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QUASI-BI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds providing some non-trivial examples. Integrability conditions of distributions associated with definition of such submanifolds have been obtained. Furthermore, some necessary and sufficient conditions for foliations determined by the above distributions to be totally geodesic have been investigated. Finally, we obtain some characterization results for totally umbilical and minimal quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

Keywords: radical distribution; screen distribution; lightlike transversal vector bundle; screen transversal vector bundle; semi-slant lightlike submanifolds.

2020 AMS Subject Classification: 53C15, 53C40, 53C50.

1. Introduction

In 1996, Duggal and Bejancu introduced the theory of lightlike submanifolds of a semi-Riemannian manifold ([6]). A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is a submanifold on which the induced metric is degenerate. Lightlike geometry has its applications in general relativity, specially in black hole theory, which motivated geometers to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. The concept of

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slant immersions in complex geometry was defined by B.Y. Chen as a natural generalization of both totally real immersions and holomorphic immersions([3, 4]) in 1990. Further, N. Papaghuic introduced the notion of semi-slant submanifolds of Kaehler manifolds in 1994. In [14, 15], Sahin studied the geometry of slant and screen-slant lightlike submanifolds of indefinite Hermitian manifolds. The theory of slant, CR lightlike submanifolds, SCR lightlike submanifolds of indefinite Kaehler manifolds and indefinite Sasakian manifolds has been studied in ([6], [7]).

In this paper, we introduce the notion of quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds. The organization of paper as follows : In section 2, we obtained some basic results. In section 3, we study quasi-bi-slant lightlike submanifolds of an indefinite Kaehler manifold with some examples. Section 4 is dedicated to the study of foliations determined by distributions on quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds. In section 5 and 6, we discuss the minimal quasi-bi-slant lightlike submanifolds and totally umbilical quasi-bi-slant lightlike submanifolds respectively and also provide some non-trivial examples.

2. Preliminaries

A lightlike submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is a submanifold in which induced metric g from $\bar{g}$ is degenerate and the rank of radical distribution $Rad(TM)$ is r, where $1 \leq r \leq m$. Now suppose that $S(TM)$ be a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, called screen distribution, i.e.

$$TM = Rad(TM) \oplus_{orth} S(TM) \tag{2.1}$$

Let $S(TM^{\perp})$ be a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$, called screen transversal vector bundle. Since for any local basis $\{\xi_i\}$ of $Rad(TM)$ there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}) \tag{2.2}$$

Then, $tr(TM)$ is a complementary which is not orthogonal vector bundle to TM in $T\bar{M}|_M$, i.e.

$$T\bar{M}|_M = TM \oplus tr(TM) \tag{2.3}$$

and therefore

$$T\bar{M}|_M = S(TM) \oplus_{orth} [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^\perp) \tag{2.4}$$

Following this, we have the four possible cases for a lightlike submanifold:

Case 1: r-lightlike if $r \leq min(m,n)$,

Case 2: co-isotropic if $r = n \leq m,\ S(TM^\perp) = \{0\}$,

Case 3: isotropic if $r = m \leq n,\ S(TM) = \{0\}$,

Case 4: totally lightlike if $r = m = n,\ S(TM) = S(TM^\perp) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V. \tag{2.6}$$

for all $X,Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(tr(TM))$. Here, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y),\ \forall\, X,Y \in \Gamma(TM). \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N),\ \forall\, N \in \Gamma(ltr(TM)). \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + D^l(X,W) + \nabla^s_X W,\ \forall\, W \in \Gamma(S(TM^\perp)). \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$, L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. Thus

h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued lightlike second fundamental form and screen second fundamental form of M respectively. On the other hand, ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively. Now by using (2.5), (2.7)-(2.9) and metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(X,Y),W)+\bar{g}(Y,D^l(X,W))=g(A_W X), \tag{2.10}$$

$$\bar{g}(D^s(X,N),W)=g(N,A_W X), \tag{2.11}$$

Suppose $\bar{P}$ is the projection of TM on $S(TM)$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y=\nabla^*_X \bar{P}Y+h^*(X,\bar{P}Y),\ \forall\, X,Y\in\Gamma(TM), \tag{2.12}$$

$$\nabla_X \xi=-A^*_{\xi}X+\nabla^{*t}_X \xi,\ \xi\in\Gamma(Rad(TM)) \tag{2.13}$$

where $\{\nabla^*_X \bar{P}Y, -A^*_{\xi}X\}$ and $\{h^*(X,\bar{P}Y),\nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $Rad(TM)$ respectively. On the other hand, h^* and A^* are called the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$ respectively, which are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued $F(M)$-bilinear forms on $\Gamma(TM)\times\Gamma(S(TM))$ and $\Gamma(Rad(TM))\times\Gamma(TM)$. Now by using the above equations, we obtain

$$\bar{g}(h^l(X,\bar{P}Y),\xi)=g(A^*_{\xi}X,\bar{P}Y), \tag{2.14}$$

$$\bar{g}(h^*(X,\bar{P}Y),N)=g(A_N X,\bar{P}Y), \tag{2.15}$$

$$\bar{g}(h^l(X,\xi),\xi)=0,\ A^*_{\xi}\xi=0. \tag{2.16}$$

Here, it is important to note that the induced connection ∇ on M is not a metric connection in general. Since $\bar{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_X g)(Y,Z)=\bar{g}(h^l(X,Y),Z)+\bar{g}(h^l(X,Z),Y) \tag{2.17}$$

A $2m$-dimensional semi-Riemannian manifold $(\bar{M},\bar{g},\bar{J})$ with constant index q, where $0 < q < 2m$, and a $(1,1)$ tensor field $\bar{J}$ on $\bar{M}$ is called an indefinite almost Hermitian manifold if the following conditions are satisfied:

$$\bar{J}^2X = -X \tag{2.18}$$

$$\bar{g}(\bar{J}X,\bar{J}Y) = \bar{g}(X,Y),\ \forall X,Y \in \Gamma(T\bar{M}). \tag{2.19}$$

An indefinite almost Hermitian manifold $(\bar{M},\bar{g};\bar{J})$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X\bar{J})Y = 0, \tag{2.20}$$

for all $X,Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to $\bar{g}$.

3. Quasi-Bi-Slant Lightlike Submanifolds

In this section, we introduce the notion of quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following lemma which was proved by Sahin[14]. We shall use this lemma in defining the notion of quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

Lemma 1.[15] Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Suppose that there exists a screen distribution $S(TM)$ such that $\bar{J}Rad(TM) \subset S(TM)$ and $\bar{J}ltr(TM) \subset S(TM)$. Then $\bar{J}Rad(TM) \cap \bar{J}ltr(TM) = \{0\}$ and any complementry distribution to $\bar{J}Rad(TM) \oplus \bar{J}ltr(TM)$ in $S(TM)$ is Riemannian.

DEFINITION 1. Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a quasi-bi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\bar{J}Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \bar{J}Rad(TM) = \{0\}$;

(ii) there exist non-degenerate orthogonal distributions D, D_1 and D_2 on M such that

$$S(TM) = (\bar{J}Rad(TM) \oplus \bar{J}ltr(TM)) \oplus_{orth} D \oplus_{orth} D_1 \oplus_{orth} D_2;$$

(iii) the distribution D is an invariant distribution, i.e. $\bar{J}D = D$;

(iv) the distribution D_1 is slant with angle θ_1, i.e. for each $x \in M$ and each non-zero vector $X \in (D_1)_x$, the angle θ_1 between $\bar{J}X$ and the vector subspace $(D_1)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_1)_x$.

(v) the distribution D_2 is slant with angle θ_2, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ_2 between $\bar{J}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

These constant angles θ_1 and θ_2 are called the slant angles of distributions D_1 and D_2 respectively. A quasi-bi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta_1 \neq \pi/2$, $\theta_2 \neq \pi/2$.

From the above definition, we have the following decomposition:

$$TM = Rad(TM) \oplus_{orth} (\bar{J}Rad(TM) \oplus \bar{J}ltr(TM)) \oplus_{orth} D \oplus_{orth} D_1 \oplus_{orth} D_2.$$

We observe that the above class of submanifolds includes slant, semi-slant, bi-slant lightlike submanifolds of indefinite Kaehler manifolds as its particular cases.

Let $(\mathbb{R}^{2m}_{2q}, \bar{g}, \bar{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by

$$\bar{g} = \frac{1}{4}(-\textstyle\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$
$$\bar{J}(\textstyle\sum_{i=1}^{m}(X_i dx_i + Y_i dy_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i),$$

where (x^i, y^i) are the Cartesian co-ordinates on $\mathbb{R}^{2m}_{2q}$. Now, we construct some examples of quasi-bi-slant lightlike submanifolds of an indefinite Kaehler manifold.

EXAMPLE 1. Let $(\mathbb{R}^{14}_{2}, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,+,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7\}$.

Suppose M is a submanifold of $\mathbb{R}^{14}_{2}$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos\beta$, $y^3 = -u_5 \cos\beta$, $x^4 = u_5 \sin\beta$, $y^4 = u_4 \sin\beta$, $x^5 = u_6 \sin u_7$, $y^5 = u_6 \cos u_7$, $x^6 = k_1 \sin u_6$, $y^6 = k_1 \cos u_6$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = k_2 \cos u_9$, $y^8 = k_2 \sin u_9$, where k_1 and k_2 are constants.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$Z_1 = 2(-\partial x_1 + \partial y_2),$$

$$Z_2 = 2(\partial x_2),\ Z_3 = 2(\partial y_1),$$

$$Z_4 = 2(\cos\beta\partial x_3 + \sin\beta\partial y_4),$$

$$Z_5 = 2(\sin\beta\partial x_4 - \cos\beta\partial y_3),$$

$$Z_6 = 2(\sin u_7\partial x_5 + \cos u_7\partial y_5 + k_1\cos u_6\partial x_6 - k_1\sin u_6\partial y_6),$$

$$Z_7 = 2(u_6\cos u_7\partial x_5 - u_6\sin u_7\partial y_5),$$

$$Z_8 = 2(\partial x_7),\ Z_9 = 2(\partial y_7 - k_2\sin u_9\partial x_8 + k_2\cos u_9\partial y_8).$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$. Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2$ and $S(TM^{\perp})$ is spanned by

$$W_1 = 2(\sin\beta\partial x_3 - \cos\beta\partial y_4),$$

$$W_2 = 2(\cos\beta\partial x_4 + \sin\beta\partial y_3),$$

$$W_3 = 2(k_1^2\sin u_7\partial x_5 + k_1^2\cos u_7\partial y_5 - k_1\cos u_6\partial x_6 + k_1\sin u_6\partial y_6),$$

$$W_4 = 2(u_6\sin u_6\partial x_6 + u_6\cos u_6\partial y_6).$$

It follows that $\bar{J}Z_1 = Z_2 + Z_3$ and $\bar{J}N = \frac{1}{2}(Z_2 - Z_3)$, which implies that $\bar{J}Rad(TM)$ and $\bar{J}ltr(TM)$ are distributions on M. On the other hand, we can see that $D = span\{Z_4, Z_5\}$ such that $\bar{J}Z_4 = Z_5$, $\bar{J}Z_5 = -Z_4$, which implies that D is invariant with respect to $\bar{J}$. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1+k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1+k_2^2})$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}_2^{14}$.

EXAMPLE 2. Let $(\mathbb{R}_2^{14}, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,+,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7\}$.

Suppose M is a submanifold of $\mathbb{R}^{14}_2$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \sin\beta$, $y^3 = -u_5 \sin\beta$, $x^4 = u_5 \cos\beta$, $y^4 = u_4 \cos\beta$, $x^5 = u_6 \cos u_7$, $y^5 = u_6 \sin u_7$, $x^6 = k_1 \cos u_6$, $y^6 = k_1 \sin u_6$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = k_2 \sin u_9$, $y^8 = k_2 \cos u_9$, where k_1 and k_2 are constants.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2),$$

$$Z_2 = 2(\partial x_2),\ Z_3 = 2(\partial y_1),$$

$$Z_4 = 2(\sin\beta\,\partial x_3 + \cos\beta\,\partial y_4),$$

$$Z_5 = 2(\cos\beta\,\partial x_4 - \sin\beta\,\partial y_3),$$

$$Z_6 = 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 - k_1 \sin u_6 \partial x_6 + k_1 \cos u_6 \partial y_6),$$

$$Z_7 = 2(-u_6 \sin u_7 \partial x_5 + u_6 \cos u_7 \partial y_5),$$

$$Z_8 = 2(\partial x_7),\ Z_9 = 2(\partial y_7 + k_2 \cos u_9 \partial x_8 - k_2 \sin u_9 \partial y_8).$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$. Now $ltr(TM)$ is spanned by $N = -\partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$W_1 = 2(\cos\beta\,\partial x_3 - \sin\beta\,\partial y_4),$$

$$W_2 = 2(\sin\beta\,\partial x_4 + \cos\beta\,\partial y_3),$$

$$W_3 = 2(k_1^2 \sin u_9 \partial x_8 + k_1^2 \cos u_9 \partial y_8),$$

$$W_4 = 2(k_2^2 \partial y_7 - k_2 \cos u_9 \partial x_8 + k_2 \sin u_9 \partial y_8).$$

It follows that $\bar{J}Z_1 = Z_2 - Z_3$ and $\bar{J}N = \frac{1}{2}(Z_2 + Z_3)$, which implies that $\bar{J}Rad(TM)$ and $\bar{J}ltr(TM)$ are distributions on M. On the other hand, we can see that $D = span\{Z_4, Z_5\}$ such that $\bar{J}Z_4 = Z_5$, $\bar{J}Z_5 = -Z_4$, which implies that D is invariant with respect to $\bar{J}$. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1+k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1+k_2^2})$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}^{14}_2$.

Now, for any vector field X tangent to M, we put

$$\bar{J}X = PX + FX \tag{3.1}$$

where PX and FX are the tangential and transversal parts of $\bar{J}X$ respectively. We denote the projections on $Rad(TM)$, $\bar{J}Rad(TM)$, $\bar{J}ltr(TM)$, D, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, P_5 and P_6 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ by Q and R respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X + P_6X, \tag{3.2}$$

Now applying $\bar{J}$ to (3.2), we have

$$\bar{J}X = \bar{J}P_1X + \bar{J}P_2X + \bar{J}P_3X + \bar{J}P_4X + \bar{J}P_5X + \bar{J}P_6X, \tag{3.3}$$

which gives

$$\bar{J}X = \bar{J}P_1X + \bar{J}P_2X + \bar{J}P_3X + \bar{J}P_4X + fP_5X + FP_5X + fP_6X + FP_6X, \tag{3.4}$$

where fP_5X and FP_5X (resp. fP_6X and FP_6X) denotes the tangential and transversal components of $\bar{J}P_5X$ (resp. $\bar{J}P_6X$). Thus we get $\bar{J}P_1X \in \Gamma(\bar{J}Rad(TM))$, $\bar{J}P_2X \in \Gamma(Rad(TM))$, $\bar{J}P_3X \in \Gamma(ltr(TM))$, $\bar{J}P_4X \in \Gamma(D)$, $fP_5X \in \Gamma(D_1)$, $fP_6X \in \Gamma(D_2)$ and $FP_5X, FP_6X \in \Gamma(S(TM^{\perp})$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = QW + RW, \tag{3.5}$$

Applying $\bar{J}$ to (3.5), we obtain

$$\bar{J}W = \bar{J}QW + \bar{J}RW, \tag{3.6}$$

which gives

$$\bar{J}W = \bar{J}QW + BR_1W + CR_1W + BR_2W + CR_2W, \tag{3.7}$$

where BR_1W (resp. CR_1W) denotes the tangential (resp. transversal) component of $\bar{J}R_1W$ and BR_2W (resp. CR_2W) denotes the tangential (resp. transversal) component of $\bar{J}R_2W$. Thus we get $\bar{J}QW \in \Gamma(\bar{J}ltr(TM))$, $BR_1W \in \Gamma(D_1)$, $CR_1W \in \Gamma(S(TM^{\perp}))$, $BR_2W \in \Gamma(D_2)$ and $CR_2W \in \Gamma(S(TM^{\perp}))$. Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $Rad(TM)$, $\bar{J}Rad(TM)$, $\bar{J}ltr(TM)$, D, D_1, D_2, $ltr(TM)$ and $S(TM^{\perp})$, we obtain

$$\begin{aligned}(3.8)\quad & P_1(\nabla_X \bar{J}P_1Y)+P_1(\nabla_X \bar{J}P_2Y)+P_1(\nabla_X \bar{J}P_4Y)+P_1(\nabla_X fP_5Y)+P_1(\nabla_X fP_6Y)\\ &= P_1(A_{FP_5Y}X)+P_1(A_{FP_6Y}X)+P_1(A_{\bar{J}P_3Y}X)+\bar{J}P_2\nabla_XY,\end{aligned}$$

$$\begin{aligned}(3.9)\quad & P_2(\nabla_X \bar{J}P_1Y)+P_2(\nabla_X \bar{J}P_2Y)+P_2(\nabla_X \bar{J}P_4Y)+P_2(\nabla_X fP_5Y)+P_2(\nabla_X fP_6Y)\\ &= P_2(A_{FP_5Y}X)+P_2(A_{FP_6Y}X)+P_2(A_{\bar{J}P_3Y}X)+\bar{J}P_1\nabla_XY,\end{aligned}$$

$$\begin{aligned}(3.10)\quad & P_3(\nabla_X \bar{J}P_1Y)+P_3(\nabla_X \bar{J}P_2Y)+P_3(\nabla_X \bar{J}P_4Y)+P_3(\nabla_X fP_5Y)+P_3(\nabla_X fP_6Y)\\ &= P_3(A_{FP_5Y}X)+P_3(A_{FP_6Y}X)+P_3(A_{\bar{J}P_3Y}X)+\bar{J}h^l(X,Y),\end{aligned}$$

$$\begin{aligned}(3.11)\quad & P_4(\nabla_X \bar{J}P_1Y)+P_4(\nabla_X \bar{J}P_2Y)+P_4(\nabla_X \bar{J}P_4Y)+P_4(\nabla_X fP_5Y)+P_4(\nabla_X fP_6Y)\\ &= P_4(A_{FP_5Y}X)+P_4(A_{FP_6Y}X)+P_4(A_{\bar{J}P_3Y}X)+\bar{J}P_4\nabla_XY,\end{aligned}$$

$$\begin{aligned}(3.12)\quad & P_5(\nabla_X \bar{J}P_1Y)+P_5(\nabla_X \bar{J}P_2Y)+P_5(\nabla_X \bar{J}P_4Y)+P_5(\nabla_X fP_5Y)+P_5(\nabla_X fP_6Y)\\ &= P_5(A_{FP_5Y}X)+P_5(A_{FP_6Y}X)+P_5(A_{\bar{J}P_3Y}X)+fP_5\nabla_XY+Bh^s(X,Y),\end{aligned}$$

$$\begin{aligned}(3.13)\quad & P_6(\nabla_X \bar{J}P_1Y)+P_6(\nabla_X \bar{J}P_2Y)+P_6(\nabla_X \bar{J}P_4Y)+P_6(\nabla_X fP_5Y)+P_6(\nabla_X fP_6Y)\\ &= P_6(A_{FP_5Y}X)+P_6(A_{FP_6Y}X)+P_6(A_{\bar{J}P_3Y}X)+fP_6\nabla_XY+Bh^s(X,Y),\end{aligned}$$

$$\begin{aligned}(3.14)\quad & h^l(X,\bar{J}P_1Y)+h^l(X,\bar{J}P_2Y)+h^l(X,\bar{J}P_4Y)+h^l(X,fP_5Y)+h^l(X,fP_6Y)\\ &= \bar{J}P_3\nabla_XY-\nabla^l_X\bar{J}P_3Y-D^l(X,FP_5Y)-D^l(X,FP_6Y),\end{aligned}$$

$$\begin{aligned}(3.15)\quad & h^s(X,\bar{J}P_1Y)+h^s(X,\bar{J}P_2Y)+h^s(X,\bar{J}P_4Y)+h^s(X,fP_5Y)+h^s(X,fP_6Y)\\ &= Ch^s(X,Y)-\nabla^s_XFP_5Y-\nabla^s_XFP_6Y-D^s(X,\bar{J}P_3Y)+FP_5\nabla_XY+FP_6\nabla_XY.\end{aligned}$$

THEOREM 1. Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then M is a quasi-bi-slant lightlike submanifold if and only if

(i) $\bar{J}Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \bar{J}Rad(TM) = \{0\}$;

(ii) the screen distribution $S(TM)$ can be split as a direct sum

$$S(TM) = (\bar{J}Rad(TM) \oplus \bar{J}ltr(TM)) \oplus D \oplus_{orth} D_1 \oplus_{orth} D_2$$

such that D is an invariant distribution on M, i.e. $\bar{J}D = D$;

(iii) there exists a constant $\lambda_1 \in [0,1)$ such that $P^2X = -\lambda_1 X$, for all $X \in \Gamma(D_1)$, where $\lambda_1 = cos^2\theta_1$ and θ_1 is the slant angle of D_1;

(iv) there exists a constant $\lambda_2 \in [0,1)$ such that $P^2X = -\lambda_2 X$, for all $X \in \Gamma(D_2)$, where $\lambda_2 = cos^2\theta_2$ and θ_2 is the slant angle of D_2.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D is invariant with respect to $\bar{J}$ and $\bar{J}Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \bar{J}Rad(TM) = \{0\}$.

For any $X \in \Gamma(D_1)$ we have $|PX| = |\bar{J}X| \cos\theta_1$, i.e.

$$\cos\theta_1 = \frac{|PX|}{|\bar{J}X|}. \tag{3.16}$$

In view of (3.16), we get $\cos^2\theta_1 = \frac{|PX|^2}{|\bar{J}X|^2} = \frac{g(PX,PX)}{g(\bar{J}X,\bar{J}X)} = \frac{g(X,P^2X)}{g(X,\bar{J}^2X)}$, which gives

$$g(X,P^2X) = \cos^2\theta_1 g(X,\bar{J}^2X). \tag{3.17}$$

Since M is a quasi-bi-slant lightlike submanifold, $\cos^2\theta_1 = \lambda_1(\text{constant}) \in [0,1)$ and therefore from (3.16) we get $g(X,P^2X) = \lambda_1 g(X,\bar{J}^2X) = g(X,\lambda_1\bar{J}^2X)$, for all $X \in \Gamma(D_1)$, which implies

$$g(X,(P^2 - \lambda_1\bar{J}^2)X) = 0 \tag{3.18}$$

Since $(P^2 - \lambda_1\bar{J}^2)X \in \Gamma(D_1)$ and the induced metric $g = g|_{D_1\times D_1}$ is non-degenerate (positive definite). From (3.18) we have $(P^2 - \lambda_1\bar{J}^2)X = 0$, which implies

$$P^2X = \lambda_1\bar{J}^2X = -\lambda_1 X,\ \forall X \in \Gamma(D_1). \tag{3.19}$$

This proves (iii).

Suppose for any $X \in \Gamma(D_2)$ we have $|PX| = |\bar{J}X| \cos\theta_2$, i.e.

$$\cos\theta_2 = \frac{|PX|}{|\bar{J}X|}. \tag{3.20}$$

Now the proof follows by using similar steps above of proof of (iii), which gives $\cos^2\theta_2 = \lambda_2$(constant). This proves (iv).

Conversely, suppose that conditions (i), (ii), (iii) and (iv) are satisfied. From (iii), we have $P^2X = \lambda_1\bar{J}^2X,\ \forall X \in \Gamma(D_1)$, where $\lambda_1 \in [0,1)$.

Now $\cos\theta_1 = \dfrac{g(\bar{J}X,PX)}{|\bar{J}X||PX|} = -\dfrac{g(X,\bar{J}PX)}{|\bar{J}X||PX|} = -\dfrac{g(X,P^2X)}{|\bar{J}X||PX|} = -\lambda_1\dfrac{g(X,\bar{J}^2X)}{|\bar{J}X||PX|} = \lambda_1\dfrac{g(\bar{J}X,\bar{J}X)}{|\bar{J}X||PX|}$.

From the above equation, we obtain

$$\cos\theta_1 = \lambda_1\frac{|\bar{J}X|}{|PX|}. \tag{3.21}$$

Therefore (3.16) and (3.21) give $\cos^2\theta_1 = \lambda_1$(constant).

Furthermore, from (iv) we have $P^2X = \lambda_2\bar{J}^2X,\ \forall X \in \Gamma(D_2)$, where $\lambda_2 \in [0,1)$. Now by using the similar steps above we get $\cos^2\theta_2 = \lambda_2$(constant). This completes the proof. Hence M is a quasi-bi-slant lightlike submanifold. □

THEOREM 2. Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then M is a quasi-bi-slant lightlike submanifold if and only if

(i) $\bar{J}Rad(TM)$ is a distribution on M such that $Rad(TM)\cap\bar{J}Rad(TM) = \{0\}$;

(ii) the screen distribution $S(TM)$ can be split as a direct sum

$$S(TM) = (\bar{J}Rad(TM)\oplus\bar{J}ltr(TM))\oplus D\oplus_{orth} D_1\oplus_{orth} D_2$$

such that D is an invariant distribution on M, i.e. $\bar{J}D = D$;

(iii) there exists a constant $\mu_1 \in [0,1)$ such that $BFX = -\mu_1X,\ \forall X \in \Gamma(D_1)$, where $\mu_1 = sin^2\theta_1$ and θ_1 is the slant angle of D_1;

(iv) there exists a constant $\mu_2 \in [0,1)$ such that $BFX = -\mu_2X,\ \forall X \in \Gamma(D_2)$, where $\mu_2 = sin^2\theta_2$ and θ_2 is the slant angle of D_2.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D is invariant with respect to $\bar{J}$ and $\bar{J}Rad(TM)$ is a distribution on M such that $Rad(TM)\cap\bar{J}Rad(TM) = \{0\}$.

Now, for any vector field $X \in \Gamma(D_1)$, we have

$$\bar{J}X = PX + FX, \tag{3.22}$$

where PX and FX are the tangential and transversal parts of $\bar{J}X$ respectively. Applying $\bar{J}$ to (3.22) and taking the tangential component, we get

$$-X = P^2X + BFX,\ \forall X \in \Gamma(D_1). \tag{3.23}$$

Since M is a quasi-bi-slant lightlike submanifold, $P^2X = -\lambda_1 X,\ \forall X \in \Gamma(D_1)$, where $\lambda_1 \in [0,1)$ and therefore from (3.23) we get

$$BFX = -\mu_1 X,\ \forall X \in \Gamma(D_1), \tag{3.24}$$

where $1-\lambda_1 = \mu_1(\text{constant}) \in (0,1]$. Now, in view of Theorem 1, we have $\lambda_1 = \cos^2\theta_1$. This proves (iii).

Suppose for any vector field $X \in \Gamma(D_2)$, we have

$$\bar{J}X = PX + FX, \tag{3.25}$$

where PX and FX are the tangential and transversal parts of $\bar{J}X$ respectively. Now the proof follows by using similar steps above of proof of (iii), which gives $1-\lambda_2 = \mu_2(\text{constant}) \in [0,1)$, where $\lambda_2 = \cos^2\theta_2$. This proves (iv).

Conversely, assume that conditions (i), (ii), (iii) and (iv) are satisfied. From (3.23) we get

$$-X = P^2X - \mu_1 X,\ \forall X \in \Gamma(D_1), \tag{3.26}$$

which implies

$$P^2X = -\lambda_1 X,\ \forall X \in \Gamma(D_1) \tag{3.27}$$

where $1-\mu_1 = \lambda_1(\text{constant}) \in [0,1)$. Furthermore, for any $X \in \Gamma(D_2)$, by using the similar steps above we have $1-\mu_2 = \lambda_2(\text{constant}) \in [0,1)$. Now the proof follows from Theorem 1. Therefore, M is a quasi-bi-slant lightlike submanifold. □

Corollary 1. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then for any slant distribution D of M with slant angle θ, we have

$$g(PX,PY) = \cos^2\theta\, g(X,Y);$$
$$g(FX,FY) = \sin^2\theta\, g(X,Y),$$

for all $X,Y \in \Gamma(D)$.

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1 of [15].

THEOREM 3. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the radical distribution $Rad(TM)$ is integrable if and only if

(i) $P_1(\nabla_X \bar{J}Y) = P_1(\nabla_Y \bar{J}X)$ and $P_4(\nabla_X \bar{J}Y) = P_4(\nabla_Y \bar{J}X)$;

(ii) $P_5(\nabla_X \bar{J}Y) = P_5(\nabla_Y \bar{J}X)$ and $P_6(\nabla_X \bar{J}Y) = P_6(\nabla_Y \bar{J}X)$;

(iii) $h^l(Y,\bar{J}X) = h^l(X,\bar{J}Y)$ and $h^s(Y,\bar{J}X) = h^s(X,\bar{J}Y)$

for all $X,Y \in \Gamma(Rad(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. From (3.8), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$P_1(\nabla_X \bar{J}Y) = \bar{J}P_2\nabla_X Y. \tag{3.28}$$

By interchanging X and Y in (3.28) we get

$$P_1(\nabla_Y \bar{J}X) = \bar{J}P_2\nabla_Y X. \tag{3.29}$$

From (3.28) and (3.29), we obtain

$$P_1(\nabla_X \bar{J}Y) - P_1(\nabla_Y \bar{J}X) = \bar{J}P_2[X,Y]. \tag{3.30}$$

From (3.11), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$P_4(\nabla_X \bar{J}Y) = \bar{J}P_4\nabla_X Y. \tag{3.31}$$

By interchanging X and Y in (3.31) we get

$$P_4(\nabla_Y \bar{J}X) = \bar{J}P_4\nabla_Y X. \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$P_4(\nabla_X \bar{J}Y) - P_4(\nabla_Y \bar{J}X) = \bar{J}P_4[X,Y]. \tag{3.33}$$

From (3.12), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$P_5(\nabla_X \bar{J}Y) = fP_5\nabla_X Y + Bh^s(X,Y). \tag{3.34}$$

By interchanging X and Y in (3.34) we get

$$P_5(\nabla_Y \bar{J}X) = fP_5\nabla_Y X + Bh^s(Y,X). \tag{3.35}$$

From (3.34) and (3.35), we obtain

$$P_5(\nabla_X \bar{J}Y) - P_5(\nabla_Y \bar{J}X) = fP_5[X,Y]. \tag{3.36}$$

From (3.13), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$P_6(\nabla_X \bar{J}Y) = fP_6\nabla_X Y + Bh^s(X,Y). \tag{3.37}$$

By interchanging X and Y in (3.37) we get

$$P_6(\nabla_Y \bar{J}X) = fP_6\nabla_Y X + Bh^s(Y,X). \tag{3.38}$$

From (3.37) and (3.38), we obtain

$$P_6(\nabla_X \bar{J}Y) - P_6(\nabla_Y \bar{J}X) = fP_6[X,Y]. \tag{3.39}$$

From (3.14), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$h^l(X,\bar{J}Y) = \bar{J}P_3\nabla_X Y. \tag{3.40}$$

Interchanging X and Y in (3.40) we get

$$h^l(Y,\bar{J}X) = \bar{J}P_3\nabla_Y X. \tag{3.41}$$

From (3.40) and (3.41) we get

$$h^l(X,\bar{J}Y) - h^l(Y,\bar{J}X) = \bar{J}P_3[X,Y]. \tag{3.42}$$

From (3.15), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$h^s(X,\bar{J}Y) = Ch^s(X,Y) + FP_5\nabla_X Y + FP_6\nabla_X Y. \tag{3.43}$$

Interchanging X and Y in (3.43) we get

$$h^s(Y,\bar{J}X) = Ch^s(Y,X) + FP_5\nabla_Y X + FP_6\nabla_Y X. \tag{3.44}$$

From (3.43) and (3.44), we obtain

$$h^s(X,\bar{J}Y)-h^s(Y,\bar{J}X)=FP_5[X,Y]+FP_6[X,Y]. \tag{3.45}$$

Now the proof follows from (3.30), (3.33), (3.36), (3.39), (3.42) and (3.45). □

THEOREM 4. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D is integrable if and only if

(i) $P_1(\nabla_X\bar{J}Y)=P_1(\nabla_Y\bar{J}X)$ and $P_2(\nabla_X\bar{J}Y)=P_2(\nabla_Y\bar{J}X)$;

(ii) $P_5(\nabla_X\bar{J}Y)=P_5(\nabla_Y\bar{J}X)$ and $P_6(\nabla_X\bar{J}Y)=P_6(\nabla_Y\bar{J}X)$;

(iii) $h^l(Y,\bar{J}X)=h^l(X,\bar{J}Y)$ and $h^s(Y,\bar{J}X)=h^s(X,\bar{J}Y)$,

for all $X,Y\in\Gamma(D)$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. From (3.8), for any $X,Y\in\Gamma(D)$, we have

$$P_1(\nabla_X\bar{J}Y)=\bar{J}P_2\nabla_XY. \tag{3.46}$$

By interchanging X and Y in (3.46) we get

$$P_1(\nabla_Y\bar{J}X)=\bar{J}P_2\nabla_YX. \tag{3.47}$$

From (3.46) and (3.47), we obtain

$$P_1(\nabla_X\bar{J}Y)-P_1(\nabla_Y\bar{J}X)=\bar{J}P_2[X,Y]. \tag{3.48}$$

From (3.9), for any $X,Y\in\Gamma(D)$, we have

$$P_2(\nabla_X\bar{J}Y)=\bar{J}P_1\nabla_XY. \tag{3.49}$$

By interchanging X and Y in (3.49) we get

$$P_2(\nabla_Y\bar{J}X)=\bar{J}P_1\nabla_YX. \tag{3.50}$$

From (3.49) and (3.50), we obtain

$$P_2(\nabla_X\bar{J}Y)-P_2(\nabla_Y\bar{J}X)=\bar{J}P_1[X,Y]. \tag{3.51}$$

From (3.12), for any $X,Y \in \Gamma(D)$, we have

$$P_5(\nabla_X \bar{J}Y) = fP_5\nabla_X Y + Bh^s(X,Y). \tag{3.52}$$

By interchanging X and Y in (3.52) we get

$$P_5(\nabla_Y \bar{J}X) = fP_5\nabla_Y X + Bh^s(Y,X). \tag{3.53}$$

From (3.52) and (3.53), we obtain

$$P_5(\nabla_X \bar{J}Y) - P_5(\nabla_Y \bar{J}X) = fP_5[X,Y]. \tag{3.54}$$

From (3.13), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$P_6(\nabla_X \bar{J}Y) = fP_6\nabla_X Y + Bh^s(X,Y). \tag{3.55}$$

By interchanging X and Y in (3.55) we get

$$P_6(\nabla_Y \bar{J}X) = fP_6\nabla_Y X + Bh^s(Y,X). \tag{3.56}$$

From (3.55) and (3.56), we obtain

$$P_6(\nabla_X \bar{J}Y) - P_6(\nabla_Y \bar{J}X) = fP_6[X,Y]. \tag{3.57}$$

From (3.14), for any $X,Y \in \Gamma(D)$, we have

$$h^l(X,\bar{J}Y) = \bar{J}P_3\nabla_X Y. \tag{3.58}$$

Interchanging X and Y in (3.58) we get

$$h^l(Y,\bar{J}X) = \bar{J}P_3\nabla_Y X. \tag{3.59}$$

From (3.58) and (3.59) we get

$$h^l(X,\bar{J}Y) - h^l(Y,\bar{J}X) = \bar{J}P_3[X,Y]. \tag{3.60}$$

From (3.15), for any $X,Y \in \Gamma(Rad(TM))$, we have

$$h^s(X,\bar{J}Y) = Ch^s(X,Y) + FP_5\nabla_X Y + FP_6\nabla_X Y. \tag{3.61}$$

Interchanging X and Y in (3.61) we get

$$h^s(Y,\bar{J}X) = Ch^s(Y,X) + FP_5\nabla_Y X + FP_6\nabla_Y X. \tag{3.62}$$

From (3.62) and (3.63), we obtain

$$h^s(X,\bar{J}Y)-h^s(Y,\bar{J}X)=FP_5[X,Y]+FP_6[X,Y]. \tag{3.63}$$

Now, in view of the equations (3.48), (3.51), (3.54), (3.57), (3.60) and (3.63), the proof follows.

□

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

DEFINITION 2. A quasi-bi-slant lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X,Y)=0$, for all $X\in\Gamma(D_1)$ and $Y\in\Gamma(D_2)$. Thus M is a mixed geodesic quasi-bi-slant lightlike submanifold if $h^l(X,Y)=0$ and $h^s(X,Y)=0$, $\forall\, X\in\Gamma(D_1)$ and $Y\in\Gamma(D_2)$.

THEOREM 5. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $Rad(TM)$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X\bar{J}P_2Z+\nabla_X\bar{J}P_4Z+\nabla_X fP_5Z+\nabla_X fP_6Z,\bar{J}Y)$$
$$=g(A_{\bar{J}P_3Z}X+A_{FP_5Z}X+A_{FP_6Z}X,\bar{J}Y),$$

for all $X\in\Gamma(Rad(TM))$ and $Z\in\Gamma(S(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. The distribution $Rad(TM)$ defines a totally geodesic foliation if and only if $\nabla_XY\in Rad(TM)$, $\forall\, X,Y\in\Gamma(Rad(TM))$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $X,Y\in\Gamma(Rad(TM))$ and $Z\in\Gamma(S(TM))$, we get

$$\bar{g}(\nabla_XY,Z)=\bar{g}((\bar{\nabla}_X\bar{J})Z-\bar{\nabla}_X\bar{J}Z,\bar{J}Y). \tag{4.1}$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\bar{\nabla}_X(\bar{J}P_2Z + \bar{J}P_3Z + \bar{J}P_4Z + fP_5Z + FP_5Z + fP_6Z + FP_6Z), \bar{J}Y). \tag{4.2}$$

In view of (2.7)-(2.9) and (4.2), for any $X,Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$, we obtain

$$\begin{aligned}\bar{g}(\nabla_X Y, Z) = g(&A_{\bar{J}P_3Z}X + A_{FP_5Z}X + A_{FP_6Z}X - \nabla_X \bar{J}P_2Z \\ &- \nabla_X \bar{J}P_4Z - \nabla_X fP_5Z - \nabla_X fP_6Z, \bar{J}Y),\end{aligned} \tag{4.3}$$

which completes the proof. □

THEOREM 6. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_2 defines a totally geodesic foliation if and only if

(i) $\bar{g}(\nabla_X fZ - A_{FZ}X, fY) = \bar{g}(h^s(X, \bar{J}Z), FY)$,

(ii) $g(fY, \nabla_X \bar{J}N) = -\bar{g}(FY, h^s(X, \bar{J}N))$,

(iii) $g(fY, A_{\bar{J}W}X) = \bar{g}(FY, D^s(X, \bar{J}W))$

for all $X,Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$, $W \in \Gamma(\bar{J}ltr(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold. To prove that the distribution D_2 defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_2)$, $\forall\, X,Y \in \Gamma(D_2)$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19) for any $X,Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$ we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X \bar{J}Y, \bar{J}Z) = -\bar{g}(\bar{\nabla}_X \bar{J}Z, \bar{J}Y) \tag{4.4}$$

From (2.7), (3.1) and (4.4) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X(fZ + FZ) + h^s(X, \bar{J}Z), fY + FY) \tag{4.5}$$

In view of (2.7)-(2.9) and (4.5) we obtain

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X fZ - A_{FZ}X, fY) - \bar{g}(h^s(X, \bar{J}Z), FY) \tag{4.6}$$

From (4.6) we get (i).

Now for any $X,Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we have

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X \bar{J}Y, \bar{J}N) = -\bar{g}(\bar{\nabla}_X \bar{J}N, \bar{J}Y) \tag{4.7}$$

From (2.7), (3.1) and (4.7) we get

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\nabla_X \bar{J}N + h^s(X, \bar{J}N), fY + FY) \tag{4.8}$$

In view of (4.8) we obtain

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\nabla_X \bar{J}N, fY) - \bar{g}(h^s(X, \bar{J}N), FY) \tag{4.9}$$

Thus from (4.9) we get the result (ii).

Now for any $X, Y \in \Gamma(D_2)$ and $W \in \Gamma(\bar{J}ltr(TM))$, we have

$$\bar{g}(\nabla_X Y, W) = \bar{g}(\bar{\nabla}_X \bar{J}Y, \bar{J}W) = -\bar{g}(\bar{\nabla}_X \bar{J}W, \bar{J}Y) \tag{4.10}$$

From (2.8), (3.1) and (4.10) we get

$$\bar{g}(\nabla_X Y, W) = -\bar{g}(-A_{\bar{J}W}X + D^s(X, \bar{J}W), fY + FY) \tag{4.11}$$

In view of (4.11) we obtain

$$\bar{g}(\nabla_X Y, W) = \bar{g}(A_{\bar{J}W}X, fY) - \bar{g}(FY, D^s(X, \bar{J}W)) \tag{4.12}$$

Thus from (4.12) we get the result (iii), which completes the proof. □

5. MINIMAL QUASI-BI-SLANT LIGHTLIKE SUBMANIFOLD

In this section, we study minimal quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds. A general notion of a minimal lightlike submanfold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal in [1] is as follows:

DEFINITION 3.[8] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if

(i) $h^s = 0$ on $Rad(TM)$,

(ii) $trace\ h = 0$, where $trace$ is written with respect to g restricted to $S(TM)$.

EXAMPLE 3. Let $(\mathbb{R}^{14}_2, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,+,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7\}$.

Suppose M is a submanifold of $\mathbb{R}^{14}_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 = y^4$, $x^4 = u_5 = -y^3$, $x^5 = u_6$, $x^6 = -u_7 = y^5$, $y^6 = -u_9$, $x^7 = u_9 = -y^6$, $y^7 = u_8$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$Z_1 = 2(-\partial x_1 + \partial y_2),$$

$$Z_2 = 2(\partial x_2),\ Z_3 = 2(\partial y_1),$$

$$Z_4 = 2(\partial x_3 + \partial y_4),\ Z_5 = 2(\partial x_4 - \partial y_3),$$

$$Z_6 = 2(\partial x_5),\ Z_7 = 2(-\partial x_6 - \partial y_5),$$

$$Z_8 = 2(\partial y_7),\ Z_9 = 2(\partial x_7 - y_6).$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$. Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$W_1 = 2(\partial x_3 - \partial y_4),\ W_2 = 2(\partial x_4 + \partial y_3),$$
$$W_3 = 2(\partial x_6 + \partial y_5),\ W_4 = 2(\partial x_7 + \partial y_6),$$

It follows that $\bar{J}Z_1 = Z_2 + Z_3$ and $\bar{J}N = \frac{1}{2}(Z_2 - Z_3)$, which implies that $\bar{J}Rad(TM)$ and $\bar{J}ltr(TM)$ are distributions on M. On the other hand, we can see that $D = span\{Z_4, Z_5\}$ such that $\bar{J}Z_4 = Z_5$, $\bar{J}Z_5 = -Z_4$, which implies that D is invariant with respect to $\bar{J}$. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \pi/4$ and $\theta_2 = \pi/4$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}^{14}_2$. Now by direct computation and using Gauss formula, we get for any $X \in \Gamma(TM)$ we have

$$\bar{\nabla}_{Z_i} Z_j = 0, \text{ where } 1 \leq i, j \leq 9$$

which implies $h^l(Z_i, Z_j) = 0$, $h^s(Z_i, Z_j) = 0$. Thus $h^s(Z_1, Z_1) = 0$, i.e. $h^s = 0$ on $Rad(TM)$. We also have $\varepsilon_1 = g(Z_1, Z_1) = 0$, $\varepsilon_2 = g(Z_2, Z_2) = 1$, $\varepsilon_3 = g(Z_3, Z_3) = -1$, $\varepsilon_4 = g(Z_4, Z_4) = 2$, $\varepsilon_5 = g(Z_5, Z_5) = 2$, $\varepsilon_6 = g(Z_6, Z_6) = 1$, $\varepsilon_7 = g(Z_7, Z_7) = 2$, $\varepsilon_8 = g(Z_8, Z_8) = 1$, $\varepsilon_9 = g(Z_9, Z_9) = 2$. Hence we get

$$trace_{g|S(TM)} h = \varepsilon_2 h(Z_2, Z_2) + \varepsilon_3 h(Z_3, Z_3) + \varepsilon_4 h(Z_4, Z_4) + \varepsilon_5 h(Z_5, Z_5) + \varepsilon_6 h(Z_6, Z_6) +$$
$$\varepsilon_7 h(Z_7, Z_7) + \varepsilon_8 h(Z_8, Z_8) + \varepsilon_9 h(Z_9, Z_9) = 0.$$

Therefore, M is a minimal quasi-bi-slant lightlike submanifold of $\mathbb{R}^{14}_2$.

Now we prove two characterization results for minimal quasi-bi-slant lightlike sub manifolds.

LEMMA 2. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Now suppose D is any slant distribution of M such that $dim(D) = dim(S(TM^{\perp})$. If $\{e_1, ..., e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{csc\theta Fe_1, ..., csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$.

Proof. Since $\{e_1, ..., e_m\}$ is a local orthonormal basis of D, which is Riemannian. So from Corollory 1, we get

$$\bar{g}(csc\theta Fe_i, csc\theta Fe_j) = csc^2\theta \sin^2\theta g(e_i, e_j) = \delta_{ij},$$

where $i, j = 1, 2, ..., m$. This proves the assertion. □

THEOREM 7. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if

$$traceA^*_{\xi_j}|_{S(TM)} = 0, traceA_{W_j}|_{S(TM)} = 0,$$
$$\bar{g}(D^l(X,W),Y) = 0, \ \forall\, X, Y \in \Gamma(Rad(TM)).$$

where $\{\xi_j\}_{j=1}^r$ is a basis of $Rad(TM)$ and $\{W_\alpha\}_{\alpha=1}^m$ is a basis of $S(TM^{\perp})$.

Proof. Since for any $X \in \Gamma(TM)$, we have $\bar{\nabla}_X X = 0$, so we get $h^l(X,X) = h^s(X,X) = 0$. Now take an orthonormal basis $\{e_1, ..., e_m\}$ of any slant distribution D. Now we know that $h^l = 0$ on $Rad(TM)$. Thus M is minimal if and only if $\sum_{i=1}^m h(e_i, e_i) = 0$ and $h^s = 0$ on $Rad(TM)$. Now using (2.10) and (2.14) we obtain

$$\textstyle\sum_{i=1}^m h(e_i, e_i) = \sum_{i=1}^m \frac{1}{r} \sum_{j=1}^r g(A^*_{\xi_j} e_i, e_i) N_j + \frac{1}{m} \sum_{\alpha=1}^m g(A^*_{W_\alpha} e_i, e_j) W_\alpha,$$

On the other hand, from(2.10), we get $h^s = 0$ on $Rad(TM)$ if

$$\bar{g}(D^l(X,W),Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. □

THEOREM 8. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Now suppose D is any slant distribution of M such that $dim(D) = dim(S(TM^{\perp})$. Then M is minimal if and only if

$$traceA^{*}_{\xi_k}|_{S(TM)} = 0, traceA_{Fe_j}|_{S(TM)} = 0,$$
$$\bar{g}(D^l(X, Fe_j), Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM))$, where $\{\xi_k\}_{k=1}^{r}$ is a basis of $\Gamma(Rad(TM))$ and $\{e_j\}_{j=1}^{m}$ is a basis of D.

Proof. From Lemma 2, $\{csc\theta Fe_1, ..., csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$. Thus we can write

$$h^s(X,X) = \sum_{i=1}^{m} A_i csc\theta Fe_i, \; \forall \, X \in \Gamma(TM)$$

for some functions $A_i, i \in \{1, ..., m\}$. Hence we obtain

$$h^s(X,X) = \sum_{i=1}^{m} csc\theta g(A_{Fe_i}X, X)Fe_i, \; \forall X \in \Gamma(\bar{J}Rad(TM) \oplus \bar{J}ltr(TM) \perp D).$$

Thus the assertion of theorem follows from Theorem 7. □

6. Totally Umbilical Quasi-Bi-Slant Lightlike Submanifolds

In this section, we study totally umbilical quasi-bi-slant lightlike submanifolds of indefinite Kaehler manifolds. A general notion of a totally umbilical lightlike submanfold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal in [1] is as follows:

DEFINITION 4.[8] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $\mathscr{H} \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,

$$h(X,Y) = \mathscr{H}\bar{g}(X,Y) \tag{6.1}$$

It is easy to see that M is totally umbilical if and only if on each coordinate neighborhood U, there exist smooth vector fields $\mathscr{H}^l \in \Gamma(ltr(TM))$ and $\mathscr{H}^s \in \Gamma(S(TM^{\perp})$, and smooth functions $\mathscr{H}^l_i \in F(ltr(TM))$ and $\mathscr{H}^s_i \in F(S(TM^{\perp}))$ such that

$$h^l(X,Y) = \mathscr{H}^l\bar{g}(X,Y), \; h^s(X,Y) = \mathscr{H}^s\bar{g}(X,Y). \tag{6.2}$$

$$h^l_i(X,Y) = \mathscr{H}^l_i\bar{g}(X,Y), \; h^s_i(X,Y) = \mathscr{H}^s_i\bar{g}(X,Y). \tag{6.3}$$

for any $X, Y \in \Gamma(TM)$.

EXAMPLE 4. Let $(\mathbb{R}_2^{14}, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,+,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7\}$.

Suppose M is a submanifold of $\mathbb{R}_2^{14}$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4$, $y^3 = u_5$, $x^4 = u_5$, $y^4 = -u_4$, $x^5 = u_6$, $y^5 = u_7$, $x^6 = \cos u_7$, $y^6 = \sin u_7$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = \sin u_9$, $y^8 = \cos u_9$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2),$$

$$Z_2 = 2(\partial x_2),\ Z_3 = 2(\partial y_1),$$

$$Z_4 = 2(\partial x_3 - \partial y_4),\ Z_5 = 2(\partial x_4 + \partial y_3),$$

$$Z_6 = 2(\partial x_5),\ Z_7 = 2(\partial y_5 - \sin u_7 \partial x_6 + \cos u_7 \partial y_6),$$

$$Z_8 = 2(\partial x_7),\ Z_9 = 2(\partial y_7 + \cos u_9 \partial x_8 - \sin u_9 \partial y_8).$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$. Now $ltr(TM)$ is spanned by $N = -\partial x_1 + \partial y_2$ and $S(TM^{\perp})$ is spanned by

$$W_1 = 2(\cos u_7 \partial x_6 + \sin u_7 \partial y_6),$$

$$W_2 = 2(\partial y_5 - \cos u_7 \partial x_6 + \sin u_7 \partial y_6),$$

$$W_3 = 2(\sin u_9 \partial x_8 + \cos u_9 \partial y_8),$$

$$W_4 = 2(\partial y_7 - \cos u_9 \partial x_8 + \sin u_9 \partial y_8).$$

It follows that $\bar{J}Z_1 = Z_2 - Z_3$ and $\bar{J}N = \frac{1}{2}(Z_2 + Z_3)$, which implies that $\bar{J}Rad(TM)$ and $\bar{J}ltr(TM)$ are distributions on M. On the other hand, we can see that $D = span\{Z_4, Z_5\}$ such that $\bar{J}Z_4 = Z_5$, $\bar{J}Z_5 = -Z_4$, which implies that D is invariant with respect to $\bar{J}$. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \pi/4$ and $\theta_2 = \pi/4$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}_2^{14}$. Now by direct computation and using Gauss formula, we get for every $X \in \Gamma(TM)$ we have

$$\bar{\nabla}_X Z_1 = \bar{\nabla}_X Z_2 = \bar{\nabla}_X Z_3 = \bar{\nabla}_X Z_4 = \bar{\nabla}_X Z_5 = \bar{\nabla}_X Z_6 = \bar{\nabla}_X Z_8 = 0.$$

Also we can see that $\bar{\nabla}_X Z_7 = 0$, for any $X \in \Gamma(TM)$ except $X = Z_7$ as

$$\bar{\nabla}_{Z_7} Z_7 = -8(\cos u_7 \partial x_6 + \sin u_7 \partial y_6) = -4W_1.$$

In the similar way we get that $\bar{\nabla}_X Z_9 = 0$, for any $X \in \Gamma(TM)$ except $X = Z_9$ as

$$\bar{\nabla}_{Z_9} Z_9 = -8(\sin u_9 \partial x_8 + \cos u_9 \partial y_8) = -4W_3.$$

Thus by (2.7) we have $h^l(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$. Also $h^s(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$ except

$$h^s(Z_7, Z_7) = -4W_1 = -2\bar{g}(Z_7, Z_7)W_1,$$
$$h^s(Z_9, Z_9) = -4W_3 = -2\bar{g}(Z_9, Z_9)W_3.$$

Therefore M is a totally umbilical quasi-bi-slant lightlike submanifold of $\mathbb{R}^{14}_2$.

The following results are important for our subsequent use.

PROPOSITION 1.[7] Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then, $h^l = 0$ on $\Gamma(Rad(TM))$.

THEOREM 9.[7] There are no minimal lightlike submanifold contained in a proper totally umbilical quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold.

Now, we prove a characterization result for totally umbilical quasi-bi-slant lightlike submanifolds of an indefinite Kaehler manifolds:

THEOREM 10. Let M be a totally umbilical quasi-bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if M is totally geodesic. **Proof:** Suppose M is a minimal submanifold of an indefinite Kaehler submanifold, then $h^s(X,Y) = 0$, for all $X,Y \in \Gamma(Rad(TM))$ and by the Proposition 6.1, we have $h^l = 0$ on $Rad(TM)$. Now we choose an orthonormal basis $\{e_1, e_2, ..., e_{m-r}\}$ of $\Gamma(S(TM))$. Then, from (6.3) we get

$$trace\ h(e_i, e_i) = \Sigma_{i=1}^{m-r} \varepsilon_i g(e_i, e_i)\mathscr{H}^l + \varepsilon_i g(e_i, e_i)\mathscr{H}^s.$$

Thus we have $trace\ h(e_i, e_i) = (m-r)\mathscr{H}^l + (m-r)\mathscr{H}^s$. Since M is minimal and $ltr(TM) \cap S(TM^{\perp}) = \{0\}$, so we get $\mathscr{H}^l = 0$ and $\mathscr{H}^s = 0$, which implies that M is totally geodesic.

Conversely, suppose that M is totally geodesic. Now, for any $x_0 \in M$ there exists $V_0 \in T_xM$ and the unique geodesic $\Gamma : x^{\alpha} = x^{\alpha}(t)$, $\alpha \in \{1, 2, ..., m\}$, such that $x^{\alpha}(0) = x_0$ and $\dfrac{dx^{\alpha}}{dt}(0) = V_0$. Since Γ is also a geodesic of $\bar{M}$, h^l and h^s vanish identically on M, which implies that $h^s = 0$

on $Rad(TM)$ and $trace\ h = 0$, where $trace$ is written with respect to g restricted to $S(TM)$. Therefore M is minimal.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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