+1,b}(f)(z)}+1}\right| < 1$$

or $\mathrm{J}_{s+1,b}(f)(z) \in \mathcal{S}^*_\alpha$, which completes the proof of the theorem. □

Theorem 2.2. *Let* $\beta = \dfrac{2-\alpha+b(1-\alpha)}{1+b(1-\alpha)}$ *and if* $\mathrm{J}_{s,b}(f)(z) \in \mathcal{R}_\beta$, *then* $\mathrm{J}_{s+1,b}(f)(z) \in \mathcal{R}_\alpha$, *for* $0 < \alpha \leq 1$.

Proof. Let $\omega(z)$ be defined by

$$\omega(z) = \frac{1}{\alpha}\left\{\frac{z\mathrm{J}'_{s+1,b}(f)(z)-1}{\mathrm{J}_{s+1,b}(f)(z)+1}\right\} \tag{2.3}$$

and $\omega(z) \neq 1$ for $z \in \mathcal{U}$. Then, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$. It is sufficient to show that $|\omega(z)| < 1$ in $\mathcal{U}$. From (2.3) we have

$$\mathrm{J}'_{s+1,b}(f)(z) = \frac{1+\alpha\omega(z)}{1-\alpha\omega(z)}.$$

Differentiating we get

$$\mathrm{J}'_{s+1,b}(f)(z) = \mathrm{J}'_{s+1,b}(f)(z) + \frac{z\mathrm{J}''_{s+1,b}(f)(z)}{b+1}$$

$$\frac{\mathrm{J}'_{s,b}(f)(z)-1}{\mathrm{J}'_{s,b}(f)(z)+1} = \frac{\mathrm{J}'_{s+1,b}(f)(z) - 1 - \dfrac{z\mathrm{J}''_{s+1,b}(f)(z)}{b+1}}{\mathrm{J}'_{s+1,b}(f)(z) + 1 + \dfrac{z\mathrm{J}''_{s+1,b}(f)(z)}{b+1}}$$

$$= \omega(z)\left\{\frac{\alpha(1+b+k)-(1+b)\alpha^2\omega(z)}{((1+b)(1-\alpha\omega(z)))+\alpha k\omega(z)}\right\}$$

Lemma 1.2 gives the existence of a point $z_0 \in \mathcal{U}$ such that $\max_{|z|<|z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Hence $z_0\omega'(z_0) = k\omega(z_0),\ k \geq 1$. Hence we obtain

$$\left|\frac{\mathrm{J}'_{s,b}(f)(z_0)-1}{\mathrm{J}'_{s,b}(f)(z_0)+1}\right| = \left|\frac{\alpha(1+b+k)-(1+b)\alpha^2 e^{i\theta}}{(1+b)+(k-(1+b))\alpha e^{i\theta}}\right|$$

$$= \frac{\alpha\left\{(1+b+k)^2+(1+b)^2\alpha^2-2\alpha(1+b)(1+b+k)\cos\theta\right\}^{\frac{1}{2}}}{\left\{(1+b)^2+(k-(1+b))^2\alpha^2+2\alpha(1+b)(k-1-c)\cos\theta\right\}} \tag{2.4}$$

$$= \phi(\cos\theta).$$

$\phi(t)$ is a decreasing function of $t = \cos\theta$ in $[-1, 1]$.

Hence from (2.4) we get

$$\left|\frac{\mathrm{J}'_{s,b}(f)(z_0)-1}{\mathrm{J}'_{s,b}(f)(z_0)+1}\right| \geq \alpha\left\{\frac{b(1-\alpha)+(2-\alpha)}{1+b(1-\alpha)}\right\} = \beta$$

which is a contradiction to our assumption that $\mathrm{J}_{s,b}(f) \in \mathcal{R}_\beta$.

Hence we must have

$$|\omega(z)| = \frac{1}{\alpha}\left|\frac{z\mathrm{J}'_{s+1,b}(f)(z)-1}{\mathrm{J}_{s+1,b}(f)(z)+1}\right| < 1$$

or $\mathrm{J}_{s+1,b}(f) \in \mathcal{R}_\alpha$ which completes the proof of the theorem. □

Remark 2.3. *For $s = 0$, we get the results in* [6].

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