



Available online at <http://scik.org>

J. Math. Comput. Sci. 11 (2021), No. 2, 1668-1675

<https://doi.org/10.28919/jmcs/5316>

ISSN: 1927-5307

## COMMON FIXED POINT THEOREMS IN KM AND GV-FUZZY METRIC SPACES

MANOJ KUMAR\*, DEEPIKA

Department of Mathematics, Baba Mastnath University, Asthal Bohar 124021, Rohtak, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Abstract.** In this paper, we prove some common fixed point theorems using (CLR) property in KM and GV-fuzzy metric spaces. Our results extend and unify several fixed point theorems present in the literature.

**Keywords:** fuzzy metric space; weakly compatible maps; (CLR) property; fixed point.

**2010 AMS Subject Classification:** 47H10, 54H25.

### 1. INTRODUCTION

Zadeh [4] introduced the concept of fuzzy sets in 1965. Fixed point theory is one of the most famous mathematical theories with application in several branches of science, especially in nonlinear programming, economics, theory of differential equations. Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [6]. He proved some fixed point theorems for fuzzy contraction mappings in metric linear space, which is a fuzzy extension of the Banach's contraction principle. The concept of fuzzy metric space was introduced by Kramosil and Michalek [3] in 1975, which opened an avenue for further development of analysis in such spaces. Further, George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [3]. Sintunavarat and Kumam [8] defined the notion of "common limit in the range" property (or (CLR) property) in fuzzy metric spaces.

---

\*Corresponding author

E-mail address: [manojantil18@gmail.com](mailto:manojantil18@gmail.com)

Received December 09, 2020

The aim of this work is to prove some common fixed point theorems for generalized contractive mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using (CLR) property. Our results does not require condition of closeness of range and so our theorems generalize, unify, and extend many results in literature.

## 2. PRELIMINARIES

Definition 2.1. [8] Let  $(X, M, *)$  be a fuzzy metric space and  $P, Q, R, S$  be self maps on  $X$ . The pairs  $(P, Q)$  and  $(R, S)$  are said to satisfy the joint common limit in the range of mappings (JCLR) property if there exists sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = \lim_{n \rightarrow \infty} Ry_n = \lim_{n \rightarrow \infty} Sy_n = Qv = Rv$ , for some  $v \in X$ .

Definition 2.2. (*t*-norm [2]) A binary operation  $*$  :  $[0, 1] \times [0, 1] \longrightarrow [0, 1]$  is a continuous *t*-norm if  $*$  satisfies the following conditions:

- (i)  $*$  is commutative and associative ;
- (ii)  $*$  is continuous ;
- (iii)  $a * 1 = a, \forall a \in [0, 1]$ ;
- (iv)  $a * b \leq c * d$ , whenever  $a \leq c$  and  $b \leq d, \forall a, b, c, d \in [0, 1]$ .

In 1975, Kramosil and Michalek [3] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy system.

Definition 2.3. (Kramosil and Michalek [3]) A fuzzy metric space is a triplet  $(X, M, *)$ , where  $X$  is a nonempty set,  $*$  is a continuous *t*-norm and  $M$  is a fuzzy set on  $X^3 \times [0, 1]$  such that the following axioms hold:

- (i)  $M(x, y, z, 0) = 0$ ;
- (ii)  $M(x, y, z, t) = 1$  if atleast two of  $x, y, z$  of  $X$  are equal;
- (iii)  $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ ;
- (iv)  $M(x, y, z, r + s + t) \geq M(x, y, w, r) * M(x, w, z, s) * M(w, y, z, t)$ ;
- (v)  $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous.

George and Veeramani [1] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek.

Definition 2.4. (George and Veeramani [1]) A fuzzy metric space is a triple  $(X, M, *)$  where  $X$

is a nonempty set,  $*$  is a continuous t-norm and  $M$  is a fuzzy set on  $X^3 \times [0, 1]$  and the following conditions are satisfied for all  $x, y \in X$  and  $t, s > 0$ :

- (i)  $M(x, y, z, 0) = 0$ ;
- (ii)  $M(x, y, z, t) = 1$  if atleast two of  $x, y, z$  of  $X$  are equal;
- (iii)  $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ ;
- (iv)  $M(x, y, z, r + s + t) \geq M(x, y, w, r) * M(x, w, z, s) * M(w, y, z, t)$ ;
- (v)  $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$  is continuous.

Example 2.5. Let  $(X, d)$  be a fuzzy metric space. Define  $a * b = ab$  and  $a \oplus b = \min\{1, a + b\}$ ,  $\forall a, b \in [0, 1]$  and  $M$  and  $N$  be fuzzy sets on  $X^3 \times [0, \infty)$  defined as follows:

$$M(x, y, z, t) = \frac{ht^n}{ht^n + md(x, y, z)},$$

for all  $h, n \in \mathbb{R}^+$ . Then  $(X, M, *, \oplus)$  is called a fuzzy metric space.

Following notion will be used in the sequel to prove our theorems:

Let  $\mathfrak{S}$  be class of all mappings  $f : [0, 1] \rightarrow [0, 1]$  satisfying the following properties:

- (1)  $f$  is continuous and non decreasing on  $[0, 1]$ ;
- (2)  $fx > x$  for all  $x \in (0, 1)$ .

### 3. COMMON FIXED POINT IN KM AND GV-FUZZY METRIC SPACES

Theorem 3.1. Let  $(X, M, *)$  be a KM-fuzzy metric space satisfying the following property:  
 $\forall x, y \in X, x \neq y, \exists t > 0 : 0 < M(x, y, t) < 1$  and let  $P, Q$  be weakly compatible self-mappings of  $X$  such that, for some  $f \in \mathfrak{S}$ ,

$$(1) \quad M(Px, Py, t) \geq f(\min\{M(Qx, Qy, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{M(Qx, Px, t_1), M(Qy, Py, t_2)\}, \\ \sup_{t_3+t_4=\frac{2t}{k}} \max\{M(Qx, Py, t_3), M(Qy, Px, t_4)\}\}),$$

where  $t > 0$  and  $1 \leq k < 2$ . If  $P$  and  $Q$  satisfy  $(CLR_Q)$  property, then  $P$  and  $Q$  have a unique common fixed point in  $X$ .

Proof. Since  $P$  and  $Q$  satisfy the  $CLR_Q$  property, there exists a sequence  $\{x_n\}$  in  $X$  such that

$$(2) \quad \lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = Qx$$

Let  $Pv \neq Qv$ . It is not difficult to prove that there exists  $t_0 > 0$  such that

$$(3) \quad M(Pv, Qv, \frac{2t_0}{k}) > M(Pv, Qv, t_0).$$

If equation (3) does not hold, then we have

$$(4) \quad M(Pv, Qv, t) = M(Pv, Qv, \frac{2t}{k}),$$

for all  $t > 0$ .

By repeating this process, we obtain

$$M(Pv, Qv, t) = M(Pv, Qv, \frac{2t}{k}) = \dots = M(Pv, Qv, (\frac{2}{k})^n t) \longrightarrow 1,$$

when  $n \longrightarrow \infty$ , which implies that  $M(Pv, Qv, t) = 1$  for all  $t > 0$ .

A contradiction to the fact that  $Pv \neq Qv$ .

Hence (3) is proved.

Substituting  $x = x_n$  and  $y = v$  in the given condition, we obtain

$$\begin{aligned} M(Px_n, Pv, t_0) &\geq f(\min\{M(Qx_n, Qv, t_0), \sup_{t_1+t_2=\frac{2t_0}{k}} \min\{M(Qx_n, Px_n, t_1), M(Qv, Pv, t_2)\}\}, \\ &\quad \sup_{t_3+t_4=\frac{2t_0}{k}} \max\{M(Qx_n, Pv, t_3), M(Qv, Px_n, t_4)\}\}) \\ &\geq f(\min\{M(Qx_n, Qv, t_0), \min\{M(Qx_n, Px_n, \varepsilon), M(Qv, Pv, \frac{2t_0}{k} - \varepsilon)\}\}, \\ &\quad \max\{M(Qx_n, Pv, \frac{2t_0}{k} - \varepsilon), M(Qv, Px_n, \varepsilon)\}\}), \end{aligned}$$

for all  $\varepsilon \in (0, \frac{2t_0}{k})$ .

Letting  $n \longrightarrow \infty$  in (1), we get

$$\begin{aligned} M(Qv, Pv, t_0) &\geq f(\min\{M(Qv, Qv, t_0), \min\{M(Qv, Qv, \varepsilon), M(Qv, Pv, \frac{2t_0}{k} - \varepsilon)\}\}, \\ &\quad \max\{M(Qv, Pv, \frac{2t_0}{k} - \varepsilon), M(Qv, Qv, \varepsilon)\}\}) \end{aligned}$$

$$= f(M(Qv, Pv, \frac{2t_0}{k} - \varepsilon))$$

$$> M(Qv, Pv, \frac{2t_0}{k} - \varepsilon).$$

When  $\varepsilon \longrightarrow 0$ , we obtain

$$M(Qv, Pv, t_0) \geq M(Qv, Pv, \frac{2t_0}{k}),$$

which is a contradiction.

Therefore,  $Qv = Pv = z$ .

Now, we claim that  $z$  is common fixed point of  $Q$  and  $P$ .

Since  $P$  and  $Q$  are weakly compatible, we have  $Pz = Qz$ .

Let us suppose that  $Pz \neq z$ .

Substituting  $Px = z$  and  $y = z$  in the given condition, we get

$$\begin{aligned} M(z, Pz, t) &\geq f(\min\{M(z, Pz, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{M(z, Pz, t_1), M(Qz, Pz, t_2)\}\}, \\ &\quad \sup_{t_3+t_4=\frac{2t}{k}} \max\{M(z, Pz, t_3), M(Pz, z, t_4)\}\}) \\ &\geq f(\min\{M(z, Pz, t), \min\{M(Qz, Pz, \varepsilon), M(z, Pz, \frac{2t}{k} - \varepsilon)\}\}, \\ &\quad \max\{M(z, Pz, \frac{2t}{k} - \varepsilon), M(Pz, z, \varepsilon)\}\}) \end{aligned}$$

for all  $\varepsilon \in (0, \frac{2t}{k})$ .

When  $\varepsilon \rightarrow 0$ , we get

$$\begin{aligned} M(z, Pz, t) &\geq f(\min\{M(z, Pz, t), M(z, Pz, \frac{2t}{k})\}) \\ &= f(M(z, Pz, t)) > M(z, Pz, t), \end{aligned}$$

which is a contradiction.

So,  $Pz = Qz = z$ . Hence,  $z$  is a common fixed point of  $P$  and  $Q$ . Uniqueness of  $z$  follows from (1).

**Theorem 3.2.** Let  $(X, M, *)$  be a GV-fuzzy metric space satisfying the following property:

$\forall x, y \in X, x \neq y, \exists t > 0 : 0 < M(x, y, t) < 1$  and let  $P, Q$  be weakly compatible self-mappings of  $X$  such that, for some  $f \in \mathfrak{F}$ ,

$$\begin{aligned} M(Px, Py, t) &\geq f(\min\{M(Qx, Qy, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{M(Qx, Px, t_1), M(Qy, Py, t_2)\}\}, \\ &\quad \sup_{t_3+t_4=\frac{2t}{k}} \max\{M(Qx, Py, t_3), M(Qy, Px, t_4)\}\}) \end{aligned}$$

where  $t > 0$  and  $1 \leq k < 2$ . If  $P$  and  $Q$  satisfy  $(CLR_Q)$  property, then  $P$  and  $Q$  have a unique common fixed point in  $X$ .

Proof. Since  $P$  and  $Q$  satisfy the  $CLR_Q$  property, there exists a sequence  $\{x_n\}$  in  $X$  such that

$$(5) \quad \lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = Qx$$

Let  $Pv \neq Qv$ . It is not difficult to prove that there exists  $t_0 > 0$  such that

$$(6) \quad M(Pv, Qv, \frac{2t_0}{k}) > M(Pv, Qv, t_0).$$

If equation (6) does not hold, then we have

$$(7) \quad M(Pv, Qv, t) = M(Pv, Qv, \frac{2t}{k}),$$

for all  $t > 0$ .

By repeating this process, we obtain

$$M(Pv, Qv, t) = M(Pv, Qv, \frac{2t}{k}) = \dots = M(Pv, Qv, (\frac{2}{k})^n t) \rightarrow 1$$

whenever  $n \rightarrow \infty$ , which implies that  $M(Pv, Qv, t) = 1$  for all  $t > 0$ , which contradicts that  $Pv \neq Qv$

Hence, (6) is proved.

Substituting  $x = x_n$  and  $y = v$  in the given condition, we obtain

$$\begin{aligned} M(Px_n, Pv, t_0) &\geq f(\min\{M(Qx_n, Qv, t_0), \sup_{t_1+t_2=\frac{2t_0}{k}} \min\{M(Qx_n, Px_n, t_1), M(Qv, Pv, t_2)\}\}, \\ &\quad \sup_{t_3+t_4=\frac{2t_0}{k}} \max\{M(Qx_n, Pv, t_3), M(Qv, Px_n, t_4)\}\}) \\ &\geq f(\min\{M(Qx_n, Qv, t_0), \min\{M(Qx_n, Px_n, \varepsilon), M(Qv, Pv, \frac{2t_0}{k} - \varepsilon)\}\}, \\ &\quad \max\{M(Qx_n, Pv, \frac{2t_0}{k} - \varepsilon), M(Qv, Px_n, \varepsilon)\}\}) \end{aligned}$$

for all  $\varepsilon \in (0, \frac{2t_0}{k})$ .

Proceeding  $n \rightarrow \infty$ , we get

$$\begin{aligned} M(Qv, Pv, t_0) &\geq f(\min\{M(Qv, Qv, t_0), \min\{M(Qv, Qv, \varepsilon), M(Qv, Pv, \frac{2t_0}{k} - \varepsilon)\}\}, \\ &\quad \max\{M(Qv, Pv, \frac{2t_0}{k} - \varepsilon), M(Qv, Qv, \varepsilon)\}\}) \\ &= \Phi(M(Qv, Pv, \frac{2t_0}{k} - \varepsilon)) \\ &> M(Qv, Pv, \frac{2t_0}{k} - \varepsilon) \end{aligned}$$

. When  $\varepsilon \rightarrow 0$ , we obtain

$$M(Qv, Pv, t_0) \geq M(Qv, Pv, \frac{2t_0}{k})$$

which is a contradiction

Therefore,  $Qv = Pv = z$

Now, we claim that  $z$  is common fixed point of  $Q$  and  $P$ .

Since  $P$  and  $Q$  are weakly compatible, we have  $Pz = Qz$ .

Let us suppose that  $Pz \neq z$ .

Substituting  $Px = z$  and  $y = z$  in the given condition, we get

$$\begin{aligned} M(z, Pz, t) &\geq f(\min\{M(z, Pz, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{M(z, Pz, t_1), M(Qz, Pz, t_2)\}, \\ &\quad \sup_{t_3+t_4=\frac{2t}{k}} \max\{M(z, Pz, t_3), M(Pz, z, t_4)\}\}) \\ &\geq f(\min\{M(z, Pz, t), \min\{M(Qz, Pz, \varepsilon), M(z, Pz, \frac{2t}{k} - \varepsilon)\}, \\ &\quad \max\{M(z, Pz, \frac{2t}{k} - \varepsilon), M(Pz, z, \varepsilon)\}\}) \end{aligned}$$

for all  $\varepsilon \in (0, \frac{2t}{k})$ .

When  $\varepsilon \rightarrow 0$ , we get

$$\begin{aligned} M(z, Pz, t) &\geq f(\min\{M(z, Pz, t), M(z, Pz, \frac{2t}{k})\}) \\ &= f(M(z, Pz, t)) > M(z, Pz, t), \end{aligned}$$

which is a contradiction.

So,  $Pz = Qz = z$ . Hence,  $z$  is a common fixed point of  $P$  and  $Q$ .

Uniqueness of  $z$  follows from (1).

**Corollary 3.3.** Let  $P$  and  $Q$  be self-mappings of a fuzzy metric space  $(X, M, *)$  satisfying the following conditions:

- (i)  $P^n(X) \subseteq Q^m(X)$ ,  $P^nQ = QP^n$  and  $PQ^m = Q^mP$

(ii)

$$M(P^n x, P^n y, t) \geq f(\min\{M(Q^m x, Q^m x, t), \\ \sup_{t_1+t_2=\frac{2t}{k}} \min\{M(Q^m x, P^n x, t_1), M(Q^m y, P^n y, t_2)\}, \\ \sup_{t_3+t_4=\frac{2t}{k}} \max\{M(Q^m x, P^n y, t_3), M(Q^m y, P^n x, t_4)\}\})$$

for all  $x, y \in X$ , for some  $n, m = 2, 3, \dots, t > 0$  and for some  $1 \leq k < 2$ . Suppose that the pair  $(P^n, Q^m)$  satisfies the property JCLR and  $(P^n, Q^m)$  is weakly compatible. Then  $P$  and  $Q$  have a unique common fixed point in  $X$ .

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

### REFERENCES

- [1] A. George, P. Veeramani, On some result in fuzzy metric space, *Fuzzy Sets Syst.* 64(3) (1994), 395-399.
- [2] B. Schweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.* 10 (1960), 313-334.
- [3] I. Kramosil, J. Michalek, Fuzzy metrics and statistical Metric Spaces, *Kybernetika*, 11(5) (1975), 326-334.
- [4] L.A. Zadeh, Fuzzy sets, *Inform. Control*, 8 (1965), 338-345.
- [5] S. Chauhan, W. Sintunavarat, P. Kumam, Common Fixed Point Theorems for Weakly Compatible Mappings in Fuzzy Metric Spaces Using (JCLR) Property, *Appl. Math.* 3 (2012), 976-982.
- [6] S. Heilpern, Fuzzy mappings and fixed point theorem, *J. Math. Anal. Appl.* 83(2) (1981), 566-569.
- [7] S. Sedghi, N. Shobe, A. Aliouche, A common fixed point theorem for weakly compatible mappings in fuzzy metric spaces, *Gen. Math.* 18 (2010), 3-12.
- [8] W. Sintunavarat, P. Kumam, Common fixed point Theorems for a pair of weakly compatible mappings in fuzzy metric space, *J. Appl. Math.* 2011 (2011), 637958.