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PARAMETRIZATION OF PERSPECTIVE SILHOUETTES ON CANAL SURFACES IN MINKOWSKI 3-SPACE

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Abstract. A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ of its centers and a radius function $r(t)$ and canal surface is parametrized through Frenet frame of the spine curve $C(t)$. In this paper, we parametrize the perspective silhouette of a canal surface in Minkowski 3-space when the spine curve $C(t)$ is a spacelike or timelike curve and then we detect all connected components of the silhouette.

Keywords: The perspective silhouette; Minkowski 3-Space; Spacelike curve; Timelike curve; Canal surface.

2000 AMS Subject Classification: 53A04; 53A05

1. Introduction

The perspective silhouette curve of a parametric surface $S(u,v)$ comprises a set of surface points which satisfy

$$N(u,v)\cdot(S(u,v)-\overrightarrow{O})=0$$

where $N(u,v)$ is the surface normal of $S(u,v)$, $\overrightarrow{O}$ is the viewpoint and "·" is the dot

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product in the Euclidean 3-space. For the silhouette curve, an alternative definition can be given below.

Silhouette curve can be defined as a nice consequence of Lambert's cosine law in optics branch of physics. Lambert's law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector N and the light vector d (Here, in the case of silhouette curve $\cos\theta = 0$, i.e., $\theta = \frac{\pi}{2}$). According to this law the intensity is irrespective of the actual viewpoint, hence the illumination is the same when viewed from any direction [11].

In computer graphics, silhouette finding and rendering has a central role in a growing number of applications. The silhouette is the simplest form of line art and is used in cartoons, technical illustrations, architectural design and medical atlases. In non-photorealistic rendering (NPR), complex models and scenes are rendered as simple line drawings by rendering silhouette edges [1].

Silhouettes are among the most important lines in describing the shape of a three-dimensional object. Also, they play a significant role in non-photorealistic rendering. More recently, Seong *et al*. [10] introduced an efficient and robust algorithm for computing the perspective silhouette of the boundary of a general swept volume and also construct the topology of connected components of the silhouette.

Kim and Lee [7] presented a method for computing the perspective silhouette of canal surfaces. They utilized the fact that both these types of surface can be decomposed into a set of circles and the normal vectors of these circles form a cone. Using the characteristics, they computed the perspective silhouettes of these surfaces.

A canal surface is the envelope of a family of one parameter spheres and is useful to represent various objects e.g. pipe, hose, rope or intestine of a body. Moreover, canal surface is an important instrument in surface modelling for CAD/CAM such as tubular surfaces, torus and Dupin cyclides.

This paper is organized as follows. Section 2 presents basic concepts about curves in Minkowski 3-space. In section 3 we observe the perspective silhouette of a canal surface in Euclidean 3-space. Finally, in section 4 we obtain the perspective silhouettes of canal surfaces in Minkowski 3-space.

2. Preliminaries

We start to introduce Minkowski 3-space. The space R_1^3 is a three dimensional real vector space endowed with the inner product

$$\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

This space is called Minkowski 3-space or Lorentz Minkowski space and denoted by E_1^3. A vector in this space is said to be spacelike, timelike and lightlike (null) if $\langle x, x \rangle > 0$ or $x = 0$, $\langle x, x \rangle < 0$ and $\langle x, x \rangle = 0$ or $x \neq 0$, respectively. Also, a regular curve $\alpha : I \longrightarrow E_1^3$ is called spacelike, timelike and lightlike if the velocity vector $\dot{\alpha}$ is spacelike, timelike and lightlike, respectively [8].

The cross product of $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in R_1^3 is defined as follows.

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2)$$

where δ_{ij} is kronecker delta, $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = -e_2$.

Let $\{t, n, b\}$ be the moving Frenet frame along the curve α with arclenght parameter s. For a spacelike curve α, the Frenet-Serret equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where $\langle t, t \rangle = 1$, $\langle n, n \rangle = \pm 1$, $\langle b, b \rangle = -\varepsilon$, $\langle t, n \rangle = \langle t, b \rangle = \langle n, b \rangle = 0$ and κ is the curvature and τ is the torsion of α. Here, ε determines the kind of spacelike curve α. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal n and timelike binormal b. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal n and spacelike binormal b [6].

If the curve α is timelike, then the Frenet-Serret equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where $\langle t,t\rangle = -1$, $\langle n,n\rangle = \langle b,b\rangle = 1$, $\langle t,n\rangle = \langle t,b\rangle = \langle n,b\rangle = 0$ [6].

Definition 1 ([9])**.** *Let v and w be spacelike vectors.*

(a) If v and w span a timelike vector subspace, then there is a unique non-negative real number $\theta \geq 0$ such that

$$\langle v,\omega\rangle = \|v\|\,\|w\| \cosh\theta.$$

(b) If v and w span a spacelike vector subspace, then there is a unique non-negative real number $\theta \geq 0$ such that

$$\langle v,\omega\rangle = \|v\|\,\|w\| \cos\theta.$$

Definition 2 ([9])**.** *Let v be a spacelike vector and w be a positive timelike vector in R_1^3. Then, there is a unique non-negative real number $\theta \geq 0$ such that*

$$\langle v,w\rangle = \|v\|\,\|w\| \sinh\theta.$$

Lemma 1. *In the Minkowski 3-space E_1^3, the following properties are satisfied.*

(i) Two timelike vectors are never orthogonal.

(ii) Two null vectors are orthogonal if and only if they are linearly dependent.

(iii) A timelike vector is never orthogonal to a null (lightlike) vector [6].

3. Canal Surface and Its Perspective Silhouette in $\mathbb{E}^3$

A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ (spine curve) of its center and a radius function $r(t)$. This moving sphere $S(t)$ is tangent to the canal surface at a characteristic circle $K(t)$. Now, we decompose and parametrize the canal surface by means of its characteristic circles. A section of the canal surface can be given as follows.

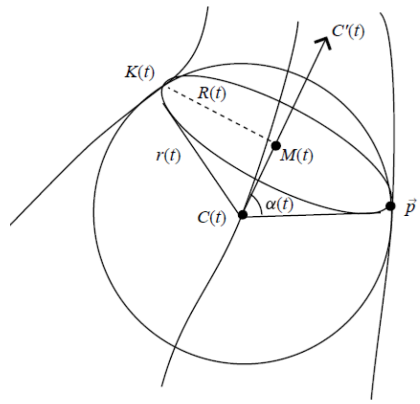

Figure 1.[7] A characteristic circle $K(t)$ on the sphere $S(t)$

In this case the canal surface point $p = K(t,\theta)$ holds the following equations.

$$\|p - C(t)\| = r(t)$$
$$(p - C(t)) \cdot C^{'}(t) + r(t)r^{'}(t) = 0.$$

For the point $p = K(t,\theta)$, the vector $\overrightarrow{C(t)M(t)}$ is orthogonal projection of $\overrightarrow{C(t)p}$ onto tangent $C^{'}(t)$ as obtained below.

$$\begin{aligned} \overrightarrow{C(t)M(t)} &= \frac{\overrightarrow{C(t)p} \cdot C^{'}(t)}{C^{'}(t) \cdot C^{'}(t)} C^{'}(t) \\ M(t) - C(t) &= \frac{(p - C(t)) \cdot C^{'}(t)}{C^{'}(t) \cdot C^{'}(t)} C^{'}(t). \end{aligned}$$

Furthermore, since $(p - C(t)) \cdot C^{'}(t) = -r(t)r^{'}(t)$ we get the center $M(t)$ and radius function $R(t)$ of characteristic circles as

$$(3.1) \quad M(t) = C(t) + r(t)\cos\alpha(t)\frac{C^{'}(t)}{\|C^{'}(t)\|}; \ \ \cos\alpha(t) = -\frac{r^{'}(t)}{\|C^{'}(t)\|}$$
$$R(t) = r(t)\sin\alpha(t) = r(t)\frac{\sqrt{\|C^{'}(t)\|^{2} - r^{'}(t)^{2}}}{\|C^{'}(t)\|}$$

where $\alpha(t)$ is the angle between $\overrightarrow{C(t)p}$ and $C^{'}(t)$. Thus, the canal surface is parametrized as follows.

$$(3.2)\quad K(t,\theta) = M(t) + R(t)\left(\cos\theta n(t) + \sin\theta b(t)\right)$$

$$K(t,\theta) = C(t) - r(t)r^{'}(t)\frac{C^{'}(t)}{\left\|C^{'}(t)\right\|^{2}} + r(t)\frac{\sqrt{\left\|C^{'}(t)\right\|^{2} - r^{'}(t)^{2}}}{\left\|C^{'}(t)\right\|}\left(\cos\theta n + \sin\theta b\right)$$

where $n(t)$ and $b(t)$ are the principal normal and binormal to $C(t)$, respectively. In other words, $n(t)$ and $b(t)$ are the basis vectors of the plane containing characteristic circle $K(t)$. Here, when $\left\|C^{'}(t)\right\|^{2} > r^{'}(t)^{2}$, the canal surface $K(t,\theta)$ is regular.

From now on, we will examine the perspective silhouette of canal surface in Euclidean 3-space [7]. For the regular canal surface $K(t,\theta)$, let $t_{\min} < t < t_{\max}$ and $N(t,\theta)$ be normal vector of $K(t,\theta)$. From a given viewpoint $\overrightarrow{O} = (O_x, O_y, O_z)$, the perspective silhouette of canal surface is the set of points which satisfy

$$(3.3)\qquad N(t,\theta)\cdot(K(t,\theta) - \overrightarrow{O}) = 0.$$

Since tangent plane at p is the same for canal surface and moving sphere, the normal $N(t,\theta)$ can be written as

$$(3.4)\qquad N(t,\theta) = K(t,\theta) - C(t).$$

If Eq (3.4) is substituted in Eq (3.3), it follows that

$$A(t)\cos\theta + B(t)\sin\theta + D(t) = 0.$$

Then, the perspective silhouette of the canal surface is parametrized by

$$(3.5)\qquad p(t) = M(t) + R(t)(c(t)n(t) + s(t)b(t))$$

where

$$\cos\theta = \frac{-A(t)D(t) \mp B(t)\sqrt{A(t)^2 + B(t)^2 - D(t)^2}}{A(t)^2 + B(t)^2} = c(t)$$

$$\sin\theta = \frac{-B(t)D(t) \mp A(t)\sqrt{A(t)^2 + B(t)^2 - D(t)^2}}{A(t)^2 + B(t)^2} = s(t)$$

and

$$\begin{aligned} A(t) &= n(t)\cdot(C(t)-\overrightarrow{O}) \\ B(t) &= b(t)\cdot(C(t)-\overrightarrow{O}) \\ D(t) &= \frac{-r'(t)C'(t)\cdot(C(t)-\overrightarrow{O})+r(t)\left\|C'(t)\right\|^2}{\|C'(t)\|\sqrt{\|C'(t)\|^2-r'(t)^2}}. \end{aligned}$$

If it is computed a set of points $p(t)$ by varying the value of the parameter t and connected them, then the components of the silhouette are traced. Since $A(t)$, $B(t)$ and $D(t)$ are continuous functions, $A(t)^2+B(t)^2-D(t)^2$ is also a continuous function. If there are two values t_0 and t_1, such that $t_{\min}\leq t_0,t_1\leq t_{\max}$ and which also satisfy

$$A(t_0)^2+B(t_0)^2-D(t_o)^2<0 \text{ and } A(t_1)^2+B(t_1)^2-D(t_1)^2>0,$$

then there exists a value t_m between t_0 and t_1 such that $A(t_m)^2+B(t_m)^2-D(t_m)^2=0$. Therefore, the solutions of t which satisfy $A(t)^2+B(t)^2-D(t)^2=0$ represent the boundary values of t for the connected components of the silhouette. Thus, if $A(t)$, $B(t)$ and $D(t)$ are substituted in the equation $A(t)^2+B(t)^2-D(t)^2=0$ and it is solved the obtained equation, the connected components of the silhouette are found.

4. The Perspective Silhouette of A Canal Surface in $\mathbb{E}_1^3$

In this section we will obtain the perspective silhouette of a canal surface in Minkowski 3-space E_1^3. Initially, let us give canal surfaces in E_1^3. A canal surface point $p=K(t,\theta)$ holds the following equations.

$$\begin{gathered} \|p-C(t)\|_L=r(t) \\ (p-C(t))\cdot C'(t)+r(t)r'(t)=0. \end{gathered}$$

In this case,

(1) For a spacelike center curve $C(t)$ with the spacelike normal, the canal surface is

parametrized by

$$(4.1)\quad K(t,\theta)=C(t)-r(t)r^{'}(t)\frac{C^{'}(t)}{\|C^{'}(t)\|^{2}}\mp r(t)\frac{\sqrt{\|C^{'}(t)\|^{2}-r^{'}(t)^{2}}}{\|C^{'}(t)\|}(\cosh\theta n+\sinh\theta b)$$

where $T=\dfrac{C^{'}(t)}{\|C^{'}(t)\|}$ [5].

(2) For a spacelike center curve $C(t)$ with the timelike normal, the canal surface is parametrized by

$$(4.2)\quad K(t,\theta)=C(t)-r(t)r^{'}(t)\frac{C^{'}(t)}{\|C^{'}(t)\|^{2}}\mp r(t)\frac{\sqrt{\|C^{'}(t)\|^{2}-r^{'}(t)^{2}}}{\|C^{'}(t)\|}(\sinh\theta n+\cosh\theta b)$$

where $T=\dfrac{C^{'}(t)}{\|C^{'}(t)\|}$ [3].

(3) For a timelike center curve $C(t)$, the canal surface is parametrized by

$$(4.3)\qquad K(t,\theta)=C(t)+r(t)r^{'}(t)\frac{C^{'}(t)}{\|C^{'}(t)\|^{2}}\mp r(t)\frac{\sqrt{\|C^{'}(t)\|^{2}+r^{'}(t)^{2}}}{\|C^{'}(t)\|}(\cos\theta n+\sin\theta b)$$

where $T=\dfrac{C^{'}(t)}{\|C^{'}(t)\|}$ [4].

In three cases above, since

$$\langle N(t,\theta),N(t,\theta)\rangle_{L}=\langle K(t,\theta)-C(t),K(t,\theta)-C(t)\rangle_{L}=r^{2}(t)>0,$$

the normal vector $N(t,\theta)$ becomes spacelike, that is, the canal surfaces which are obtained become timelike. For this reason, the perspective silhouette of canal surface in E_1^3 can be spacelike or timelike. For the cases (1) and (2), because

$$\langle p-C(t),p-C(t)\rangle_{L}=r^{2}(t)>0 \text{ and } \left\langle C^{'}(t),C^{'}(t)\right\rangle>0,$$

from the Definition 1(a) the angle between $p-C(t)$ and $C^{'}(t)$ is $\cos\alpha(t)=-\dfrac{r^{'}(t)}{\|C^{'}(t)\|}$.

Also, according to the Definition 1(b) the angle between $p-C(t)$ and $C^{'}(t)$ is

$$\cosh\alpha(t)=-\frac{r^{'}(t)}{\|C^{'}(t)\|}.$$

Then, the center and radius of characteristic circles are as follows.

$$\begin{aligned} M(t) &= C(t) - r(t)r'(t)\frac{C'(t)}{\|C'(t)\|^2} \\ R(t) &= \sqrt{r^2(t) - \|M(t) - C(t)\|^2} \\ &= r(t)\frac{\sqrt{\|C'(t)\|^2 - r'(t)^2}}{\|C'(t)\|}. \end{aligned}$$

Now, we will parametrize the perspective silhouette curve for three cases.

(1) For Eq (4.1), if we firstly substitute $N(t,\theta)$ and $K(t,\theta)$ below

$$N(t,\theta) \cdot (K(t,\theta) - \overrightarrow{O}) = 0$$

$$(K(t,\theta) - C(t)) \cdot (K(t,\theta) - \overrightarrow{O}) = 0$$

$$\begin{gathered} [\frac{-r(t)r'(t)}{\|C'(t)\|^2}C'(t) + R(t)(\cosh\theta n + \sinh\theta b)] \cdot \\ [(C(t) - \overrightarrow{O}) - \frac{r(t)r'(t)}{\|C'(t)\|^2}C'(t) + R(t)(\cosh\theta n + \sinh\theta b)] = 0 \\ \frac{-r(t)r'(t)}{\|C'(t)\|^2}[C'(t) \cdot (C(t) - \overrightarrow{O})] + R(t)\cosh\theta[n \cdot (C(t) - \overrightarrow{O})] \\ +R(t)\sinh\theta[b \cdot (C(t) - \overrightarrow{O})] + R(t)^2 + \frac{r(t)^2 r'(t)^2}{\|C'(t)\|^2} = 0 \end{gathered}$$

and then we multiply the last equation by $\frac{1}{R(t)}$, we obtain

$$n \cdot (C(t) - \overrightarrow{O})\cosh\theta + b \cdot (C(t) - \overrightarrow{O})\sinh\theta + \frac{-r'C'(t) \cdot (C(t) - \overrightarrow{O}) + r\left\|C'(t)\right\|^2}{\|C'(t)\|\sqrt{\|C'(t)\|^2 - r'(t)^2}} = 0.$$

By taking

$$\begin{aligned} A(t) &= n(t) \cdot (C(t) - \overrightarrow{O}) \\ B(t) &= b(t) \cdot (C(t) - \overrightarrow{O}) \\ D(t) &= \frac{-r'(t)C'(t) \cdot (C(t) - \overrightarrow{O}) + r(t)\left\|C'(t)\right\|^2}{\|C'(t)\|\sqrt{\|C'(t)\|^2 - r'(t)^2}} \end{aligned}$$

we obtain

$$A(t)\cosh\theta + B(t)\sinh\theta + D(t) = 0.$$

Since $\cosh\theta = \sqrt{1+\sinh^2\theta}$, we get the quadratic equation with unknown $\sinh\theta$

$$(A(t)^2 - B(t)^2)\sinh^2\theta - 2B(t)D(t)\sinh\theta + A(t)^2 - D(t)^2 = 0.$$

Solutions of this quadratic equation are

$$\sinh\theta = \frac{B(t)D(t) \mp A(t)\sqrt{B(t)^2 + D(t)^2 - A(t)^2}}{A(t)^2 - B(t)^2} = sh(t).$$

So we have

$$\cosh\theta = \frac{-A(t)D(t) \pm A(t)\sqrt{B(t)^2 + D(t)^2 - A(t)^2}}{A(t)^2 - B(t)^2} = ch(t).$$

Then, the perspective silhouette can be parametrized by

$$p(t) = M(t) + R(t)(ch(t)n(t) + sh(t)b(t)). \tag{4.4}$$

(2) For Eq (4.2), using $N(t,\theta) = K(t,\theta) - C(t)$ and $N(t,\theta)\cdot(K(t,\theta) - \overrightarrow{O}) = 0$, we get

$$n\cdot(C(t) - \overrightarrow{O})\sinh\theta + b\cdot(C(t) - \overrightarrow{O})\cosh\theta + \frac{-r'C'(t)\cdot(C(t) - \overrightarrow{O}) + r\left\|C'(t)\right\|^2}{\|C'(t)\|\sqrt{\left\|C'(t)\right\|^2 - r'(t)^2}} = 0.$$

If we take

$$\begin{aligned} A(t) &= n(t)\cdot(C(t) - \overrightarrow{O}) \\ B(t) &= b(t)\cdot(C(t) - \overrightarrow{O}) \\ D(t) &= \frac{-r'(t)C'(t)\cdot(C(t) - \overrightarrow{O}) + r(t)\left\|C'(t)\right\|^2}{\|C'(t)\|\sqrt{\left\|C'(t)\right\|^2 - r'(t)^2}} \end{aligned}$$

we obtain

$$A(t)\sinh\theta + B(t)\cosh\theta + D(t) = 0.$$

Since $\cosh\theta = \sqrt{1+\sinh^2\theta}$, we get the quadratic equation with unknown $\sinh\theta$

$$(A(t)^2 - B(t)^2)\sinh^2\theta + 2A(t)D(t)\sinh\theta + D(t)^2 - B(t)^2 = 0.$$

Solutions of this quadratic equation are

$$\sinh\theta = \frac{-A(t)D(t) \mp B(t)\sqrt{A(t)^2 - B(t)^2 + D(t)^2}}{A(t)^2 - B(t)^2} = sh(t).$$

Hence we have

$$\cosh\theta = \frac{-B(t)D(t) \mp A(t)\sqrt{A(t)^2 - B(t)^2 + D(t)^2}}{A(t)^2 - B(t)^2} = ch(t).$$

Then, the perspective silhouette can be parametrized by

$$p(t) = M(t) + R(t)(sh(t)n(t) + ch(t)b(t)). \tag{4.5}$$

(3) In the third case, since $\left\langle C^{'}(t), C^{'}(t)\right\rangle < 0$ and $\langle p - C(t), p - C(t)\rangle_L = r^2(t) > 0$, from Definition (2) $\sinh\alpha(t) = \dfrac{r^{'}(t)}{\|C'(t)\|}$. Then

$$\begin{aligned} M(t) &= C(t) + r(t)r^{'}(t)\frac{C^{'}(t)}{\|C'(t)\|^2} \\ R(t) &= \sqrt{r^2(t) - \|M(t) - C(t)\|^2} \\ &= r(t)\frac{\sqrt{\|C'(t)\|^2 + r'(t)^2}}{\|C'(t)\|}. \end{aligned}$$

For Eq (4.3), applying $N(t,\theta)\cdot(K(t,\theta) - \overrightarrow{O}) = 0$ we obtain

$$n\cdot(C(t) - \overrightarrow{O})\cos\theta + b\cdot(C(t) - \overrightarrow{O})\sin\theta + \frac{r^{'}C^{'}(t)\cdot(C(t) - \overrightarrow{O}) + r\left\|C^{'}(t)\right\|^2}{\|C'(t)\|\sqrt{\|C'(t)\|^2 + r'(t)^2}} = 0.$$

If we say

$$\begin{aligned} A(t) &= n(t)\cdot(C(t) - \overrightarrow{O}) \\ B(t) &= b(t)\cdot(C(t) - \overrightarrow{O}) \\ D(t) &= \frac{r^{'}(t)C^{'}(t)\cdot(C(t) - \overrightarrow{O}) + r(t)\left\|C^{'}(t)\right\|^2}{\|C'(t)\|\sqrt{\|C'(t)\|^2 + r'(t)^2}} \end{aligned}$$

we get

$$A(t)\cos\theta + B(t)\sin\theta + D(t) = 0.$$

Therefore it concludes that

$$\cos\theta = \frac{-A(t)D(t) \mp B(t)\sqrt{A(t)^2 + B(t)^2 - D(t)^2}}{A(t)^2 + B(t)^2} = c(t)$$

and

$$\sin\theta = \frac{-B(t)D(t) \mp A(t)\sqrt{A(t)^2 + B(t)^2 - D(t)^2}}{A(t)^2 + B(t)^2} = s(t).$$

In this case, the perspective silhouette can be parametrized by

$$p(t) = M(t) + R(t)(c(t)n(t) + s(t)b(t)). \tag{4.6}$$

For the cases (1) and (2) the solutions of t which satisfy $B(t)^2 + D(t)^2 - A(t)^2 = 0$ and $A(t)^2 - B(t)^2 + D(t)^2 = 0$ determine the boundary values of t for the connected components of the silhouette, respectively. Again, for the case (3) the solutions of t which satisfy $A(t)^2 + B(t)^2 - D(t)^2 = 0$ determine the boundary values of t for the connected components of the silhouette. Finally, by solving these equations, each connected component of the perspective silhouettes are obtained.

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