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ON SOME ORDER STATISTICS PROPERTIES OF THE MUKHERJEE-ISLAM DISTRIBUTION

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Abstract. In this paper, we study the distribution of order statistics of the Mukherjee-Islam distribution. We establish some recurrence relation for single and product moments of order statistics from Mukherjee-Islam distribution. These recurrence relations will enable the computation of the means, variances and covariances of all order statistics for all sample sizes in a simple and efficient recursive manner. The exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Mukherjee-Islam distribution are derived.

Keywords: Mukherjee-Islam distribution; Order statistics; Moments; Single and product moment of order statistics; Entropy; Residual entropy.

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1. Introduction

Order statistics have been used in wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distribution, goodness of fit-tests,

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quality control, analysis of censored sample. The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold et al. [2], Malik et al. [9]). For improved form of these results, Samuel and Thomes [13], Arnold et al. [2], and Ali and Khan [1] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Very recently, Dar and Abdullah [4] have studied the sampling distribution of order statistics of the two parametric Lomax distribution and derived the exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Lomax distribution.

Definition 1.1. (See Mukherjee and Islam [10]) A random variable X with range of values $(0,\theta)$ is said to have the Mukherjee-Islam distribution, from now onwards we use MI to abbreviate such a distribution, if its probability density function (pdf) is given by

$$f(x) = \frac{p}{\theta^p}\, x^{p-1}, \quad 0 < x \leq \theta,\ p, \theta > 0. \tag{1}$$

Here θ is the scale parameter and p is the shape parameter. The above distribution is monotonically decreasing and highly skewed to the right. The cumulative distribution function (cdf) and survival function (sf) associated with equation (1) is given, respectively, by

$$F(x) = \left(\frac{x}{\theta}\right)^p, \quad 0 < x \leq \theta,\ p, \theta > 0. \tag{2}$$

$$\bar{F}(x) = 1 - \left(\frac{x}{\theta}\right)^p, \quad 0 < x \leq \theta,\ p, \theta > 0. \tag{3}$$

The MI distribution mainly appears as the inverse distribution of the Pareto distribution. The MI distribution was originally introduced as a life testing model, but it has been used in many areas of applications. For example, it has been used as a queuing model. For various properties and applications of MI distribution, one should refer to Mukherjee and Islam [10], Siddiqui and Shiva [14], Siddiqui [15], Siddiqui and Manish [16] and Saxena et. al [17].

The following functional relationship exists between the pdf and cdf of the MI distribution and will be very useful in simplification the findings.

$$f(x) = px^{-1}F(x), \quad 0 < x \leq \theta,\ p, \theta > 0. \tag{4}$$

Definition 1.2. (see Shannon [18]) An entropy of a continuous random variable X with density function $f_X(x)$ is defined as

$$H(X) = -\int_0^\infty f_X(x)\log f_X(x)dx. \tag{5}$$

Analytical expression for univerate distribution are discussed in references such as Laz and Rathie [8], Nadarajah and Zagrafos [11]. Also the information properties of order statistics have been studied by Wong and Chen [19], Park [12], and Ebrahimi et al. [7]. The measure given in (5) is not suitable for measuring the uncertainty of a component with information only about its current age. A more realistic approach which make the use of the age into account is described by Ebrahimi [6] and is defined as follows:

$$H(X,t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}dx. \tag{6}$$

It is obvious that for $t = \infty$, equation (6) is reduced to equation (5).

In many realistic situation uncertainty is not necessarily related to future but can also refer to past. Based on this idea, Crescenzo and Longobardi [3] develop the concept of past entropy over $(0,t)$.

Definition 1.3. If X denote the lifetime of a component, then the past entropy of X is defined by

$$H^0(X;t) = -\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx. \tag{7}$$

It is obvious that for $t = 0$, equation (7) is reduced to equation (5).

2. Distribution of order statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the MI distribution and let $X_{1:n}, \cdots, X_{n:n}$ denote the corresponding order statistics. Then, the pdf of $X_{r:n}$, $1 \le r \le n$, is given by (see David and Nagaraja [5] and Arnold et al. [2])

$$f_{r:n}(x) = C_{r,s:n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad 0 < x < \infty, \tag{8}$$

where $C_{r,n} = [B(r, n-r+1)]^{-1}$, with $B(a,b)$ being the complete beta function.

The probability density functions of smallest $(r = 1)$ and largest $(r = n)$ order statistics can be easily obtained from equation (8) and are given, respectively, by

$$f_{1,n}(x) = n[1-F(x)]^{n-1}f(x).$$
$$f_{n,n}(x) = n[F(x)]^{n-1}f(x).$$

Using equations (1) and (2), and taking $r = 1$ in equation (8) yields the pdf of the minimum order statistics for the MI distribution.

$$f_{1:n}(x) = \frac{np}{\theta^{np}}(\theta^p - x^p)^{n-1}x^{p-1}. \tag{9}$$

Similarly using equations (1) and (2), and taking $r = n$ in equation (8) yields the pdf of the largest order statistics for the MI distribution

$$f_{n:n}(x) = \frac{np}{\theta^{np}}x^{np-1}. \tag{10}$$

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \le r < s \le n$ is given by (see Arnold et al. [2])

$$f_{r,s:n}(x) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), \tag{11}$$

where $C_{r,s:n} = n!/((r-1)!(s-r-1)!(n-s)!)$ and for $-\infty < x < y < \infty$.

Theorem 2.1. Let $F(x)$ and $f(x)$ be the cdf and pdf of the MI distribution. Then the density function of the rth order statistics, say $f_{r:n}(x)$, is given by

$$f_{r:n}(x) = pC_{r:n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\frac{x^{pr+pi-1}}{\theta^{pr+pi}}. \tag{12}$$

Proof. First it should be noted that equation (8) can be written as

$$f_{r:n}(x) = C_{r:n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i[F(x)]^{r+i-1}f(x). \tag{13}$$

The proof follows by substituting equations (1) and (2) into equation (13).

Theorem 2.2. Let $X_{r:n}$ and $X_{s:n}$ for $1 \le r < s \le n$ be the rth and sth order statistics from the MI distribution. Then the joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r;s:n}(x) = p^2C_{r:s;n}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\binom{s-r-1}{i}\binom{n-s}{j}(-1)^{i+j}\left[\frac{x^{p(r+i)-1}y^{p(s-r-i+j)-1}}{\theta^{p(s+j)}}\right].$$

Proof. Another form of representing equation (11) is as follows:

$$\begin{aligned} f_{r;s:n}(x) &= C_{r:n}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\binom{s-r-1}{i}\binom{n-s}{j}(-1)^{i+j} \\ &\quad [F(y)]^{s-r-1-i+j}[F(x)]^{r+i-1}f(x)f(y). \end{aligned} \tag{14}$$

The proof immediately follows by substituting equations (1) and (2) into equation (14).

3. Single and product moments

In this section, we derive explicit expressions for both of the single and product moments of order statistics from the MI distribution.

Theorem 3.1. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the MI distribution, and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then the kth moment of the rth order statistic for $k = 1, 2, \cdots$, denoted by $\mu_{r:n}^{(k)}$, is given by

$$\mu_{r:n}^{(k)} = \theta^k C_{r:n} B\left(\frac{k}{p} + r; n+1-r\right), \tag{15}$$

where $B(.,.)$ is the beta function.

Proof. We know that

$$\begin{aligned} \mu_{r:n}^{(k)} &= \int_0^\infty x^k f_{r:n}(x)dx. \\ &= C_{r:n}\int_0^\infty x^k [F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx. \end{aligned} \tag{16}$$

Now substituting equations (1) and (2) into equation (16), yields equation (15).

Theorem 3.1 can be exploited to drive the mean and the variance of the rth order statistics. For example, when $k = 1$ we can obtain the mean of the rth order statistics as follows:

$$\mu_{r:n} = \theta C_{r:n} B\left(\frac{1}{p} + r; n+1-r\right). \tag{17}$$

For $k = 2$, one can get the second order moment of the rth order statistics as

$$\mu_{r:n}^{(2)} = \theta^2 C_{r:n} B\left(\frac{2}{p} + r; n+1-r\right). \tag{18}$$

Therefore, the variance of the rth order statistics can be obtained easily by using the relation

$$V(X_{r:n}) = \mu_{r:n}^{(2)} - \mu_{r:n}^2 = C_{r:n}\theta^2\left[B\left(\frac{2}{p} + r; n+1-r\right) - C_{r:n}B^2\left(\frac{1}{p} + r; n+1-r\right)\right].$$

Similarly the third and fourth order moments of the rth order statistic, $\mu_{r:n}^{(3)}$ and $\mu_{r:n}^{(4)}$, can be obtained in similar ways. The mean, variance and other statistical measures of the extreme order statistics are always of great interest. Taking $r = 1$, one can obtain the mean of smallest order statistics:

$$\mu_{1:n} = n\theta B\left(\frac{1}{p}+1;n\right).$$

Also, second order moment of the smallest order statistics can be obtained as follows:

$$\mu_{1:n}^{(2)} = n\theta^2 B\left(\frac{2}{p}+1;n\right).$$

Therefore, the variance of the smallest order statistics is

$$V(X_{1:n}) \;=\; \mu_{1:n}^{(2)} - \mu_{1:n}^{2} = n\theta^2\left[B\left(\frac{2}{p}+1;n\right) - nB^2\left(\frac{1}{p}+1;n\right)\right].$$

Similarly, the mean, the second order moment and hence the variance of the largest order statistics $(r = n)$ is given by

$$\mu_{n:n} = \frac{np\theta}{1+np}, \quad \mu_{n:n}^{(2)} = \frac{np\theta^2}{2+np}, \quad \text{and} \quad V(X_{n:n}) = np\theta^2\left(\frac{1}{2+np} - \frac{np}{(1+np)^2}\right).$$

To show the computability of the moments we give the means and variances of some order statistics for some values of θ and p in Tables 1 and 2, respectively.

Theorem 3.2. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the MI distribution and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then for $1 \le r \le n$, we have the following moment relation:

$$\mu_{r:n}^{(k)} = \frac{pr}{k+pr}\mu_{r+1:n}^{(k)}.$$

Proof. Using equations (4) and (16) gives

$$\mu_{r:n}^{(k)} = pC_{r:n}\int_0^{\theta} x^{k-1}[F(x)]^r[1-F(x)]^{n-r}dx.$$

By using integration by parts, we easily obtain the desired result.

Theorem 3.3. For $1 \le r \le s \le n$, and $n \in N$, we have

$$\frac{k_2 + p(n-s+1)}{k_2}\mu_{r;s:n}^{(k_1,k_2)} = \frac{np}{k_2}\left(\mu_{r;s:n-1}^{(k_1,k_2)} - \mu_{r;s-1:n-1}^{(k_1,k_2)}\right) + \frac{p(n-s-1)}{k_2}\mu_{r;s-1:n}^{(k_1,k_2)}.$$

Proof. We start by noting that

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n} \int_0^\theta \int_x^\theta x^{k_1} y^{k_2} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dy dx.$$

or

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n} \int_0^\theta x^{k_1} [F(x)]^{r-1} f(x) I_X dx,$$

where

$$I_X = \int_x^\theta y^{k_2} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) dy. \tag{19}$$

Applying equation (4) gives

$$\begin{aligned} I_X &= p\bigg[\int_x^\theta y^{k_2-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} dy \\ &\quad - \int_x^\theta y^{k_2-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy.\bigg]. \end{aligned}$$

Now, integrating by parts and then substituting I_X into equation (19) gives directly the desired result.

4. Entropy based on order statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a distribution $F_X(x)$ with density function $f(x)$ and let $Y_1 < Y_2 < \cdots < Y_n$ denote the corresponding order statistics. Then the pdf of Y_r, $1 \le r \le n$, is given by

$$f_{Y_r}(y) = C_{r:n} [F_X(y)]^{r-1} [1 - F_X(y)]^{n-r} f(x), \quad 0 < x < \infty,$$

where $C_{r:n} = 1/B(r, n-r+1)$ and $B(.,.)$ is the beta function as before. Further, let U is the uniform distribution defined over the unit interval. The order statistics of a sample taken randomly from uniform distribution $U_1, U_2, \cdots, U_n$ are denoted by $W_1 < W_2 < \cdots < W_n$. The random variable W_r, $r = 1, 2, \cdots, n$ has a Beta distribution with density function

$$g_r(w) = C_{r:n} [w]^{r-1} [1 - w]^{n-r}, \quad 0 < w < 1.$$

In the following subsections, we derive the exact form of entropy, residual entropy and past residual entropy for the MI distribution based on order statistics.

4.1. Entropy

Using the transformation $W_r = F_X(Y_r)$, the entropies of order statistics can be computed by

$$H(Y_r) = H_n(W_r) - E_{g_r}[\log f_X(F_X^{-1} W_r)]. \tag{20}$$

where $H_n(W_r)$ denotes the entropy of the Beta distribution and is given by

$$\begin{aligned} H_n(W_r) &= \log B(r, n-r+1) - (r-1)[\psi(r) - \psi(n+1)] \\ &\quad -(n-r)[\psi(n-r+1) - \psi(n+1)], \end{aligned} \tag{21}$$

where ψ is the digamma function and is defined by $\psi(\theta) = (d/d\theta)\log\Gamma(\theta)$.

Remark 4.1. For $r = 1$, i.e smallest order statistics and for $r = n$, i.e largest order statistics, it can be easily shown that

$$H_n(W_1) = H_n(W_n) = 1 - \log(n) - 1/n. \tag{22}$$

Remark 4.2. It should be noted that $\psi(n+1) - \psi(n) = 1/n$.

Theorem 4.1. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from MI distribution with distribution function given in (2) and let $Y_1 < Y_2 < \cdots < Y_n$ denote the corresponding order statistics. Then the entropy of the rth order statistics for the MI distribution is given by

$$\begin{aligned} H(Y_r) &= \log B(r, n-r+1) - (r-1)[\psi(r) \\ &\quad -\psi(n+1)] - (n-r)[\psi(n-r+1) - \psi(n+1)] \\ &\quad -\log(p/\theta) + (1-p)/p\,(\psi(r) - \psi(n+1)). \end{aligned} \tag{23}$$

Proof. Using equation (2) and the probability integral transformation $Y_r = F_X^{-1}(W_r)$, one can easily arrive at

$$F_X^{-1}(W_r) = \theta (W_r)^{1/p}.$$

Therefore, after applying equation (20) we get the following:

$$\begin{aligned} E_{g_r}[\log f_X(F_X^{-1} W_r)] &= E_{g_r}[\log f_X(\theta (W_r)^{1/p})]. \\ &= \log(p/\theta) + (p-1)/p\,[\psi(r) - \psi(n+1)]. \end{aligned} \tag{24}$$

Substituting equations (24) and (21) into equation (20) gives the required result.

Corollary 4.1. For $r = 1$, i.e smallest order statistics, we have

$$H(Y_1) = 1 - 1/n - \log(np/) + (p-1)/p(\psi(n+1) + \gamma).$$

where $-\psi(1) = \gamma = 0.5772$ is the Eulers constant.

Corollary 4.2. For $r = n$, i.e largest order statistics, we have

$$H(Y_n) = 1 - 1/np - \log(np/\theta).$$

4.2. Residual Entropy

Analogous to relation (6), the residual entropy of order statistics $X_{r,n}$ is given by

$$H(X_{r,n};t) = -\int_t^\infty \frac{f_{r,n}(x)}{\bar{F}_{r,n}(t)} \log \frac{f_{r,n}(x)}{\bar{F}_{r,n}(t)} dx. \tag{25}$$

Clearly the residual entropy of first order statistics is obtained by substituting $r = 1$ and using the probability integral transformation $U = F_X(x)$ in equation (25). Then, we have

$$\begin{aligned} H(X_{1,n};t) &= (n-1)/n - \log(n) + \log(\bar{F}(t)) \\ &\quad - \frac{n}{\bar{F}^n(t)} \int_{F(t)}^1 (1-u)^{n-1} \log[f(F^{-1}(u))]du. \end{aligned} \tag{26}$$

The residual entropy of the first order statistics for MI distribution can be easily obtained by using equations (1), (2), and (3), and then put $f(F^{-1}(u)) = pu/\theta$ into equation (26).

$$\begin{aligned} H(X_{1,n};t) &= (n-1)/n - \log(np/\theta) - \log(\bar{F}(t)) - \log(F(t)(1 - 1/(\bar{F}^n(t))) \\ &\quad \times \left[\sum_{i=1}^n \binom{n}{i} (-1)^i (1/i)(1 - F^i(t)) \right], \end{aligned}$$

where $F(t)$ and $\bar{F}(t)$ are the cumulative distribution function and survival function for MI distribution given by equations (2) and (3) respectively. The case for $r = n$ follows on similar lines.

4.3. Past Residual Entropy

Analogous to relation (7), the past residual entropy of the rth order statistics is defined as

$$H^0(X_{r,n};t) = -\int_0^t \frac{f_{r,n}(x)}{F_{r,n}(t)} \log \frac{f_{r,n}(x)}{F_{r,n}(t)} dx. \tag{27}$$

The past residual entropy of nth order statistics is obtained by substituting $r = n$ and using the probability integral transformation $U = F_X(x)$ in equation (27), we have

$$\begin{aligned} H^0(X_{n,n};t) &= (n-1)/n - \log(n) + \log(F(t)) \\ &\quad - \frac{n}{F^n(t)} \int_0^{F(t)} u^{n-1} \log[f(F^{-1}(u))]du. \end{aligned} \tag{28}$$

The past residual entropy of the nth order statistics for MI distribution can be easily obtained by using (1), (2), (3) and $f(F^{-1}(u)) = pu/\theta$ in equation (28).

$$H^0(X_{n,n};t) = 1 - \log(np/\theta).$$

The case for $r = 1$ follows on similar lines.

5. Conclusion

In this paper we study the sampling distribution from the order statistics of MI distribution. Also we consider the single and product moment of order statistics from MI distribution. We establish recurrence relation for single moments of order statistics. Also, we have derived the entropy, residual and past residual entropies for order statistics of the MI distribution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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TABLE 1. Expected values, second moments and variances of the rth order statistic from MI distribution for $n = 1, 2, \cdots, 9$, $\theta = 0.5, 1$ and $p = 0.5$

θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$	θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$
0.5	1	1	0.166667	0.050000	0.022222	1	1	1	0.333333	0.200000	0.088889
		2	0.083333	0.016667	0.009722			2	0.166667	0.066667	0.038889
		3	0.050000	0.007143	0.004643			3	0.100000	0.028571	0.018571
		4	0.033333	0.003571	0.002460			4	0.066667	0.014286	0.009841
		5	0.023810	0.001984	0.001417			5	0.047619	0.007937	0.005669
		6	0.017857	0.001190	0.000872			6	0.035714	0.004762	0.003486
		7	0.013889	0.000758	0.000565			7	0.027778	0.003030	0.002259
		8	0.011111	0.000505	0.000382			8	0.022222	0.002020	0.001526
		9	0.009091	0.000350	0.000267			9	0.018182	0.001399	0.001068
	2	2	0.250000	0.083333	0.020833		2	2	0.500000	0.333333	0.083333
		3	0.150000	0.035714	0.013214			3	0.300000	0.142857	0.052857
		4	0.100000	0.017857	0.007857			4	0.200000	0.071429	0.031429
		5	0.071429	0.009921	0.004819			5	0.142857	0.039683	0.019274
		6	0.053571	0.005952	0.003082			6	0.107143	0.023810	0.012330
		7	0.041667	0.003788	0.002052			7	0.083333	0.015152	0.008207
		8	0.033333	0.002525	0.001414			8	0.066667	0.010101	0.005657
		9	0.027273	0.001748	0.001004			9	0.054545	0.006993	0.004018
	3	3	0.300000	0.107143	0.017143		3	3	0.600000	0.428571	0.068571
		4	0.200000	0.053571	0.013571			4	0.400000	0.214286	0.054286
		5	0.142857	0.029762	0.009354			5	0.285714	0.119048	0.037415
		6	0.107143	0.017857	0.006378			6	0.214286	0.071429	0.025510
		7	0.083333	0.011364	0.004419			7	0.166667	0.045455	0.017677
		8	0.066667	0.007576	0.003131			8	0.133333	0.030303	0.012525
		9	0.054545	0.005245	0.002270			9	0.109091	0.020979	0.009078
	4	4	0.333333	0.125000	0.013889		4	4	0.666667	0.500000	0.055556
		5	0.238095	0.069444	0.012755			5	0.476190	0.277778	0.051020
		6	0.178571	0.041667	0.009779			6	0.357143	0.166667	0.039116
		7	0.138889	0.026515	0.007225			7	0.277778	0.106061	0.028900
		8	0.111111	0.017677	0.005331			8	0.222222	0.070707	0.021324
		9	0.090909	0.012238	0.003973			9	0.181818	0.048951	0.015893
	5	5	0.357143	0.138889	0.011338		5	5	0.714286	0.555556	0.045351
		6	0.267857	0.083333	0.011586			6	0.535714	0.333333	0.046344
		7	0.208333	0.053030	0.009628			7	0.416667	0.212121	0.038510
		8	0.166667	0.035354	0.007576			8	0.333333	0.141414	0.030303
		9	0.136364	0.024476	0.005880			9	0.272727	0.097902	0.023522
	6	6	0.375000	0.150000	0.009375		6	6	0.750000	0.600000	0.037500
		7	0.291667	0.095455	0.010385			7	0.583333	0.381818	0.041540
		8	0.233333	0.063636	0.009192			8	0.466667	0.254545	0.036768
		9	0.190909	0.044056	0.007610			9	0.381818	0.176224	0.030439
	7	7	0.388889	0.159091	0.007856		7	7	0.777778	0.636364	0.031425
		8	0.311111	0.106061	0.009270			8	0.622222	0.424242	0.037082
		9	0.254545	0.073427	0.008633			9	0.509091	0.293706	0.034533
	8	8	0.400000	0.166667	0.006667		8	8	0.800000	0.666667	0.026667
		9	0.327273	0.115385	0.008277			9	0.654545	0.461538	0.033109
	9	9	0.409091	0.173077	0.005722		9	9	0.818182	0.692308	0.022886

TABLE 2. Expected values, second moments and variances of the rth order statistic from MI distribution for $n = 1, 2, \cdots, 9$, $\theta = 0.5, 1$ and $p = 1.5$

θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$	θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$
0.5	1	1	0.300000	0.107143	0.017143	1	1	1	0.600000	0.428571	0.068571
		2	0.225000	0.064286	0.013661			2	0.450000	0.257143	0.054643
		3	0.184091	0.044505	0.010616			3	0.368182	0.178022	0.042464
		4	0.157792	0.033379	0.008481			4	0.315584	0.133516	0.033923
		5	0.139228	0.026352	0.006967			5	0.278457	0.105408	0.027870
		6	0.125306	0.021561	0.005859			6	0.250611	0.086243	0.023437
		7	0.114409	0.018111	0.005021			7	0.228819	0.072444	0.020086
		8	0.105609	0.015524	0.004370			8	0.211217	0.062095	0.017482
		9	0.098325	0.013521	0.003853			9	0.196651	0.054083	0.015411
	2	2	0.375000	0.150000	0.009375		2	2	0.750000	0.600000	0.037500
		3	0.306818	0.103846	0.009709			3	0.613636	0.415385	0.038835
		4	0.262987	0.077885	0.008722			4	0.525974	0.311538	0.034890
		5	0.232047	0.061488	0.007642			5	0.464095	0.245951	0.030567
		6	0.208843	0.050308	0.006693			6	0.417685	0.201233	0.026772
		7	0.190682	0.042259	0.005899			7	0.381365	0.169036	0.023597
		8	0.176015	0.036222	0.005241			8	0.352029	0.144888	0.020963
		9	0.163876	0.031548	0.004693			9	0.327751	0.126193	0.018772
	3	3	0.409091	0.173077	0.005722		3	3	0.818182	0.692308	0.022886
		4	0.350649	0.129808	0.006853			4	0.701299	0.519231	0.027411
		5	0.309396	0.102480	0.006754			5	0.618793	0.409919	0.027014
		6	0.278457	0.083847	0.006309			6	0.556914	0.335388	0.025235
		7	0.254243	0.070432	0.005792			7	0.508486	0.281726	0.023168
		8	0.234686	0.060370	0.005292			8	0.469372	0.241480	0.021169
		9	0.218501	0.052580	0.004838			9	0.437002	0.210321	0.019351
	4	4	0.428571	0.187500	0.003827		4	4	0.857143	0.750000	0.015306
		5	0.378151	0.148026	0.005028			5	0.756303	0.592105	0.020112
		6	0.340336	0.121112	0.005284			6	0.680672	0.484450	0.021135
		7	0.310742	0.101734	0.005174			7	0.621483	0.406938	0.020696
		8	0.286838	0.087201	0.004925			8	0.573677	0.348804	0.019699
		9	0.267057	0.075949	0.004630			9	0.534113	0.303797	0.018520
	5	5	0.441176	0.197368	0.002732		5	5	0.882353	0.789474	0.010927
		6	0.397059	0.161483	0.003828			6	0.794118	0.645933	0.015310
		7	0.362532	0.135646	0.004217			7	0.725064	0.542584	0.016866
		8	0.334645	0.116268	0.004281			8	0.669290	0.465072	0.017123
		9	0.311566	0.101266	0.004192			9	0.623132	0.405063	0.016769
	6	6	0.450000	0.204545	0.002045		6	6	0.900000	0.818182	0.008182
		7	0.410870	0.171818	0.003004			7	0.821739	0.687273	0.012018
		8	0.379264	0.147273	0.003431			8	0.758528	0.589091	0.013726
		9	0.353108	0.128270	0.003584			9	0.706216	0.513079	0.014338
	7	7	0.456522	0.210000	0.001588		7	7	0.913043	0.840000	0.006352
		8	0.421405	0.180000	0.002418			8	0.842809	0.720000	0.009672
		9	0.392342	0.156774	0.002842			9	0.784685	0.627097	0.011367
	8	8	0.461538	0.214286	0.001268		8	8	0.923077	0.857143	0.005072
		9	0.429708	0.186636	0.001987			9	0.859416	0.746544	0.007947
	9	9	0.465517	0.217742	0.001036		9	9	0.931034	0.870968	0.004143