Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 5514-5524
https://doi.org/10.28919/jmcs/6004
ISSN: 1927-5307

A STABILIZATION APPROACH FOR A NOVEL CHAOTIC FRACTIONAL-ORDER DISCRETE NEURAL NETWORK

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Abstract. This paper aims to make a contribution to the topic of the stabilization of chaotic dynamics in fractional-order discrete systems by presenting a novel three-dimensional fractional-order discrete neural network formulated by the h-fractional difference operator. A novel theorem is illustrated with the aim of stabilizing the chaotic trajectories of the three-dimensional fractional-order discrete neural network at zero through proposing a quite simple linear control laws. Finally, certain simulation results are carried out to highlight the effectiveness of the stabilization approach proposed herein.

Keywords: fractional discrete calculus; neural networks; stabilization; Lyapunov direct method.

2010 AMS Subject Classification: 92B20, 37B25.

1. Introduction

For a long period of time, the consideration and expansion of fractional calculus were only limited to its case formulated by continuous-time [1]. This formulation plays a major role in many sciences and fields. During the early modern era, many researchers give appropriate

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Received May 11, 2021

solutions for lots of applications in the field of neural network [2, 3, 4, 5], such as encryption and information security [6, 7, 8, 9]. However, this field has also been extended to involve the fractional-order derivatives in state of the integer-order ones, and consequently it was called the field of Fractional-order Neural Networks (FoNNs). Many researchers see that the use of the FoNN can decrease the mathematical cost [10, 11]. Moreover, this kind of networks has some important features, like its parallel computation and its notability of learning capacity, a great interest has been latterly devoted by many researchers to explain several major aspects like, e.g., control and stabilization [12, 13, 14, 15, 16].

Diaz and Osier are the first researchers introduced the discrete fractional calculus and that is in the year 1974. Also, they carried out many major research works concerning to this new field [2]. Lately, many researchers focus their studies on the FoNN in its discrete-time form not only on its continuous-time form [17, 18, 19]. Afterwards, many categories of applications related to the Fractional-order Discrete Neural Networks (FoDNNs), like bidirectional associative memory and others are studied in [20]. Besides, certain synchronization between specific types of FoDNNs with/without time delays is discussed in [13]. However, here is how this article is arranged. Some preliminaries and notations associated with the fractional h-difference operators are presented in the second section. In the third section, A the FoDNN model as well as its equivalent form are formulated in view of the fractional h-difference operator. Section four illustrates the proposed stabilization approach, while the fifth section contains the main conclusions of this study.

2. PRELIMINARIES

Due to the fact that the discrete fractional calculus is considered a relatively new topic that has not yet settled, some preliminaries and notations associated with this topic are presented here for completeness.

Definition 1. [21] *Let $X : (h\mathbb{N})_a \to \mathbb{R}$ 0n which a is a starting point. Then, the ν^{th}–order h–sum is given by:*

(1) $${}_h\Delta_a^{-\nu} X(t) = \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu} (t - \sigma(sh))^{(\nu-1)} x(sh), \quad \sigma(sh) = (s+1)h, \ \ a \in \mathbb{R}, \ \ t \in (h\mathbb{N})_{a+\nu h}$$

where $\nu > 0$ and the h–falling factorial function is defined as:

$$t_h^{(\nu)} = h^{\nu} \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\nu\right)}, \quad t,\nu \in \mathbb{R},$$

where $(h\mathbb{N})_{a+(1-\nu)h} = \{a+(1-\nu)h, a+(2-\nu)h, \cdots\}$.

Definition 2. [10] *For the function $x(t)$ defined on $(h\mathbb{N})_a$, the Caputo–like difference operator is defined by:*

$$ {}_h^C\Delta_a^{\nu} X(t) = \Delta_a^{-(n-\nu)} \Delta^n X(t), \; t \in (h\mathbb{N})_{a+(n-\nu)h}, \tag{2}$$

where $\Delta X(t) = \frac{X(t+h)-X(t)}{h}$ and $n = \lceil \nu \rceil + 1$ in which $\nu > 0$ and $\nu \notin \mathbb{N}$.

Next, we briefly illustrate a significant result reported in [22] with the aim of identifying some convenient stability conditions of the zero equilibrium point for the following nonlinear fractional-order difference system:

$$ {}_h^C\Delta_a^{\nu} X(t) = f(t+\nu h, X(t+\nu h)), \quad t\,(h\mathbb{N})_{a+(1-\nu)h}. \tag{3}$$

Theorem 1. *Let $x = 0$ be an equilibrium point of the nonlinear fractional-order discrete system (3). If there exists a positive definite and decrescent scalar function $V(t, X(t))$, such that ${}_h^C\Delta_a^{\nu} V(t, X(t)) \leq 0$, then the equilibrium point is asymptotically stable.*

In the following content, we state an extremely useful technical inequality associated with identifying a proper Lyapunov's function which is considered the heart of Lyapunov's direct method that will be used later on.

Lemma 1. [22] *For any discrete time $t \in (h\mathbb{N})_{a+(1-\nu)h}$, the following inequality holds:*

$$ {}_h^C\Delta_a^{\nu} X^2(t) \leq 2X(t+\nu h)\, {}_h^C\Delta_a^{\nu} X(t), \quad 0 < \nu \leq 1. \tag{4}$$

3. THE CHAOTIC FRACTIONAL-ORDER DISCRETE NEURAL NETWORK

In this part, a novel version of the fractional-order discrete neural network, which has been recently proposed in [8], is established by using the fractional-order h–difference operator. In particular, the authors in [8] have operated the ν–Caputo operator to formulate a three-dimensional continuous fractional-order Hopfield neural network model which was formerly proposed by

Zhang et al. in [23]. Herein, in view of the Caputo h–difference operator, we can formulate the following three-dimensional fractional-order discrete neural network model:

(5)
$$\begin{cases} {}_{h}^{C}\Delta_a^{\nu} x_1(t) = -x_1(t+\nu h) + 2\tanh(x_1(t+\nu h)) + 1.2\tanh(x_2(t+\nu h)), \\ {}_{h}^{C}\Delta_a^{\nu} x_2(t) = -x_2(t+\nu h) + 2\tanh(x_1(t+\nu h)) + 1.71\tanh(x_2(t+\nu h)) + 1.15\tanh(x_3(t+\nu h)), \\ {}_{h}^{C}\Delta_a^{\nu} x_3(t) = -x_3(t+\nu h) - 4.75\tanh(x_1(t+\nu h)) + 1.1\tanh(x_3(t+\nu h)), \end{cases}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$ and

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 1.2 & 0 \\ 2 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}.$$

Based on a key result reported in [24], system (5) can be converted into an equivalent form consisting of the following three discrete integral equation:

(6)
$$\begin{cases} x_1(n) = x_1(0) + \frac{h^{\nu}}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-x_1(j) + 2\tanh(x_1(j)) + 1.2\tanh(x_2(j))\right], \\ x_2(n) = x_2(0) + \frac{h^{\nu}}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-x_2(j) + 2\tanh(x_1(j)) + 1.71\tanh(x_2(j)) + 1.15\tanh(x_3(j))\right], \\ x_3(n) = x_3(0) + \frac{h^{\nu}}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-x_3(j) - 4.75\tanh(x_1(j)) + 1.1\tanh(x_3(j))\right]. \end{cases}$$

The Predictor-corrector method is adopted here to transfer the above three formulas given in system (6) into other explicit forms to be used in presenting the dynamic of the proposed neural network model by performing some numerical simulations. In other words, by taking the initial state values $(x_1(0), x_2(0), x_3(0) = 0.8) = (0.02, -0.5, 0.8)$, the fractional-order value $\nu = 0.9$ and the system's parameters $(a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{31}, a_{33}) = (2, -1.2, 2, 1.71, 1.15, -4.75, 1.1)$, we can notice that system (5) displaying a clear attractor exhibited in Figure 1. On the other hand, the bifurcation diagram and the Lyapunov exponents are both shown in Figure 2.

4. Stabilization Approach

In this section, a novel stability result through using the Lyapunov direct method is stated and derived with the aim of achieving a proper control for the chaos which occurred to the

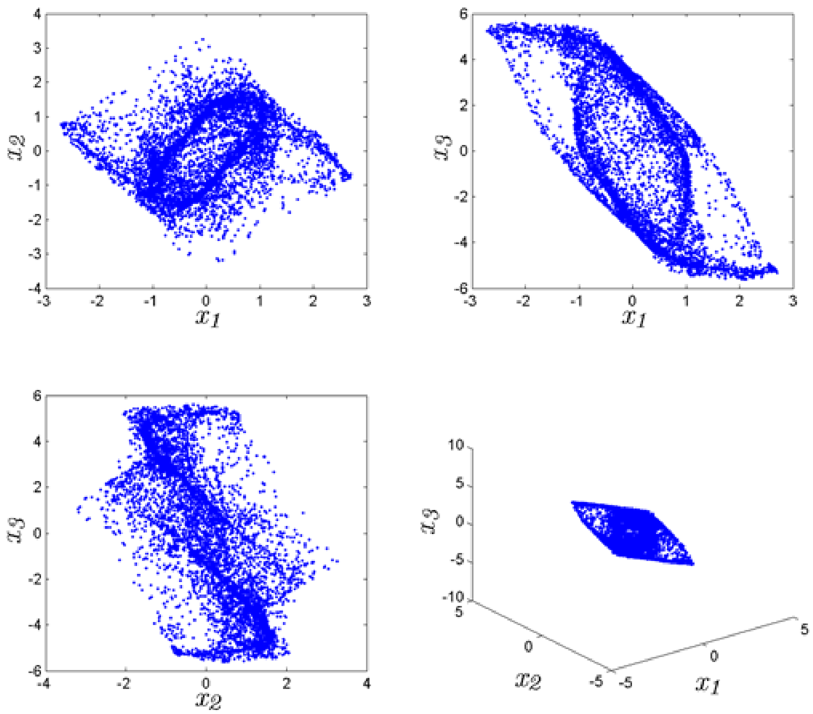

FIGURE 1. Chaotic attractor of fractional map (5) with fractional order value $\nu = 0.9$ (without control).

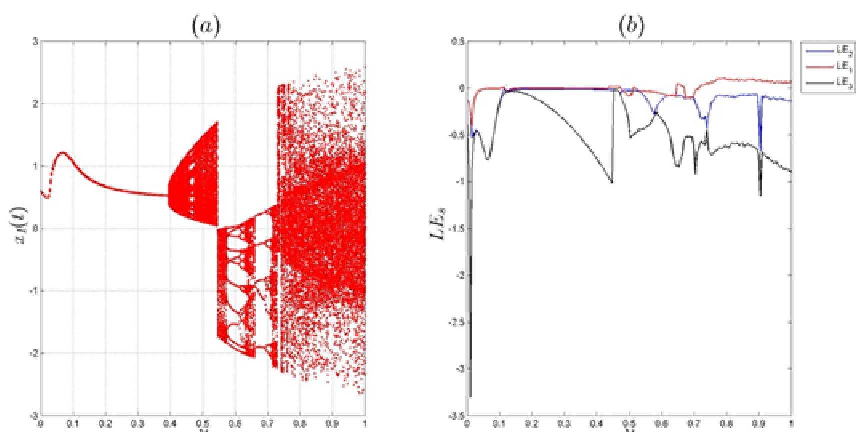

FIGURE 2. (a) Bifurcation diagram (b) Lyapunov exponents as a function of the fractional-order ν.

fractional-order discrete neural network (5). In other words, we will endeavor to propose some proper linear control laws that would stabilize all the three states of system (5) at the origin. Afterwards, we will verify whether this task is successfully carried out by performing several numerical simulations. For this purpose, let us begins with the following theorem that represents the main result of our work.

Theorem 2. *The fractional-order discrete neural network (5) can be stabilized under the following linear control law:*

(7)
$$\begin{cases} \mathbf{C}_1 = -sgn\left(x_1(t)\right) \times \left(|a_{11}| + |a_{12}|\right), \\ \mathbf{C}_2 = -sgn\left(x_2(t)\right) \times \left(|a_{21}| + |a_{22}| + |a_{23}|\right), \\ \mathbf{C}_2 = -sgn\left(x_3(t)\right) \times \left(|a_{31}| + |a_{33}|\right), \end{cases}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$ and whereas $sgn(\cdot)$ represents the signum function that can be outlined as:

(8)
$$sgn\left(x\right) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Proof. The controlled system, which can be inferred through including the three controllers given in (5) into system (5), can be described as:

(9)
$$\begin{cases} {}^C_h\Delta^\nu_a x_1(t) = -x_1(t+\nu h) + a_{11}\tanh(x_1(t+\nu h)) + a_{12}\tanh(x_2(t+\nu h)) + \mathbf{C}_1, \\ {}^C_h\Delta^\nu_a x_2(t) = -x_2(t+\nu h) + a_{21}\tanh(x_1(t+\nu h)) + a_{22}\tanh(x_2(t+\nu h)) + a_{23}\tanh(x_3(t+\nu h)) + \mathbf{C}_2, \\ {}^C_h\Delta^\nu_a x_3(t) = -x_3(t+\nu h) + a_{31}\tanh(x_1(t+\nu h)) + a_{33}\tanh(x_3(t+\nu h)) + \mathbf{C}_3. \end{cases}$$

The controlled system (9) can be consequently re-expressed by the following form:

(10)
$$\begin{cases} {}^C_h\Delta^\nu_a x_1(t) = -x_1(t+\nu h) + 2\tanh(x_1(t+\nu h)) + 1.2\tanh(x_2(t+\nu h)) \\ \qquad\qquad - sgn\left(x_1(t+\nu h)\right) \times \left(|a_{11}| + |a_{12}|\right), \\ {}^C_h\Delta^\nu_a x_2(t) = -x_2(t+\nu h) + 2\tanh(x_1(t+\nu h)) + 1.71\tanh(x_2(t+\nu h)) \\ \qquad\qquad + 1.15\tanh(x_3(t+\nu h)) - sgn\left(x_2(t+\nu h)\right) \times \left(|a_{21}| + |a_{22}| + |a_{23}|\right), \\ {}^C_h\Delta^\nu_a x_3(t) = -x_3(t+\nu h) - 4.75\tanh(x_1(t+\nu h)) + 1.1\tanh(x_3(t+\nu h)) \\ \qquad\qquad - sgn\left(x_3(t+\nu h)\right) \times \left(|a_{31}| + |a_{33}|\right). \end{cases}$$

One can utilize the Lyapunov's direct method by first letting the Lyapunov function, $V(t)$, in the form:

$$V = \frac{1}{2}\left(x_1^2(t) + x_2^2(t) + x_3^2(t)\right). \tag{11}$$

Actually, the adoption of the Caputo h-difference operator implies the following assertion:

$${}_h^C\Delta_a^\nu V(t) = \frac{1}{2}{}_h^C\Delta_a^\nu x_1^2(t) + \frac{1}{2}{}_h^C\Delta_a^\nu x_2^2(t) + \frac{1}{2}x_3^2(t). \tag{12}$$

By using Lemma 1, it follows that:

$$\begin{aligned}
{}_h^C\Delta_a^\nu V \leq\ & x_1(t+\nu h){}_h^C\Delta_a^\nu x_1(t) + x_2(t+\nu h){}_h^C\Delta_a^\nu x_2(t) + x_3(t+\nu h){}_h^C\Delta_a^\nu x_3(t) \\
=\ & -x_1^2(t+\nu h) + x_1(t+\nu h)\left[a_{11}\tanh(x_1(t+\nu h)) + a_{12}\tanh(x_2(t+\nu h))\right] \\
& -sgn\left(x_1(t+\nu h)\right) \times (|a_{11}| + |a_{12}|) \times x_1(t+\nu h) - x_2^2(t+\nu h) \\
& +x_2(t+\nu h)\left[a_{21}\tanh(x_1(t+\nu h)) + a_{22}\tanh(x_2(t+\nu h)) + a_{23}\tanh(x_3(t+\nu h))\right] \\
& -sgn\left(x_2(t+\nu h)\right) \times (|a_{21}| + |a_{22}| + |a_{23}|) \times x_2(t+\nu h) \\
& -x_3^2(t+\nu h) + x_3(t+\nu h)\left[a_{31}\tanh(x_1(t+\nu h)) + a_{33}\tanh(x_3(t+\nu h))\right] \\
& -sgn\left(x_3(t+\nu h)\right) \times (|a_{31}| + |a_{33}|) \times x_3(t+\nu h) \\
\leq\ & -x_1^2(t+\nu h) + |x_1(t+\nu h)|\left(|a_{11}|\,|\tanh(x_1(t+\nu h))| + |a_{12}|\,|\tanh(x_2(t+\nu h))|\right) \\
& -sgn\left(x_1(t+\nu h)\right) \times (|a_{11}| + |a_{12}|) \times x_1(t+\nu h) - x_2^2(t+\nu h) \\
& +|x_2(t+\nu h)|\left(|a_{21}|\,|\tanh(x_1(t+\nu h))| + |a_{22}|\,|\tanh(x_2(t+\nu h))| + |a_{23}|\,|\tanh(x_3(t+\nu h))|\right) \\
& -sgn\left(x_2(t+\nu h)\right) \times (|a_{21}| + |a_{22}| + |a_{23}|) \times x_2(t+\nu h) - x_3^2(t+\nu h) \\
& +|x_3(t+\nu h)|\left(|a_{31}|\,|\tanh(x_1(t+\nu h))| + |a_{33}|\,|\tanh(x_3(t+\nu h))|\right) \\
& -sgn\left(x_3(t+\nu h)\right) \times (|a_{31}| + |a_{33}|) \times x_3(t+\nu h).
\end{aligned}$$

That is;

$$\begin{aligned}
{}_h^C\Delta_a^\nu V \leq\ & -x_1^2(t+\nu h) - x_2^2(t+\nu h) - x_3^2(t+\nu h) \\
& +(|a_{11}| + |a_{12}|) \times |x_1(t+\nu h)| - sgn\left(x_1(t+\nu h)\right) \times (|a_{11}| + |a_{12}|) \times x_1(t+\nu h) \\
& +(|a_{21}| + |a_{22}| + |a_{23}|) \times |x_2(t+\nu h)| - sgn\left(x_2(t+\nu h)\right) \times (|a_{21}| + |a_{22}| + |a_{23}|) \times x_2(t+\nu h) \\
& +(|a_{31}| + |a_{33}|) \times |x_3(t+\nu h)| \times -sgn\left(x_3(t+\nu h)\right) \times (|a_{31}| + |a_{33}|) \times x_3(t+\nu h) \\
=\ & -x_1^2(t+\nu h) - x_2^2(t+\nu h) - x_3^2(t+\nu h) < 0
\end{aligned}$$

From Theorem 1, it can be concluded that the zero equilibrium of (10) is asymptotically stable, which proves that the dynamics of the proposed three-dimensional fractional-order discrete neural network (5) are stabilized by the linear control law (7). □

For performing specific numerical simulations that confirm the concluded result reported in the previous theorem, the parameters' values of the neural network model are taken to be as $(a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{31}, a_{33}) = (2, -1.2, 2, 1.71, 1.15, -4.75, 1.1)$, while the value of the fractional-order derivative is taken to be as $\nu = 0.9$. However, Figure 3 and Figure 4 depict, respectively, the evolution of the states and the phase space for the controlled system (9). Actually, these plots clearly show that the chaotic dynamics are effectively converged to the origin as $t \to 20$, which confirm the effectiveness of the designed control law (7).

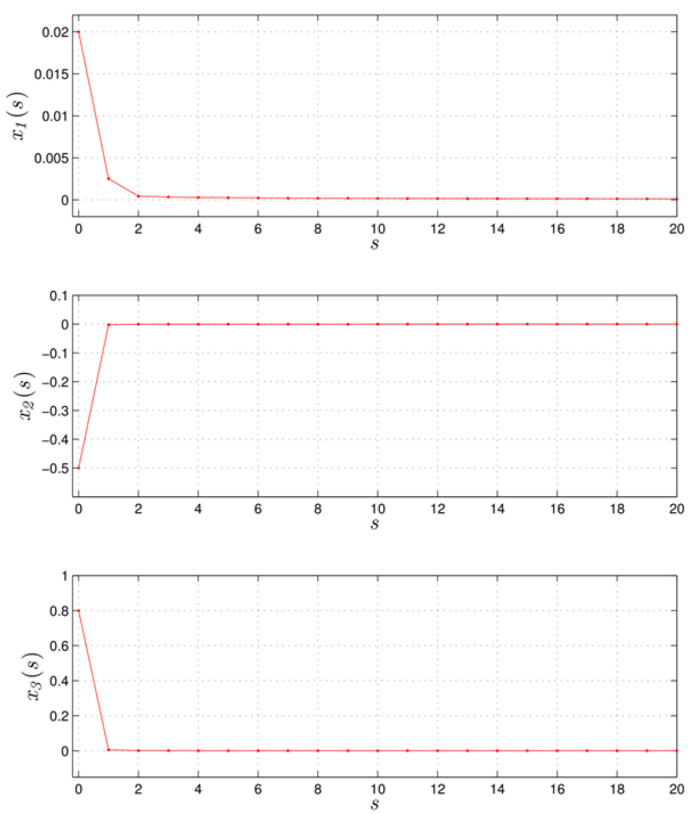

FIGURE 3. The evolution of the states of the controlled system (9).

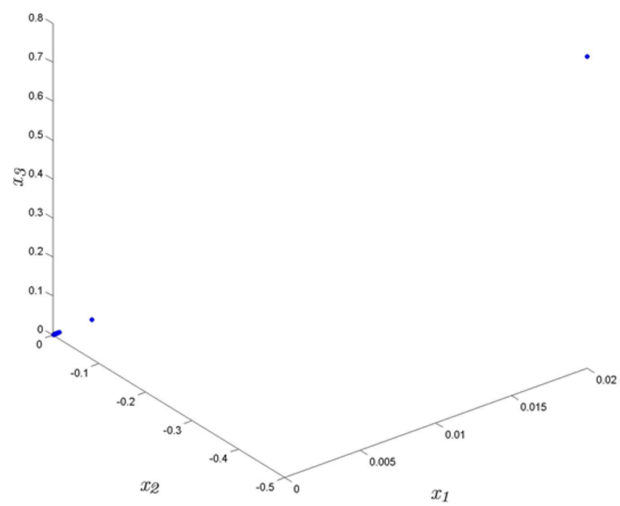

FIGURE 4. The Phase space of the controlled system (9).

5. CONCLUSION

In this paper, a novel three-dimensional fractional-order discrete neural network has been formulated using the h-fractional difference operator. For the purpose of stabilizing the chaotic trajectories of such proposed model at zero, a novel theorem, which proposes certain simple linear control laws, has been stated and derived. Several numerical simulations have been carried out to verify the theoretical analysis.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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