Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 3, 891-904
ISSN: 1927-5307

SOME REMARKS ON OPERATIONS ON GRAPHS

S. N. DAOUD[1,2,*] AND O. A. EMBABY[1,3]

[1]Department of Applied Mathematics, Faculty of Applied Science, Taibah University, Al-Madinah, K.S.A.

[2]Department of Mathematics, Faculty of Science, El-Minufiya University, ShebeenEl-Kom, Egypt

[3]Department of Mathematics, Faculty of Science, Tanta University, Tanta Egypt

Abstract. In this paper we investigate the closedness of some known operations on certain kinds of graphs. We show that the operations, Cartesian product and Tensor product are closed on Hamiltonian, Eulerian, and perfect graphs while they are not closed on triangulated graphs, it also the operation, join product, is shown to be not closed on Eulerian graphs but is closed on Hamiltonian, perfect and triangulated graphs. Other operations and graphs are also investigated.

Keywords: Cartesian product, Join product, Tensor product, Normal product, Composition product, Triangulated, Perfect, Hamiltonian, Eulerian graph.

Mathematics Subject Classification: 05C05, 05C50.

1- Introduction

By a simple graph *G*, we mean that a graph with no loops or multiple edges.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs. Then

(1) The simple graph $G = (V, E)$, where $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$ is called the union of G_1 and G_2, and is denoted by $G_1 \bigcup G_2$, [2,5].

When G_1 and G_2 are vertex disjoint, $G_1 \bigcup G_2$ is denoted by $G_1 + G_2$, and is called the sum of the graphs G_1 and G_2.

(2) If G_1 and G_2 are vertex-disjoint graphs. Then the join, $G_1 \vee G_2$, is the super-graph of $G_1 + G_2$, in which each vertex of G_1 is adjacent to every vertex of G_2, [2,6].

---

*Corresponding author

Received April 2, 2013

(3) The Cartesian product, $G_1 \times G_2$, is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \bigcup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ iff either:

(i) $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or

(ii) u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, [1,7].

(4) The composition, or lexicographic product, $G_1[G_2]$, is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2), (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2.

The graph $G_1[G_2]$ need not to be isomorphic to $G_2[G_1]$, [2,8].

(5) The normal product, or the strong product, $G_1 \circ G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ iff either:

(i) $u_1 = v_1$ and u_2 is adjacent to v_2, or

(ii) u_1 is adjacent to v_1 and $u_2 = v_2$, or

(iii) u_1 is adjacent to v_1 and u_2 is adjacent to v_2, [2,9].

(6) The tensor product, or Kronecher product, $G_1 \otimes G_2$, is a simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ iff u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

Note that $G_1 \circ G_2 = (G_1 \times G_2) \bigcup (G_1 \otimes G_2)$, [2,10].

(7) The *k*th power G^k of a simple graph *G* has $V(G^k) = V(G)$ where *u* and *v* are adjacent in G^k wherever $d_G(u, v) \le k$, where $d_G(u, v)$ is the length of a shortest $u - v$ path in *G*, [2].

(8) The closure of a graph *G*, denoted $cl(G)$, is that graph obtained from *G* by recursively joining pairs of nonadjacent vertices whose degree sum is at least *n* until no such pair remains, [3].

2- Main results

(2-1) Regular graphs

A graph *G* is said to be regular if all the vertices of *G* have the same degree, [1].

Lemma (2.1.1) A finite graph *G* has at least two vertices of the same degree.

Lemma (2.1.2) If G is a graph with n vertices which is regular of degree d, then $x(G) \geq \frac{n}{n-d}$.

Lemma (2.1.3) If $G^k = cl(G)$, then G is a regular graph.

Lemma (2.1.4) Complement of a regular graph is regular with degree $d' = n - d - 1$.

Lemma (2.1.5) IfG is a regular graph, then G^k and $cl(G)$ are regular graphs.

Theorem (2.1.6) Let G_1 and G_2 be regular graphs such that $d_{G_1} = d_1$and $d_{G_2} = d_2$, then:

(i) $G_1 \cup G_2$is not regular graph.

(ii) $G_1 \times G_2$is a regular graph with $d_{G_1 \times G_2} = d_1 + d_2$.

(iii) $G_1 \vee G_2$is a regular graph with $d_{G_1 \vee G_2} = d_1 + d_2 + 1$.

(iv) $G_1 \otimes G_2$is a regular with $d_{G_1 \otimes G_2} = d_1 d_2$.

(v) $G_1 \circ G_2$is a regular with $d_{G_1 \circ G_2} = d_1 + d_2 + d_1 d_2$.

(vi) $G_1[G_2]$ is a regular graph with $d_{G_1[G_2]} = n_2 d_1 + d_2$, and $G_2[G_1]$ is a regular graph with $d_{G_2[G_1]} = n_1 d_2 + d_1$, where n_1, n_2 are the number of vertices of G_1, G_2 respectively.

(2-2) Complete graphs

A complete graph is a graph in which every two distinct vertices are joined by exactly one edge, i.e., the complete graph is necessary simple graph. The complete graph with n vertices is denoted by K_n, [1].

Lemma (2.2.1) If G is a complete graph, then $G^k = cl(G) = G$ but the converse is not true.

G $\qquad$ $G^2 = cl(G)$

Fig. (1)

Lemma (2.2.2) If G_1 and G_2 are complete graphs, then $G_1[G_2] = G_1 \circ G_2$.

Lemma (2.2.3) Complement of a complete graph k_n is a null graph N_n.

Theorem (2.2.4) Let G_1, G_2 be complete graphs such that $G_1 = K_n, G_2 = K_m$ then:

(i) $G_1 \cup G_2$ is not complete graph.

(ii) $G_1 \times G_2$ is not complete graph.

(iii) $G_1 \otimes G_2$ is not complete graph.

(iv) $G_1 \vee G_2$ is a complete graph, $(K_n \vee K_m = K_{m+n})$.

(v) $G_1 \circ G_2 = G_1[G_2]$ is complete graph, $(K_n \circ K_m = K_n[K_m] = K_{nm})$.

(2-3) Bipartite graphs

A graph is bipartite if its vertex set can be partitioned into nonempty subsets *X* and *Y* such that each edge of *G* has one end in *X* and the other in *Y*. the pair (X, Y) is called a bipartition of the bipartite graph *G*. The bipartite graph *G* with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is complete if each vertex of *X* is adjacent to all the vertics of *Y*. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. It is noticed that $K_{p,q}$ has $p+q$ vertices (p vertices of degree p and q vertices of degree p) and pq edges, $K_{p,q} = K_{q,p}$ and $k_{1,q}$ is called a star graph, [14].

Lemma (2.3.1) For a simple bipartite graph $m \le \frac{n^2}{4}$.

Lemma (2.3.2) A bipartite graph $G(p, q)$ is complete iff $q = \binom{p}{2}$.

Lemma (2.3.3) If a bipartite graph $G(X, Y)$ is regular, then $|X| = |Y|$.

Lemma (2.3.4) Every tree is a bipartite graph.

Lemma (2.3.5) A connected graph *G* is complete bipartite iff no induced subgraph of *G* is a K_3 or P_4.

Lemma (2.3.6) If every cycle of a graph has an even number of edges, then the graph is bipartite.

Lemma (2.3.7) $N_p \vee N_q = K_{p,q}$.

Lemma (2.3.8) $(K_{p,q})^2 = K_{p+q}$ and $(K_{p,q})^k = K_{p,q}$ for $k > 2$.

Lemma (2.3.9) $cl(k_{p,p}) = k_{2p}$, $p \geq 2$.

Lemma (2.3.10) Cyclomatic number, $\mu(k_{p,q}) = (p-1)(q-1)$.

(2-4) Hamiltonian graphs

A connected graph *G* is Hamiltonian if there is a cycle which includes every vertex of *G*, such a cycle is called a Hamiltonian cycle.

Dirace's Theorem, [2] Let G be a simple graph with n vertices, where $n \geq 3$. If $\deg v \geq \frac{1}{2}n$ for each vertex *v*, then *G* is Hamiltonian.

Ore's Theorem, [2] Let G be a simple graph with n vertices, where $n \geq 3$. If $\deg v + \deg w \geq n$ for each pair of non-adjacent vertices *v* and *w*, then *G* is Hamiltonian.

Lemma (2.4.1) No tree can be Hamiltonian graph.

Lemma(2.4.2) Any bipartite graph with odd number of vertices cannot be Hamiltonian.

Lemma (2.4.3) $K_{p,q}$ is Hamiltonian if $p = q \geq 2$ and K_n is Hamiltonian if $n \geq 3$.

Lemma (2.4.4) If *G* is a (p, q) graph with $q \geq \binom{p-1}{2} + 3$, then *G* is Hamiltonian.

Lemma (2.4.5) $cl(G)$ is a Hamiltonian graph iff *G* is Hamiltonian.

Lemma (2.4.6) If *G* is a Hamiltonian graph, then G^k is a Hamiltonian graph but the converse is not true.

$G = K_{2,3}$
Not

$G^2 = K_5$
Hamiltoni

Fig. (2)

Theorem (2.4.7)

Let G_1 and G_2 be Hamiltonian graphs, then:

(i) $G_1 \times G_2$ is a Hamiltonian graph.

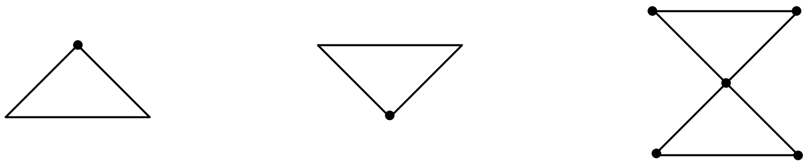
(ii) $G_1 \vee G_2$ is a Hamiltonian graph.

(iii) $G_1 \circ G_2$ is a Hamiltonian graph.

(iv) $G_1 [G_2]$ is a Hamiltonian graph.

(v) $G_1 \otimes G_2$ is a Hamiltonian graph..

(vi) $G_1 \cup G_2$ is not necessary a Hamiltonian graph.



G_1 G_2 $G_1 \cup G_2$

Fig. (3)

(2-5) Eulerian graphs

A connected graph *G* is Eulerian if there is a closed trail which includes every edge of *G*, such a trail is called Eulerian trail, [2]. A graph is Eulerianiff every vertex of *G* has even degree, [1]. A graph *G* is Eulerianiff each edge *e* of *G* belongs to an odd number of cycles of *G*, [4] i.e., a graph is Eulerianiff it has an odd number of cycle decomposition, [2].

Lemma (2.5.1) $K_{p,q}$ is an Eulerian if p, q are even, $p, q \geq 2$.

Lemma (2.5.2) If G is an Eulerian graph, then G^k and $cl(G)$ are not necessary Eulerian graphs.

G Eulerian graph $\qquad$ $G^2 = cl(G)$ Not Eulerian graph

Fig. (4)

Lemma (2.5.3) An Eulerian regular graph is a Hamiltonian graph.

Lemma (2.5.4) $K_{2n+1},\ n \geq 1$ is an Eulerian graph.

Theorem (2.5.5) Let $G_1,\ G_2$ be Eulerian graphs, then:

(i) $G_1 \cup G_2$ is not necessary Eulerian graph.

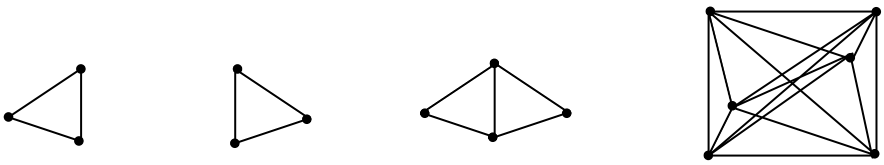
(ii) $G_1 \vee G_2$ is not necessary Eulerian graph.

(iii) Cartesian product $G_1 \times G_2$ is an Eulerian graph.

(iv) $G_1 \circ G_2$ is an Eulerian graph.

(v) $G_1[G_2]$ is an Eulerian graph.

(vi) $G_1 \otimes G_2$ is an Eulerian graph.



G_1 $\qquad$ G_2 $\qquad$ $G_1 \cup G_2$ $\qquad$ $G_1 \vee G_2$

Eulerain graphs $\qquad$ Not Eulerian graphs

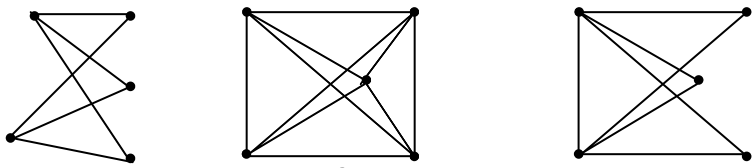
Fig. (5)

(2-6) Triangulated graphs

A simple graph G is called triangulated if every cycle of length at least four has a chord, that is an edge joining two non-adjacent vertices of the cycle, [2].

Lemma (2.6.1) If G is a triangulated graph, then G^k is a triangulated graph but the converse is not true.

Lemma (2.6.2) If G is a triangulated graph, then $cl(G)$ is a triangulated but the converse is not true.



G Not triangulated $\quad$ G^2 Triangulated $\quad$ $cl(G)$ Triangulated

Fig. (6)

Lemma (2.6.3) A complete graph K_n, $n \geq 3$ is a triangulated graph but the converse is not true.

Lemma (2.6.4) If $G_1 \cong G_2$, G_1 is triangulated, then G_2 is triangulated.

Theorem (2.6.5) Let G_1, G_2 be triangulated graphs, then:

(i) $G_1 \cup G_2$ is not necessary triangulated graph.

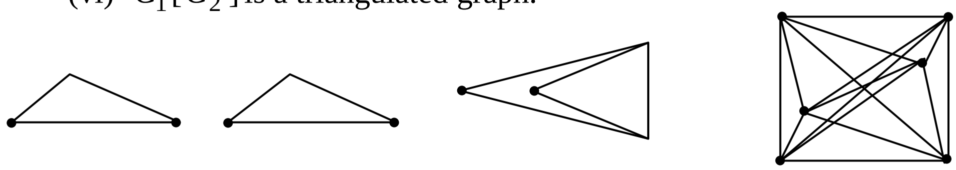
(ii) $G_1 \times G_2$ is not triangulated graph.

(iii) $G_1 \vee G_2$ is a triangulated graph.

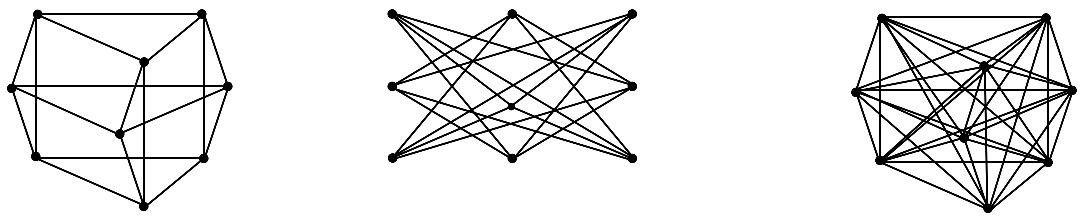
(iv) $G_1 \otimes G_2$ is not triangulated graph.

(v) $G_1 \circ G_2$ is a triangulated graph.

(vi) $G_1[G_2]$ is a triangulated graph.



G_1 $\quad$ G_2 $\quad$ $G_1 \cup G_2$ $\quad$ $G_1 \vee G_2$



$G_1 \times G_2$ $\quad$ $G_1 \otimes G_2$ $\quad$ $G_1[G_2] = G_1 \circ G_2$

Fig. (7)

(2-7) Perfect graphs

A clique of a graph *G* is a complete subgraph of *G*. A clique of *G* is a maximal clique of *G* if it is not properly contained in other clique, [2]. The maximum order of a complete subgraph of *G* is called the clique number of *G* and is denoted by $\omega(G)$. Clearly $\omega(G) \leq x(G)$, the chromatic number of *G* which is the smallest number *n* for which *G* is *n*-colorable. A graph *G* is perfect if *G* and each induced subgraphs have the property that $\omega(G) = x(G)$, [2].

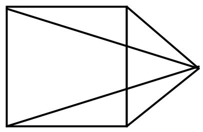
Lemma (2.7.1) The complement of any bipartite graph is perfect.

Lemma (2.7.2) The complement of a null graph N_n is perfect.

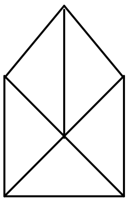
Lemma (2.7.3) If *G* is a perfect graph, then G^k is perfect.

Lemma (2.7.4) If *G* is a perfect graph, then $cl(G)$ is perfect.

Lemma (2.7.5) A perfect graph is not necessary triangulated graph and a triangulated graph is not necessary a perfect graph.



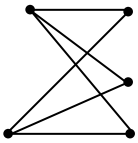
The graph is perfect but not triangulated



Weel graph, triangulated but not perfect

Fig. (8)

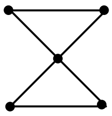
Lemma (2.7.6) An Eulerian graph is a perfect graph but the converse is not true.



The graph is perfect but not Eulerian

Fig. (9)

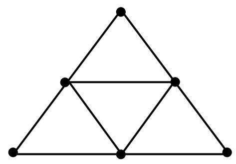
Lemma (2.7.7) Hamiltonian graph is perfect graph but the converse is not true.



The graph is perfect but not Hamiltonian

Fig. (10)

Lemma (2.7.8) A regular graph is perfect graph but the converse is not true.



Perfect graph but not regular

Fig. (11)

Theorem (2.7.9) Let $G_1,\ G_2$ be perfect graphs, then:

(i) $G_1 \vee G_2$ is a perfect graph,

$$\chi(G_1 \vee G_2) = \omega(G_1 \vee G_2) = \chi(G_1) + \chi(G_2) = \omega(G_1) + \omega(G_2).$$

(ii) $G_1 \times G_2$ is a perfect graph,

$$\chi(G_1 \times G_2) = \omega(G_1 \times G_2) = \max(\chi(G_1), \chi(G_2)) = \max(\omega(G_1), \omega(G_2)).$$

(iii) $G_1 \otimes G_2$ is a perfect graph.,

$$\chi(G_1 \otimes G_2) = \omega(G_1 \otimes G_2) = \min(\chi(G_1), \chi(G_2)) = \min(\omega(G_1), \omega(G_2)).$$

(v) $G_1 \circ G_2$ is a perfect graph,

$$\chi(G_1 \circ G_2) = \omega(G_1 \circ G_2) = \chi(G_1)\ \chi(G_2) = \omega(G_1)\ \omega(G_2).$$

(vi) $G_1[G_2]$ is a perfect graph,

$$G_1[G_2] = \omega(G_1[G_2]) = \chi(G_1)\ \chi(G_2) = \omega(G_1)\ \omega(G_2).$$

(2-8) Line graphs

The line graph $L(G)$ of a graph G is the graph obtained by taking the edges of G as vertices and joining two of these vertices whenever the corresponding edges of G have vertex in common, [1].

Lemma (2.8.1) $L(C_n) \cong C_n,\ n \geq 3$.

Lemma (2.8.2) The line graph of $K_{p,q}$ is regular of degree $p+q-2$.

Lemma (2.8.3) The line graph of K_n is regular of degree $2n-4$.

Lemma (2.8.4) L (Tetrahedron) = Octahedron.

Tetrahedron Octahedron $= L$ (Tetrahedron)

Fig. (12)

Lemma (2.8.5) The line graph of the star graph $K_{1,n}$ is the complete graph K_n.

Lemma (2.8.6) The line graph of an Eulerian graph is not necessary an Eulerian graph.

Lemma (2.8.7) The line graph of a tree is a triangulated graph.

Lemma (2.8.8) The line graph of a triangulated graph is not necessary a triangulated graph.

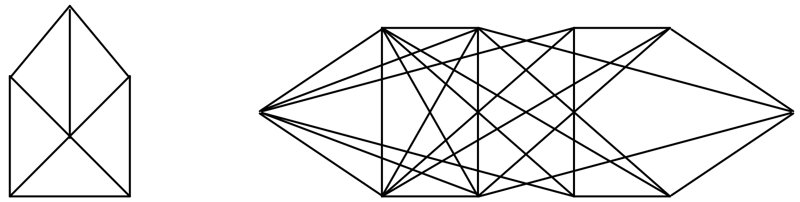
G, triangulated graph $L(G)$, not triangulated graph

Fig. (13)

Lemma (2.8.9) The line graph of a Hamiltonian graph is a Hamiltonian graph.

Lemma (2.8.10) Line graph of a perfect graph is a perfect graph but the converse is not true.

For example W_5, wheel graph, is not perfect but its line graph is perfect.



W_5 $L(W_5)$

Fig. (14)

Lemma (2.8.11) If G is not null graph. Then $\chi'(G) = \chi(L(G))$.

Lemma (2.8.12) Line graph of a regular graph is a regular graph, $d_{L(G)} = 2(d_G - 1)$.

Lemma (2.8.13) $L(P_n) = P_{n-1}$.

(2-9) Clique graphs

A clique graph $K(G)$ of a graph G is the intersection of the family of maximal cliques of G. i.e., the vertices of $K(G)$ are maximal cliques of G and two vertices of $K(G)$ are adjacent in $K(G)$ iff the corresponding cliques of G has nonempty intersection, [1].

Lemma (2.9.1) The clique graph $K(G)$ of a graph G is the same to its line graph $L(G)$ in case of G is:

(i) The star graph $K_{1,n}$.

(ii) Complete bipartite graph $K_{p,q}$.

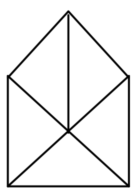
(iii) Cycle graph $C_{n,}$ $n \geq 3$.

(iv) Path graph P_n.

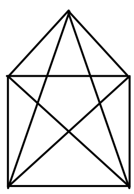
(v) Tree, T.

Lemma (2.9.2) The clique graph of $K_{p,q}$ is regular of degree $p+q-2$.

Lemma (2.9.3) The Clique graph of a perfect graph is perfect but the converse is not true.



W_5
not perfect



$K(W_5) = K_5$
Perfect

Fig. (15)

Lemma (2.9.4) The clique graph of a wheel graph W_n is a complete graph K_n.

Lemma (2.9.5) The clique graph of a regular graph is not necessary regular.

Lemma (2.9.6) The clique graph of an Eulerian graph is not necessary an Eulerian graph.

G Regular and Eulerian

$K(G)$ Not regular and not Eulerian

Lemma (2.9.7) The clique graph of a Hamiltonian graph is not necessary Hamiltonian.

G Hamiltonian

$K(G)$ not Hamiltonian

Fig. (17)

Lemma (2.9.8) The clique graph of a triangulated graph is triangulated, but the converse is not true.

G not triangulated

$K(G)$ triangulated

Fig. (18)

(2.10) Euler characteristic

Theorem (2.10.1) Let G_1, G_2 be two finite connected graphs with number of vertices and edges are n_1, n_2 and m_1, m_2 respectively, then

(i) $\eta(G_1 \cup G_2) = \eta(G_1) + \eta(G_2) - \eta(G_1 \cap G_2)$.

(ii) $\eta(G_1 \vee G_2) = \eta(G_1) + \eta(G_2) - m_1 m_2$.

(iii) $\eta(G_1 \times G_2) = \eta(G_1)\eta(G_2) - m_1 m_2$.

(iv) $\eta(G_1 \otimes G_2) = \eta(G_1)\eta(G_2) + n_1 m_2 + n_2 m_1 - 3m_1 m_2$.

(v) $\eta(G_1 \circ G_2) = \eta(G_1)\eta(G_2) - 3m_1 m_2$.

(iv) $\eta(G_1[G_2]) = \eta(G_1)\eta(G_2) + n_2 m_1 (1 - n_2) - m_1 m_2$.

3. Acknowledgements

The authors are deeply indebted and thankful to the deanship of the scientific research for his helpful and for distinct team of employees at Taibah university, Al-Madinah Al-Munawarah, K.S.A. for their continuous helps and the encouragement us to finalize this work. This research work was supported by a grant No. (3021/1434) from the deanship of the scientific research at Taibah university, Al-Madinah Al-Munawwarah, K.S.A.

REFERENCES

[1] Gross J. L. and Tucker T. W., Topological graph theory, John Wiely& Sons, Inc. New York, U. S. A., (1987).

[2] Balakrishnan R. andRanganathan, Textbook of graph theory, Springer-Verlag, Inst. New York, (2000).

[3] Wilson R. J. and Watkins J. J., Graphs an introductory approach, John Wiley & Sons, Inc. New York, U. S. A., (1981).

[4] Ronald G. , Graph theory, the Benjamin/Cummings Publishing Company, Inc. California, (1988).

[5]Axler S. and Ribet K.A.,Combinatorics and Graph Theory,Springer-Verlag, Inst. New York (2008).

[6]JeanC., Graph Theory and Applications, Wiley(2009).

[7] ReinhardD., Graph Theory , Springer-Verlag New York, (2005).

[8] BangJ. , Digraphs Theory, Algorithms and Applications, Springer-Verlag London Limited (2009).

[9] MichealD., Random Trees , Springer-Wien Inst. New York (2009).

[10]HartmutE. and Gregor E ., Graph Transformations, Springer-Verlag, Inst. New York (2005).