Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 6557-6567
https://doi.org/10.28919/jmcs/6030
ISSN: 1927-5307

LOCATION OF ZEROS OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

C. GANGADHAR[1], P. RAMULU[2,*], G.L. REDDY[3]

[1]Department of Mathematics, National Institute of Technology, Andhra Pradesh, India

[2]Department of Mathematics, M.V.S. Govt. Arts and Science College(Autonomous), Mahabubnagar, Telangana, India 509001

[3]Department of Mathematics and Statistics, University of Hyderabad, India 500046

Abstract. In this paper we extended Eneström-Kakeya theorem (Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_n$ then all the zeros of P(z) lie in $|z| \leq 1$) and our results [1] by relaxing the hypothesis in different ways by considering complex coefficients we get various other results which in term generalizes.

Keywords: zeros of polynomial; Eneström-Kakeya theorem.

2010 AMS Subject Classification: 30C10, 30C15.

1. Introduction

Location of zeros of a polynomial is a long standing classical problem [1,3-5,8,10-11]. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in literature. Some results on the location of zeros of polynomial propced by taking real coefficients. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the

*Corresponding author
E-mail address: ramulu.purra@gmail.com
Received May 14, 2021

coefficient,there is always need for refinement of results in this subject. The well known result in theory of the distribution of zeros of polynomials is the following.

2. PRELIMINARIES

Theorem 2.1. [2, 7] : *Let $P(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree n such that* $0<a_0\leq a_1\leq \ldots\leq a_{n-1}\leq a_n$. *Then all the zeros of P(z) lie in* $|z|\leq 1$.

A.Joyal, G.Labelle and Q. I. Rahman [6] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem 2.2. *Let $P(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree n such that $a_0\leq a_1\leq \ldots\leq a_{n-1}\leq a_n$. Then all the zeros of P(z) lie in $|z|\leq \frac{1}{|a_n|}\{a_n-a_0+|a_0|\}$.*

Theorem 2.3. *Let $P(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree n with real coefficients such that*

$$a_0\leq a_1\leq \ldots\leq a_{m-1}\leq a_m\geq a_{m+1}\geq \ldots\geq a_{n-1}\geq a_n.$$

Then all the zeros of P(z) lie in

$$|z|\leq \frac{1}{|a_n|}\left[2a_m+|a_0|-(a_0+|a_n|)\right].$$

Theorem 2.4. [9] : *Let $P(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree n with real coefficients such that*

$$a_0\geq a_1\geq \ldots\geq a_{m-1}\geq a_m\leq a_{m+1}\leq \ldots\leq a_{n-1}\leq a_n.$$

Then all the zeros of P(z) lie in

$$|z|\leq \frac{1}{|a_n|}\left[|a_0|+a_0+a_n-2a_m\right].$$

In this paper We want to prove the following results.

3. MAIN RESULTS

Theorem 3.1. *Let $P(z)=\sum_{i=0}^{n}\alpha_iz^i$ be a polynomial of degree n with $Re(\alpha_i)=a_i$ and $Im(\alpha_i)=b_i$ for $i=0,1,2,\ldots,n$ such that for some $k\geq 1,\ l\geq 1,\ 0<r\leq 1,\ 0<s\leq 1,\ \delta\geq 0,\ \eta\geq 0,\ a_m\neq 0,\ b_m\neq 0$,*

$$a_0-\delta\leq a_1\leq \ldots\leq a_{m-1}\leq ka_m\geq a_{m+1}\geq \ldots\geq a_{n-1}\geq ra_n \quad and$$

$$b_0-\eta \le b_1 \le \dots \le b_{m-1} \le lb_m \ge b_{m+1} \ge \dots \ge b_{n-1} \ge sb_n.$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|\alpha_n|}\Big[(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m| \\ +2(l-1)|b_m|-\big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Big].$$

Corollary 3.1.1. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,\dots,n$ *such that for some* $k\ge 1,\ 0<r\le 1,\ \delta\ge 0,\ a_m\ne 0,\ b_m\ne 0,$

$$a_0-\delta \le a_1 \le \dots \le a_{m-1} \le ka_m \ge a_{m+1} \ge \dots \ge a_{n-1} \ge ra_n \quad \textit{and}$$

$$b_0-\delta \le b_1 \le \dots \le b_{m-1} \le kb_m \ge b_{m+1} \ge \dots \ge b_{n-1} \ge rb_n.$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|\alpha_n|}\Big[(|a_n|+|b_n|)+k(a_m+b_m+|a_m|+|b_m|)+|a_0|+|b_0| \\ +2(k-1)(|a_m|+|b_m|)-\big[a_0+b_0+r(a_n+b_n+|a_n|+|b_n|)\big]+4\delta\Big].$$

Corollary 3.1.2. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,\dots,n$ *such that*

$$a_0 \le a_1 \le \dots \le a_{m-1} \le a_m \ge a_{m+1} \ge \dots \ge a_{n-1} \ge a_n \quad \textit{and}$$

$$b_0 \le b_1 \le \dots \le b_{m-1} \le b_m \ge b_{m+1} \ge \dots \ge b_{n-1} \ge b_n.$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|\alpha_n|}\Big[|a_0|+|b_0|+a_m+b_m+|a_m|+|b_m|-[a_0+b_0+a_n+b_n]\Big].$$

Corollary 3.1.3. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i)=a_i>0$ *and* $Im(\alpha_i)=b_i>0$ *for* $i=0,1,2,\dots,n$ *such that for some* $k\ge 1,\ l\ge 1,\ 0<r\le 1,\ 0<s\le 1,\ \delta\ge 0,\ \eta\ge 0,$

$$a_0-\delta \le a_1 \le \dots \le a_{m-1} \le ka_m \ge a_{m+1} \ge \dots \ge a_{n-1} \ge ra_n \quad \textit{and}$$

$$b_0-\eta \le b_1 \le \dots \le b_{m-1} \le lb_m \ge b_{m+1} \ge \dots \ge b_{n-1} \ge sb_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[(2r+1)a_n+(2l+1)b_n+(4k-1)a_m+(4s-1)b_m+2\delta+2\eta\Bigg].$$

Corollary 3.1.4. *Let* $P(z) = \sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0,1,2,...,n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0< r \leq 1,\ 0< s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0-\delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_0-\eta \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq lb_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_n|+l(b_n+|b_n|)+k(a_m+|a_m|)+|a_0|+|b_0| \\ +2(k-1)|a_m|-\big[a_0+b_0+r(a_n+|a_n|)\big]+2\delta+2\eta\Bigg].$$

Remark 3.1.1. *By taking* $k = 1,\ r = 1$ *and* $\delta = 0$ *and* $b_i = 0$ *in Theorem 3.1, it reduces to Theorem 2.3.*

Remark 3.1.2. *By taking* $l = k,\ s = r$ *and* $\delta = \eta$ *in Theorem 3.1, it reduces to Corollary 3.1.1.*

Remark 3.1.3. *By taking* $l = k = 1,\ s = r = 1$ *and* $\delta = \eta = 0$*in Theorem 3.1, it reduces to Corollary 3.1.2.*

Remark 3.1.4. *By taking* $a_i > 0$ *and* $b_i > 0$ *in Theorem 3.1, it reduces to Corollary 3.1.3.*

Remark 3.1.5. *By taking* $l = 1,\ s = 1$ *and* $\eta = 0$ *in Theorem 3.1, it reduces to Corollary 3.1.4.*

Theorem 3.2. *Let* $P(z) = \sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0,1,2,...,n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0< r \leq 1,\ 0< s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0+\delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0+\eta \geq b_1 \geq ... \geq b_{m-1} \geq sb_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq lb_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+a_0+b_0+2(1-r)|a_m|$$
$$+2(1-s)|b_m|-\big[(|a_n|+|b_n|)+2ra_m+2sb_m\big]+2\delta+2\eta\Big].$$

Corollary 3.2.1. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1,\ 0< r\leq 1,\ \delta\geq 0,\ \ a_m\neq 0,\ b_m\neq 0,$

$$a_0+\delta\geq a_1\geq ...\geq a_{m-1}\geq ra_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \textit{and}$$

$$b_0+\delta\geq b_1\geq ...\geq b_{m-1}\geq rb_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq kb_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|+a_0+b_0$$
$$+2(1-r)[|a_m||b_m|]-\big[(|a_n|+|b_n|)+2r(a_m+b_m)\big]+4\delta\Big].$$

Corollary 3.2.2. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that*

$$a_0\geq a_1\geq ...\geq a_{m-1}\geq a_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq a_n \quad \textit{and}$$

$$b_0\geq b_1\geq ...\geq b_{m-1}\geq b_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq b_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[(a_n+b_n)-2[a_m+b_m]+|a_0|+|b_0|+a_0+b_0\Big].$$

Corollary 3.2.3. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i>0$, *and* $Im(\alpha_i)=b_i>0$, *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1,\ l\geq 1,\ 0< r\leq 1,\ 0< s\leq 1,\ \delta\geq 0,\ \eta\geq 0,\ a_m\neq 0,\ b_m\neq 0,$

$$a_0+\delta\geq a_1\geq ...\geq a_{m-1}\geq ra_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \textit{and}$$

$$b_0+\eta\geq b_1\geq ...\geq b_{m-1}\geq sb_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq lb_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\left[2[ka_n + lb_n a_0 + b_0] + 2(1-2r)a_m + 2(1-2s)b_m - (a_n + b_n) + 2\delta + 2\eta\right].$$

Corollary 3.2.4. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0,1,2,...,n$ *such that for some* $k \geq 1,\ 0< r \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,$

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad \text{and}$$

$$b_0 \geq b_1 \geq ... \geq b_{m-1} \geq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\left[k(a_n + |a_n|) + b_n + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m| - \left[|a_n| + 2ra_m + 2b_m\right] + 2\delta\right].$$

Remark 3.2.1. *By taking* $k = 1,\ r = 1$ *and* $\delta = 0$ *and* $b_i = 0$ *in Theorem 3.2, it reduces to Theorem 2.4.*

Remark 3.2.2. *By taking* $l = k,\ s = r$ *and* $\delta = \eta$ *in Theorem 3.2, it reduces to Corollary 3.2.1.*

Remark 3.2.3. *By taking* $l = k = 1,\ s = r = 1$ *and* $\delta = \eta = 0$ *in Theorem 3.2, it reduces to Corollary 3.2.2.*

Remark 3.2.4. *By taking* $a_i > 0$ *and* $b_i > 0$ *in Theorem 3.2, it reduces to Corollary 3.2.3.*

Remark 3.2.5. *By taking* $l = 1,\ s = 1$ *and* $\eta = 0$ *in Theorem 3.2, it reduces to Corollary 3.2.4.*

By rearranig coefficients in above Thorems 3.1 and Thorems 3.2 we get the follwoing Thorem 3.3 and Thorem 3.4.

Theorem 3.3. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0,1,2,...,n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0< r \leq 1,\ 0< s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad \text{and}$$

$$b_n \leq b_{n-1} \leq ... \leq b_1 \leq b_0.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_0|+|b_0|+|a_n|+k(a_m+|a_m|)+2(k-1)|a_m|-\big[a_0+r(a_n+|a_n|)+b_n\big]+2\delta\Bigg].$$

Theorem 3.4. *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i)=a_i$ and $Im(\alpha_i)=b_i$ for $i=0,1,2,...,n$ such that for some $k\geq 1,\ 0< r\leq 1,\ \delta\geq 0,\ \eta\geq 0,\ a_m\neq 0,$*

$$a_0+\delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0 \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[k(a_n+|a_n|)+b_n+|a_0|+|b_0|+a_0+2(1-r)|a_m|-[|a_n|+2ra_m+b_0]+2\delta\Bigg].$$

4. PROOF OF THE THEOREMS

Proof of Theorem 3.1.

Let $P(z)=\alpha_n z^n+\alpha_{n-1}z^{n-1}+...+\alpha_{m-1}z^{m-1}+\alpha_m z^m+\alpha_{m+1}z^{m+1}+...+\alpha_1 z+\alpha_0$ be a polynomial of degree n. Then consider the polynomial

$$\begin{aligned}
Q(z) =&(1-z)P(z)\\
=&-\alpha_n z^{n+1}+(\alpha_n-\alpha_{n-1})z^n+...+(\alpha_{m+1}-\alpha_m)z^{m+1}+(\alpha_m-\alpha_{m-1})z^m+...+(\alpha_1-\alpha_0)z+\alpha_0.\\
=&-\alpha_n z^{n+1}+(a_n-a_{n-1})z^n+...+(a_{m+1}-a_m)z^{m+1}+(a_m-a_{m-1})z^m+...+(a_1-a_0)z+a_0+\\
&+i\big\{(b_n-b_{n-1})z^n+...+(b_{m+1}-b_m)z^{m+1}+(b_m-b_{m-1})z^m+...+(b_1-b_0)z+b_0\big\}.
\end{aligned}$$

$$Also\ if\ |z|>1\ then\ \frac{1}{|z|^{n-i}}<1\ for\ i=0,1,2,...,n-1.\ Now$$

$$\begin{aligned}
|Q(z)| \geq&|\alpha_n||z|^{n+1}-\Bigg\{\big(|a_n-a_{n-1}||z|^n+...+|a_{m+1}-a_m||z|^{m+1}+|a_m-a_{m-1}||z|^m+...+|a_1-a_0||z|+a_0\big)\\
&+\big(|b_n-b_{n-1}||z|^n+...+|b_{m+1}-b_m||z|^{m+1}+|b_m-b_{m-1}||z|^m+...+|b_1-b_0||z|+b_0\big)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{&\Big(|a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+...+\frac{|a_{m+1}-a_m|}{|z|^{n-m-1}}\\
&+\frac{|a_m-a_{m-1}|}{|z|^{n-m}}+...+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\Big)\\
&+\Big(|b_n-b_{n-1}|+\frac{|b_{n-1}-b_{n-2}|}{|z|}+...+\frac{|b_{m+1}-b_m|}{|z|^{n-m-1}}\\
&+\frac{|b_m-b_{m-1}|}{|z|^{n-m}}+...+\frac{|b_1-b_0|}{|z|^{n-1}}+\frac{|b_0|}{|z|^n}\Big)\Bigg\}\Bigg]
\end{aligned}$$

$$\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Bigg\{\big(|ra_n-a_{n-1}-ra_n+a_n|+|a_{n-1}-a_{n-2}|+...+|a_{m+1}-ka_m+ka_m-a_m|$$
$$+|a_m-ka_m+ka_m-a_{m-1}|+...+|a_1+\delta-a_0-\delta|+|a_0|\big)+\big(|sb_n-b_{n-1}-sb_n+b_n|$$
$$+|b_{n-1}-b_{n-2}|+...+|b_{m+1}-lb_m+lb_m-b_m|+|b_m-lb_m+lb_m-b_{m-1}|$$
$$+...+|b_1+\eta-b_0-\eta|+|b_0|\big)\Bigg\}\Bigg]$$
$$\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{\big[(a_{n-1}-ra_n)+(1-r)|a_n|+(a_{n-2}-a_{n-1})+...+(ka_m-a_{m+1})$$
$$+(k-1)|a_m|+(ka_m-a_{m-1})+(k-1)|a_m|+..+(a_1+\delta-a_0)+\delta+|a_0|\big]$$
$$+\big[(b_{n-1}-sb_n)+(1-s)|b_n|+(b_{n-2}-b_{n-1})+...+(lb_m-b_{m+1})+(l-1)|b_m|$$
$$+(lb_m-b_{m-1})+(l-1)|b_m|+..+(b_1+\eta-b_0)+\eta+|b_0|\big]\Bigg\}\Bigg]$$
$$=|a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m|$$
$$+2(l-1)|b_m|-\big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Bigg\}\Bigg]>0$$
$$if\ |z|>\frac{1}{|\alpha_n|}\Bigg[(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m|$$
$$+2(l-1)|b_m|-\big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Bigg].$$

This shows that if $|z|>1,\ Q(z)>0$

$$provided\ |z|>\frac{1}{|\alpha_n|}\Bigg[(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m|$$
$$+2(l-1)|b_m|-\big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Bigg].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z|\leq\frac{1}{|\alpha_n|}\Bigg[(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m|$$
$$+2(l-1)|b_m|-\big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Bigg].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 3.1.

Proof of Theorem 3.2.

Let $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + ... + \alpha_{m-1}z^{m-1} + \alpha_m z^m + \alpha_{m+1}z^{m+1} + ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n. Then consider the polynomial

$$\begin{aligned}
Q(z) =&(1-z)P(z)\\
=&-\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{m+1} - \alpha_m)z^{m+1} + (\alpha_m - \alpha_{m-1})z^m + ... + (\alpha_1 - \alpha_0)z + \alpha_0.\\
=&-\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... + (a_1 - a_0)z + a_0 +\\
&+ i\{(b_n - b_{n-1})z^n + ... + (b_{m+1} - b_m)z^{m+1} + (b_m - b_{m-1})z^m + ... + (b_1 - b_0)z + b_0\}.
\end{aligned}$$

$$Also\ if\ |z|>1\ then\ \frac{1}{|z|^{n-i}}<1\ for\ i = 0,1,2,...,n-1.\ Now$$

$$\begin{aligned}
|Q(z)| \geq&|\alpha_n||z|^{n+1} - \Big\{\big(|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m + ... + |a_1 - a_0||z| + a_0\big)\\
&+ \big(|b_n - b_{n-1}||z|^n + ... + |b_{m+1} - b_m||z|^{m+1} + |b_m - b_{m-1}||z|^m + ... + |b_1 - b_0||z| + b_0\big)\Big\}\\
\geq |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Big\{&\big(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}}\\
&+ \frac{|a_m - a_{m-1}|}{|z|^{n-m}} + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big)\\
&+ \big(|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + ... + \frac{|b_{m+1} - b_m|}{|z|^{n-m-1}}\\
&+ \frac{|b_m - b_{m-1}|}{|z|^{n-m}} + ... + \frac{|b_1 - b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\big)\Big\}\Bigg]\\
\geq |a_n||z|^n\Bigg[|z| - \frac{1}{|a_n|}\Big\{&\big(|ka_n - a_{n-1} - ka_n + a_n| + |a_{n-1} - a_{n-2}| + ... + |a_{m+1} - ra_m + ra_m - a_m|\\
&+ |a_m - ra_m + ra_m - a_{m-1}| + ... + |a_1 - \delta - a_0 + \delta| + |a_0|\big) + \big(|lb_n - b_{n-1} - lb_n + b_n|\\
&+ |b_{n-1} - b_{n-2}| + ... + |b_{m+1} - sb_m + sb_m - b_m| + |b_m - lb_m + sb_m - b_{m-1}|\\
&+ ... + |b_1 - \eta - b_0 + \eta| + |b_0|\big)\Big\}\Bigg]
\end{aligned}$$

$$\geq |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\big[(ka_n - a_{n-1}) + (k-1)|a_n| + (a_{n-1} - a_{n-2}) + \ldots + (a_{m+1} - ra_m)$$
$$+ (1-r)|a_m| + (a_{m-1} - ra_m) + (1-r)|a_m| + .. + (a_0 - a_1 + \delta) + \delta + |a_0|\big]$$
$$+ \big[(lb_n - b_{n-1}) + (l-1)|b_n| + (b_{n-1} - b_{n-2}) + \ldots + (b_{m+1} - sb_m) + (1-s)|b_m|$$
$$+ (b_{m-1} - sb_m) + (1-s)|b_m| + .. + (b_0 - b_1 + \eta) + \eta + |b_0|\big]\Bigg\}\Bigg]$$
$$= |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$
$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Bigg\}\Bigg] > 0$$
$$if\ |z| > \frac{1}{|\alpha_n|}\Bigg[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$
$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Bigg].$$

This shows that if $|z| > 1,\ Q(z) > 0$

$$provided\ |z| > \frac{1}{|\alpha_n|}\Bigg[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$
$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Bigg].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$
$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Bigg].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 3.2.

Proof of Theorem 3.3.

Proof of Theorem 3.3 is similar to the proof of Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.4.

Proof of Theorem 3.4 is similar to the proof of Theorem 3.1 and Theorem 3.2.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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