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ON SEMISIMPLE AND LEFT QUASI-REGULAR ELEMENTS OF ORDERED SEMIGROUPS

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Abstract. We prove, among others, that an ordered semigroup contains a left (resp. right) quasi-regular element if and only it contains a left (resp. right) regular element. It has a semisimple element if and only if it has an intra-regular element. An element a of an ordered semigroup S is a semisimple element of S if and only if there exists an intra-regular element b of S such that $I(a) = I(b)$. The element a is a left (resp. right) quasi-regular element of S if and only if there exists a left (resp. right) regular element b of S such that $L(a) = L(b)$ (resp. $R(a) = R(b)$). As a consequence, if the ideal $I(a)$ generated by an element a of S has an intra-regular generator, then a is semisimple. If the principal left (resp. right) ideal $L(a)$ (resp. $R(a)$) of an element a of S has a left (resp. right) regular generator, then a is a left (resp. right) quasi-regular element of S.

Keywords: ordered semigroup; semisimple element; left (right) quasi-regular element; intra-regular, left (right) regular element; π-semisimple, left (right) quasi π-regular element.

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1. Introduction and prerequisites

Let S be an ordered semigroup. A nonempty subset T of S is called a subsemigroup of S if $T^2 \subseteq T$. For a subset H of S, we denote by $(H]$ the subset of S defined by $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. A nonempty subset A of S is called a left (resp.

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right) ideal of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$. A nonempty subset A of S is called an ideal of S if it is both a left and a right ideal of S. The left (resp. right) ideals, and so the ideals of S are clearly subsemigroups of S. For an element a of S, $I(a)$, $L(a)$, $R(a)$, denotes the ideal, left ideal, and the right ideal of S, respectively, generated by a ($a \in S$), and we have $I(a) = (a \cup Sa \cup aS \cup SaS]$, $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$ [2]. As usual, for an element b of S, denote by $< b >$ the subsemigroup of S generated by b, that is, the smallest (with respect to the inclusion relation) subsemigroup of S containing b. We have $< b >= \{b, b^2, b^3, ..., b^n \mid n \in N\}$ ($N = \{1, 2, ...\}$ is the set of natural numbers). An element a of S is called intra-regular if $a \in (Sa^2S]$, equivalently, if $a \leq xa^2y$ for some $x, y \in S$ (cf., for example, [4]). It is called left (resp. right) regular if $a \in (Sa^2]$ (resp. $a \in (a^2S]$), that is, if $a \leq xa^2$ (resp. $a \leq a^2x$) for some $x \in S$ [3]. An element a of an ordered semigroup S is called semisimple if $a \in (SaSaS]$ [5]; it is called left (resp. right) quasi-regular if $a \in (SaSa]$ (resp. $a \in (aSaS]$) [5,6]. So the element a is semisimple, left quasi-regular or right quasi-regular if $a \leq xayaz$, $a \leq xaya$, $a \leq axay$ for some $x, y, z \in S$, respectively. An ordered semigroup S is called intra-regular, left (right) regular, semisimple or left (right) quasi-regular if every element of S is so.

Semisimple and left (right) quasi-regular elements of semigroups (without order) as well as π-semisimple and left quasi π-regular semigroups have been studied in [1]. An ordered semigroup S is a semilattice of left strongly simple semigroups if and only if every left ideal of S is a semisimple subsemigroup of S (cf. [7; Theorem 9]). It is a semilattice of simple and regular semigroups if and only if every bi-ideal of S is a semisimple subsemigroup of S, equivalently, if every left ideal of S is a right quasi-regular subsemigroup of S [8]. Characterization of left (right) quasi-regular and semisimple ordered semigroups in terms of fuzzy sets has been given in [5]. In the present paper we prove, among others, that an ordered semigroup contains a left (resp. right) quasi-regular element if and only if it contains a left (resp. right) regular element. It contains a semisimple element if and only if it contains an intra-regular element. Moreover we prove that an element a of an ordered semigroup S is a semisimple element of S if and only if there exists an intra-regular element b of S such that $I(a) = I(b)$. An element a of an ordered semigroup S is a left (resp. right) quasi-regular element of S if and only if there exists a left (resp. right) regular element b of S such that $L(a) = L(b)$ (resp. $R(a) = R(b)$). As a consequence, if S is an ordered semigroup, $a \in S$ and b an intra-regular element of S such that $I(a) =< b >$, then a is semisimple. If b is a left (resp. right) regular element of S and $L(a) =< b >$ (resp. $R(a) =< b >$), then a is left (resp. right) quasi-regular. We use the terminology semisimple, π-semisimple instead of the terminology intra quasi-regular, intra quasi π-regular given for semigroups (without order) in [1]. For an ordered semigroup S, we denote by $IQReg(S)$ (or $Sems(S)$), $LQReg(S)$, $RQReg(S)$ the set of semisimple (intra quasi-regular according to [1]), left quasi-regular and right quasi-regular elements of S, respectively. It has been proved in [1] that a semigroup S is left quasi π-regular if and only if it is π-semisimple and $IQReg(S) = LQReg(S)$. If an ordered semigroup S is π-semisimple and $IQReg(S) = LQReg(S)$, then it is left quasi π-regular, but the converse statement does not seem to be true, in general. However, for an ordered semigroup S we have $LQReg(S) \subseteq IQReg(S)$, and if S is left quasi π-regular, then it is π-semisimple.

2. Main results

Proposition 1. *Let S be an ordered semigroup. If a is an intra-regular element of S, then a is a semisimple element of S. If a is a left (resp. right) regular element of S, then a is a left (resp. right) quasi-regular element of S.*

Proof. Let a be an intra-regular element of S and let $x, y \in S$ such that $a \leq xa^2y$. Then we have $a \leq x(xa^2y)ay = (x^2a)(ay)(ay) \in SaSaS$, so $a \in (SaSaS]$, and a is semisimple. Let a be a left regular element of S and $x \in S$ such that $a \leq xa^2$. Then we have $a \leq x(xa^2)a = x^2aaa \in SaSa$, so $a \in (SaSa]$, and a is a left quasi-regular element of S. Finally, if a is a right regular element of S and $y \in S$ such that $a \leq a^2y$, then $a \leq a(a^2y)y = aaay^2 \in aSaS$, so $a \in (aSaS]$, and a is a right quasi-regular element of S. $\square$

Proposition 2. *Let S be an ordered semigroup and $a \in S$. If $a \leq xayaz$ for some $x, y, z \in S$, then the element yaz is an intra-regular element of S. If $a \leq xaya$ for some $x, y \in S$, then the element ya is a left regular element of S. If $a \leq axay$ for some $x, y \in S$, then the element ax is a right regular element of S.*

Proof. Let $a \leq xayaz$ for some $x, y, z \in S$. Then we have

$$\begin{aligned} yaz &\leq y(xayaz)z = (yx)a(yaz^2) \leq (yx)(xayaz)(yaz^2) \\ &= (yx^2a)(yaz)^2z, \text{ where } yx^2a, z \in S, \end{aligned}$$

so the element yaz is an intra-regular element of S.

Let $a \leq xaya$ for some $x, y \in S$. Then we have

$$\begin{aligned} ya &\leq y(xaya) = (yx)a(ya) \leq (yx)(xaya)(ya) \\ &= (yx^2a)(ya)^2, \text{ where } yx^2a \in S, \end{aligned}$$

so the element ya is a left regular element of S.

Similarly, if $a \leq axay$ for some $x, y \in S$, then

$$\begin{aligned} ax &\leq (axay)x = (ax)a(yx) \leq (ax)(axay)(yx) \\ &= (ax)^2(ay^2x), \text{ where } ay^2x \in S, \end{aligned}$$

so the element ax is a right regular element of S. $\square$

By Propositions 1 and 2 we have the following

Theorem 3. (cf. also [5]) *An ordered semigroup has a semisimple element if and only if it has an intra-regular element. An ordered semigroup contains a left (resp. right) quasi-regular element if and only if it contains a left (resp. right) regular element.*

Remark 4. If a is an intra-regular element of an ordered semigroup S, then $I(a) = (SaS]$. In fact: First of all, $(SaS] \subseteq (a \cup Sa \cup aS \cup SaS] = I(a)$. Let now $t \in I(a)$. Then $t \leq a$ or $t \leq pa$ or $t \leq aq$ or $t \leq paq$ for some $p, q \in S$. On the other hand, since a is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. If $t \leq a$, then $t \leq xa^2y \in SaS$, and $t \in (SaS]$. If $t \leq pa$ for some $p \in S$, then $t \leq pxa^2y \in SaS$, so $t \in (SaS]$. If $t \leq aq$ for some

$q \in S$, then $t \leq xa^2yq \in SaS$, so $t \in (SaS]$. Finally, if $t \leq paq$ for some $p, q \in S$, then $t \leq pxa^2yq \in SaS$, and $t \in (SaS]$.

Theorem 5. *Let S be an ordered semigroup. An element a of S is a semisimple element of S if and only if there exists an intra-regular element b of S such that $I(a) = I(b)$.*

Proof. $\Longrightarrow$. Let a be a semisimple element of S. Then $a \leq xayaz$ for some $x, y, z \in S$. By Proposition 2, the element yaz is an intra-regular element of S. In addition, we have $I(a) = I(yaz)$. In fact: First of all,

$$I(yaz) = (yaz \cup Syaz \cup yazS \cup SyazS] \subseteq (SaS] \subseteq I(a).$$

Let now $t \in I(a)$. Then $t \leq a$ or $t \leq pa$ or $t \leq aq$ or $t \leq paq$ for some $p, q \in S$. If $t \leq a$, then $t \leq xayaz \in Syaz$. Then $t \in (Syaz] \subseteq I(yaz)$, so $t \in I(yaz)$. If $t \leq pa$ for some $p \in S$, then $t \leq p(xayaz) \in Syaz$, and $t \in (Syaz] \subseteq I(yaz)$, so $t \in I(yaz)$. If $t \leq aq$ for some $q \in S$, then $t \leq (xayaz)q \in S(yaz)S$, and $t \in (S(yaz)S] \subseteq I(yaz)$, $t \in I(yaz)$. If $t \leq paq$ for some $p, q \in S$, then $t \leq p(xayaz)q \in S(yaz)S$, so $t \in (S(yaz)S] \subseteq I(yaz)$, and $t \in I(yaz)$.

$\Longleftarrow$. Let $a \in S$ and b an intra-regular element of S such that $I(a) = I(b)$. Then $I(a) = I(b)$ and $b \leq xb^2y$ for some $x, y \in S$. Then we have

$$a \in I(b) \subseteq I(xb^2y) = (xb^2y \cup Sxb^2y \cup xb^2yS \cup Sxb^2yS] \subseteq (Sb^2S].$$

Since $b \in I(a)$, we have $b \leq a$ or $b \leq pa$ or $b \leq qa$ or $b \leq paq$ for some $p, q \in S$. If $b \leq a$, then $b^2 \leq a^2$, $a \in (Sb^2S] \subseteq (Sa^2S]$, so $a \in (Sa^2S]$, and a is an intra-regular element of S. Then, by Proposition 1, a is a semisimple element of S. If $b \leq pa$ for some $p \in S$, then $b^2 \leq papa$, then $a \in (Sb^2S] \subseteq (SpapaS] \subseteq (SaSaS]$, so $a \in (SaSaS]$, and a is semisimple. If $b \leq aq$ for some $q \in S$, then $b^2 \leq aqaq$, $a \in (Sb^2S] \subseteq (SaqaqS] \subseteq (SaSaS]$, $a \in (SaSaS]$, and a is semisimple. If $b \leq paq$ for some $p, q \in S$, then $a \in (Sb^2S] \subseteq (SpaqpaqS] \subseteq (SaSaS]$, and a is semisimple. $\square$

Theorem 6. *Let S be an ordered semigroup. An element a of S is a left quasi-regular element of S if and only if there exists a left regular element b of S such that $L(a) = L(b)$.*

Proof. $\Longrightarrow$. Let a be a left quasi-regular element of S. Then $a \leq xaya$ for some $x, y \in S$. By Proposition 2, the element ya is a left regular element of S. Moreover we have $L(a) = L(ya)$. In fact: First of all,

$$L(ya) = (ya \cup Sya] \subseteq (Sa] \subseteq (a \cup Sa] = L(a).$$

Let now $t \in L(a)$. Then $t \leq a$ or $t \leq pa$ for some $p \in S$. If $t \leq a$, then $t \leq xaya \in Sya$. Then $t \in (Sya] \subseteq (ya \cup Sya] = L(ya)$, and $t \in L(ya)$. If $t \leq pa$ for some $p \in S$, then $t \leq p(xaya) \in Sya$, and $t \in L(ya)$.

$\Longleftarrow$. Let $a \in S$, b a left regular element of S such that $L(a) = L(b)$, and $x \in S$ such that $b \leq xb^2$. Then we have $a \in L(b) \subseteq L(xb^2) = (xb^2 \cup Sxb^2] \subseteq (Sb^2]$. Since $b \in L(a)$, we have $b \leq a$ or $b \leq pa$ for some $p \in S$. If $b \leq a$, then $a \in (Sb^2] \subseteq (Sa^2]$, $a \in (Sa^2]$, and a is a left regular element of S. By Proposition 1, a is a left quasi-regular element of S. If $b \leq pa$ for some $p \in S$, then $a \in (Sb^2] \subseteq (Spapa] \subseteq (SaSa]$, and a is again a left quasi-regular element of S. $\square$

The right analogue of Theorem 6 also holds, and we have

Theorem 7. *Let S be an ordered semigroup. An element a of S is a right quasi-regular element of S if and only if there exists a right regular element b of S such that $R(a) = R(b)$.*

Theorem 8. *Let S be an ordered semigroup and $a, b \in S$. If $I(a) =< b >$, then $I(a) = I(b)$. If $L(a) =< b >$ (resp. $R(a) =< b >$), then $L(a) = L(b)$ (resp. $R(a) = R(b)$).*

Proof. Let $I(a) =< b >$. As $I(b)$ is an ideal of S containing b, it is a subsemigroup of S containing b. On the other hand, $< b >$ is the smallest subsemigroup of S containing b, so $< b > \subseteq I(b)$, and $I(a) \subseteq I(b)$. Let now $t \in I(b)$. Then $t \leq b$ or $t \leq xb$ or $t \leq by$ or $t \leq xby$ for some $x, y \in S$. If $t \leq b$, then $t \leq b \in < b >= I(a)$, and $t \in I(a)$. If $t \leq xb$ for some $x \in S$, then $t \leq xb \in S < b >= SI(a) \subseteq I(a)$, so $t \in I(a)$. If $t \leq by$ for some $y \in S$, then $t \leq by \in < b > S = I(a)S \subseteq I(a)$, and $t \in I(a)$. If $t \leq xby$ for some $x, y \in S$, then $t \leq xby \in S < b > S = SI(a)S \subseteq I(a)$, and $t \in I(a)$.

Let now $L(a) =< b >$. $L(b)$ as a left ideal, it is a subsemigroup of S, thus

$$L(a) =< b > \subseteq L(b).$$

Let now $t \in L(b)$. Then $t \leq b$ or $t \leq xb$ for some $x \in S$. If $t \leq b$, then $t \leq b \in < b >= L(a)$, and $t \in L(a)$. If $t \leq xb$ for some $x \in S$, then $t \leq xb \in S < b >= SL(a) \subseteq L(a)$, and $t \in L(a)$. In a similar way we prove that $R(a) =< b >$ implies $R(a) = R(b)$. □

By Theorems 5,6,7 and 8 we have the following

Theorem 9. *Let S be an ordered semigroup and $a \in S$. If b is an intra-regular element of S such that $I(a) =< b >$, then a is a semisimple. If b is a left (resp. right) regular element of S and $L(a) =< b >$ (resp. $R(a) =< b >$), then a is left (resp. right) quasi-regular.*

For the sake of completeness, we keep the notation given for semigroups (without order) in [1] and, for an ordered semigroup S, we denote by $IQReg(S)$, $LQReg(S)$, $RQReg(S)$ the set of semisimple, left quasi-regular and right quasi-regular elements of S, respectively.

Remark 10. We have $LQReg(S) \subseteq IQReg(S)$ and $RQReg(S) \subseteq IQReg(S)$. In fact, if $a \in LQReg(S)$, then $a \in (SaSa]$ i.e. $a \leq xaya$ for some $x, y \in S$, then $a \leq xay(xaya) \in SaSaS$, so $a \in (SaSaS]$, and $a \in IQReg(S)$. If $a \in RQReg(S)$, then $a \leq axay \leq (axay)xay \in SaSaS$, and $a \in IQReg(S)$.

Proposition 11. *Let S be an ordered semigroup. If $a \in IQReg(S)$, then there exist $x, y \in S$ such that $a \leq (xay)^n az^n$ for every $n \in N$.*

Proof. Let $a \in IQReg(S)$. Then there exist $x, y, z \in S$ such that $a \leq xayaz$. Then we have

$$\begin{aligned} a &\leq (xay)az = (xay)(xayaz)z = (xay)^2az^2 \\ &\leq (xay)^2(xayaz)z^2 = (xay)^3az^3 \\ &\leq \text{......} \leq (xay)^n az^n \end{aligned}$$

for every $n \in N$. □

Definition 12. An element a of an ordered semigroup S is called *π-semisimple* if there exists $n \in N$ such that the element a^n is a semisimple element of S, that is, if a power of a is semisimple. An element a of an ordered semigroup S is called *left* (resp. *right*) *quasi π-regular* if there exists $n \in N$ such that the element a^n is a left (resp. right) quasi-regular element of S i.e. if a power of a is left (resp. right) quasi-regular. An ordered semigroup

S is called π-semisimple, left quasi π-regular or right quasi π-regular, respectively, if every element of S is so.

Theorem 13. *Let S be an ordered semigroup and $a \in S$. If a is left quasi π-regular (or right quasi π-regular), then it is π-semisimple as well. "Conversely", if a is π-semisimple and $IQReg(S) = LQReg(S)$, then a is a left quasi π-regular element of S. If a is π-semisimple and $IQReg(S) = RQReg(S)$, then a is a right quasi π-regular element of S.*

Proof. Let a be left quasi π-regular element of S. Then there exists $n \in N$ such that $a^n \in (Sa^nSa^n]$. Then we have

$$a^n \in (Sa^nS(Sa^nSa^n]] = (Sa^nS(Sa^nSa^n)] \subseteq (Sa^nSa^nS],$$

so a is a π-semisimple element of S. Let now a be a π-semisimple element of S and $IQReg(S) = LQReg(S)$. Since a is π-semisimple, there exists $n \in N$ such that $a^n \in (Sa^nSa^nS]$. That is $a^n \in IQReg(S)\,(= LQReg(S))$. Then $a^n \in LQReg(S)$ which means that the element a is a left quasi π-regular element of S. Finally, let a be π-semisimple element of S, $IQReg(S) = RQReg(S)$, and $a^n \in (Sa^nSa^nS]$ for some $n \in N$. Then $a^n \in IQReg(S)\,(= RQReg(S))$, and a is a right quasi π-regular element of S. $\square$

Theorem 14. *If an ordered semigroup S is left quasi π-regular (or right quasi π-regular), then it is π-semisimple. "Conversely", if S is π-semisimple and $IQReg(S) = LQReg(S)$ (resp. $IQReg(S) = RQReg(S)$), then it is left (resp. right) quasi π-regular.*

Problem. Find an example of a left quasi π-regular ordered semigroup having a semisimple element which is not left quasi-regular.

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