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ZWEIER I-CONVERGENT DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this paper, we introduce the sequence spaces $\mathscr{Z}_0^I(F,\triangle)$ and $\mathscr{Z}_\infty^I(F,\triangle)$ for the sequence of modulii $F=(f_k)$ and study some inclusion relations that arise on the said spaces.

Keywords: difference sequence spaces, sequence of moduli, I-Convergence, Zweier sequences.

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1. Introduction

Let N,R and C be the sets of all natural, real and complex numbers respectively.
We write

$$\omega=\{x=(x_k):x_k\in R\ or\ C\ \},$$

the space of all real or complex sequences.
Let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences respectively, normed by

$$||x||_\infty=\sup_k|x_k|,\ \text{where } k\in N.$$

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The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen[1], Başar and Altay[2], Malkowsky[11], Ng and Lee[13], and Wang[15].

Şengönül[14] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, \ (i = k), \\ 1-p, \ (i-1 = k), (i,k \in N) \\ 0 \ otherwise \end{cases}$$

Following Başar and Altay[2], Şengönül[14] introduced the Zweier sequence spaces $\mathscr{Z}$ and $\mathscr{Z}_0$ as follows

$$\mathscr{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathscr{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

The idea of difference sequence spaces was introduced by Kizmaz [10] as

$$\ell_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_\infty\},$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$

and

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\},$$

where $\triangle x = (x_k - x_{k+1})$ and $\triangle^0 x = (x_k)$.

These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

The idea of modulus was structured by Nakano[12].

A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if

(1)f(t) = 0 if and only if t = 0,

(2) f(t+u)$\leq$ f(t)+ f(u) for all t, u$\geq$0,

(3) f is increasing, and

(4) f is continuous from the right at zero.

If X be a non- empty set, then a family of set $I \subset P(X)(P(X)$ denoting the power set of X) is called an ideal in X if and only if

(a) $\phi \in I$;

(b) For each $A,B \in I$, we have $A \cup B \in I$;

(c) For each $A \in I$ and $B \subset A$ we have $B \in I$.

If X be a non- empty set. A non- empty family of sets $F \subset P(X)(P(X)$ denoting the power set of X) is called a filter on X if and only if

(a) $\phi \notin F$;

(b) For each $A,B \in F$, we have $A \cap B \in F$;

(c) For each $A \in F$ and $A \subset B$ we have $B \in F$.

Recently Khan, Ebadullah and Yasmeen[9] introduced the following classes of sequences.

$$\mathscr{Z}_0^I = \{x = (x_k) \in \omega : I - \lim Z^p x = 0\},$$

$$\mathscr{Z}^I = \{x = (x_k) \in \omega : I - \lim Z^p x = L \text{ for some L}\in \text{C}\},$$

$$\mathscr{Z}_\infty^I = \{x = (x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

In [6] for a modulus function f

$$\mathscr{Z}_0^I(f) = \{(x_k) \in \omega : \text{for a given } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k^/|) \geq \varepsilon\} \in I\},$$

$$\mathscr{Z}^I(f) = \{(x_k) \in \omega : \exists L \in C \text{ such that for a given } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k^/ - L|) \geq \varepsilon\} \in I\},$$

$$\mathscr{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k^/|) \geq M\} \in I, \text{ for each fixed M>0}\}.$$

where $(x_k^{/}) = (Z^p x)$

In [8] for a sequence of modulii $F = (f_k)$

$$\mathscr{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k^{/}|) \geq \varepsilon\} \in I\},$$

$$\mathscr{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k^{/} - L|) \geq \varepsilon, \text{ for some L} \in C\} \in I\},$$

$$\mathscr{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k^{/}|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We need the following results in order to establish some of the results of this article.

Lemma 1.1.[3, Lemma 1.2.] The condition $\sup_k f_k(t) < \infty$, $t > 0$ holds if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.

Lemma 1.2.[3, Lemma 1.3.] The condition $\inf_k f_k(t) > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.

Theorem 1.3.[14, Theorem 2.2.] The sequence spaces $\mathscr{Z}$ and $\mathscr{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathscr{Z} \cong c$ and $\mathscr{Z}_0 \cong c_0$

Theorem 1.4.[14, Theorem 2.3.] The inclusions $\mathscr{Z}_0 \subset \mathscr{Z}$ strictly hold for $p \neq 1$.

c.f. ([3],[4],[5],[7],[9]).

2. MAIN RESULTS.

In this article we introduce the following classes of sequence spaces.

$$\mathscr{Z}_0^I(F, \triangle) = \{x = (x_k) \in \omega : \text{I-lim} f_k(|\triangle x_k^{/}|) = 0\};$$

$$\mathscr{Z}_\infty^I(F, \triangle) = \{x = (x_k) \in \omega : \sup_k f_k(|\triangle x_k^{/}|) < \infty\}.$$

where $(x_k^{/}) = (Z^p x)$

Theorem 2.1. For a sequence $F=(f_k)$ of moduli, the following statements are equivalent:

(a) $\mathscr{Z}^I_\infty(\triangle)\subseteq\mathscr{Z}^I_\infty(F,\triangle)$

(b) $\mathscr{Z}^I_0(\triangle)\subset\mathscr{Z}^I_\infty(F,\triangle)$

(c) $\sup_k f_k(t)<\infty,\ (t>0)$

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $\mathscr{Z}^I_0(\triangle)\subset\mathscr{Z}^I_\infty(F,\triangle)$.

Suppose that (c) is not true.

Then by Lemma 1.1 $\sup_k f_k(t)=\infty$ for all $t>0$, and, therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i})>i \text{ for i=1,2,3.....} \qquad [2.1]$$

Define $x=(x_k)$ as follows

$$x_k=\begin{cases}\frac{1}{i},\ \text{if } k=k_i, i=1,2,3.......;\\ 0,\ otherwise.\end{cases}$$

Then $x\in\mathscr{Z}^I_0(\triangle)$ but by [2.1], $x\notin\mathscr{Z}^I_\infty(F,\triangle)$ which contradicts (b).

Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and $x\in\mathscr{Z}^I_\infty(\triangle)$.

If we suppose that $x\notin\mathscr{Z}^I_\infty(F,\triangle)$ then

$$\sup_k f_k(|\triangle x_k|)=\infty \text{ for } \triangle x\in\mathscr{Z}^I_\infty$$

If we take t = $|\triangle x|$ then $\sup_k f_k(t) = \infty$ which contradicts (c).
Hence $\mathscr{Z}_\infty^I(\triangle) \subseteq \mathscr{Z}_\infty^I(F,\triangle)$.

Theorem 2.2. If $F = (f_k)$ is a sequence of moduli, then the following statements are equivalent:

(a) $\mathscr{Z}_0^I(F,\triangle) \subseteq \mathscr{Z}_0^I(\triangle)$,

(b) $\mathscr{Z}_0^I(F,\triangle) \subset \mathscr{Z}_\infty^I(\triangle)$,

(c) $\inf_k f_k(t) > 0,\ (t > 0)$.

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $\mathscr{Z}_0^I(F,\triangle) \subset \mathscr{Z}_\infty^I(\triangle)$.
Suppose that (c) does not hold.
Then, by lemma 1.2 ,

$$\inf_k f_k(t) = 0, (t > 0), \qquad [2.2]$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \text{ for } i = 1,2,.........$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} i^2, \text{if } k = k_i,\ i = 1,2,3.......; \\ 0,\ otherwise. \end{cases}$$

By [2.2] $x \in \mathscr{Z}_0^I(F,\triangle)$ but $x \notin \mathscr{Z}_\infty^I(\triangle)$ which contradicts (b).

Hence (c) must hold.

(c) implies (a). Let (c) holds and $x \in \mathscr{Z}_0^I(F,\triangle)$ that is

$$\lim_k f_k(|\triangle x_k|) = 0$$

.

Suppose that $x \notin \mathscr{Z}_0^I(\triangle)$.

Then for some $\varepsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| \leq \varepsilon_0$ for $k \geq k_o$.

Therefore $f_k(\varepsilon_0) \geq f_k(|\triangle x_k|)$ for $k \geq k_0$ and hence $\lim_k f_k(\varepsilon_0) > 0$ which contradicts $x \notin \mathscr{Z}_0^I(\triangle)$.

Thus $\mathscr{Z}_0^I(F,\triangle) \subseteq \mathscr{Z}_0^I(\triangle)$.

Theorem 2.3. The inclusion $\mathscr{Z}_\infty^I(F,\triangle) \subseteq \mathscr{Z}_0^I(\triangle)$ holds if and only if

$$\lim_k f_k(t) = \infty \text{ for } t > 0. \qquad [2.3]$$

Proof. Let $\mathscr{Z}_\infty^I(F,\triangle) \subseteq \mathscr{Z}_0^I(\triangle)$ such that $\lim_k f_k(t) = \infty$ for t >0 does not hold.

Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$f_{k_i}(t_0) \leq M < \infty. \qquad [2.4]$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, \text{ if } k = k_i, \ i = 1,2,3.......; \\ 0, \ otherwise. \end{cases}$$

Thus $x \in \mathscr{Z}_\infty^I(F,\triangle)$, by [2.4].

But $x \notin \mathscr{Z}_0^I(\triangle)$, so that [2.3] must hold If $\mathscr{Z}_\infty^I(F,\triangle) \subseteq \mathscr{Z}_0^I(\triangle)$.

Conversely, let [2.3] hold.

If $x \in \mathscr{Z}_\infty^I(F,\triangle)$, then $f_k(|\triangle x_k|) \leq M < \infty$

for k = 1,2,3.......Suppose that $x \notin \mathscr{Z}_0^I(\triangle)$.

Then for some $\varepsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| < \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(|\triangle x_k|) \leq M$ for k$\geq$ k_0 which contradicts [2.3].

Hence $x \in \mathscr{Z}_0^I(\triangle)$.

Theorem 2.4. The inclusion $\mathscr{Z}_\infty^I(\triangle) \subseteq \mathscr{Z}_0^I(F,\triangle)$ holds, if and only if

$$\lim_k f_k(t) = 0 \text{ for } t > 0. \qquad [2.5]$$

Proof. Suppose that $\mathscr{Z}_\infty^I(\triangle) \subseteq \mathscr{Z}_0^I(F,\triangle)$ but [2.5] does not hold.

Then

$$\lim_k f_k(t_0) = l \neq 0. \qquad [2.6]$$

for some $t_0 > 0$.

Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{v=0}^{k-1} (-1) \begin{bmatrix} k-v \\ k-v \end{bmatrix}$$

for k = 1,2,3..........

Then $x \notin \mathscr{Z}_0^I(F,\triangle)$, by [2.6].

Hence [2.5] must hold.

Conversly , let $x \in \mathscr{Z}_\infty^I(\triangle)$ and suppose that [2.5] holds.

Then $|\triangle x_k| \leq M < \infty$ for k = 1,2,3.....

Therefore $f_k(|\triangle x_k|) \leq f_k(M)$ for k = 1,2,3..... and

$\lim_k f_k(|\triangle x_k|) \leq \lim_k f_k(M) = 0$, by [2.5].

Hence $x \in \mathscr{Z}_0^I(F,\triangle)$.

Conflict of Interests

The author declare that there is no conflict of interests.

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