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NUMERICAL SOLUTIONS OF FRACTIONAL CONVECTION-DIFFUSION EQUATION USING FINITE-DIFFERENCE AND FINITE-VOLUME SCHEMES

RANIA SAADEH*

Department of Mathematics, Zarqa University, Zarqa 13110, Jordan

Abstract: Many natural phenomena in physics and engineering can be modeled by linear and nonlinear partial differential equations, which are constructed using derivatives of fractional order. The main purpose of this work is to facilitate the implementation of space finite-volume and finite-difference schemes to solve fractional convection-diffusion equation of order $\beta \in (0,1]$ without source term along with appropriate initial conditions. The fractional derivative is described in Riemann-Liouville sense. The highlight of the proposed methods is to introduce an alternative way to discretize the space-fractional derivative utilizing the fractional Grünwald formula. Numerical results are provided to examine the accuracy of the proposed scheme and to compare it in different conditions. The obtained results show that the proposed techniques are simple, accurate, and applicable to a wide range of space-fractional models that arise in the natural sciences.

Keywords: finite volume method; finite difference method; convection-diffusion equation; Riemann-Liouville fractional derivative; order of convergence.

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*Corresponding author
E-mail address: rsaadeh@zu.edu.jo
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1. INTRODUCTION

Fractional computations are a generalization of natural order calculations and have been a vital and major branch of mathematics in recent decades. Anyhow, fractional partial differential equations (FPDEs) are generalized classical differential equations that have many applications in electrical engineering, physics, biology, hydrology, viscoelasticity, financial mathematics, and so on, including the fractional diffusion, which represents the anomalous emission of a high-velocity particle [1-4]. Recently, many mathematicians and physicists have analyzed and applied partial differential equations of fractional order using classical calculus in various fields of science to find solutions to various types of real-world problems, including fluid mechanics, chemistry, medicine, and engineering. This is due to the tremendous ability of fractional differentiation to deal with the complexities found in natural phenomena involving genetic memory and the non-local characteristic [5-10].

Furthermore, several real-world observations in fluid flows, quantum mechanics, signal process, fractals, nonlinear fiber optics, neural networks, process identification, elastic materials, bifurcation, and polymers are well applied by the partial differential equation of fractional order [11-14]. In the literature, there are many local and non-local fractional differential operators, including the Riemann-Liouville, Atangana-Baleanu-Caputo, Riesz, Caputo, Katugampola, Grünwald, Caputo-Katugampola, and Caputo-Fabrizio [15-21]. Although non-local differential operators are more interesting because the long-term effect of physical applications depends on nonlocality and memory features, there is a shortage of mathematical tools like chain rule, quotient rule, Leibniz rule, and semi-group property. Completely different FPDEs are solved in the literature, such as the fractional telegraph model [22], fractional convection-diffusion model [23], fractional advection-dispersion model [24], fractional heat- wave-like model [25], fractional massive Thirring model [26], Fitzhugh-Nagumo neurons model [27], and wave interaction equations [28].

Investigation of closed-form solutions of both linear and nonlinear FPDEs is rare. Not much work has been done for nonlinear models, and only a few numerical techniques have been proposed to solve such FPDEs. The most common numerical methods used to obtain approximate analytical solutions for these FPDEs are the Adomian decomposition method, variational iteration method, homotopy perturbation method, residual power series method, differential transform method, and [29-32]. Numerous numerical strategies are projected for finding the partial differential equations of fractional order, like finite difference technique [33], finite volume technique [34], finite element technique [35], reproducing kernel technique [36], and fractional sub-equation technique [37]. Furthermore, after the approximation process by any local method (finite differences, finite volumes, finite elements etc) the fractional differential equation leads to another challenge, due the nonlocal nature of the involved operators. Indeed, the coefficient matrices of the associated linear systems are dense and hence a further branch of research has started in order to cope with the difficulties related to computational issues, see [38-48] and references therein. In this work, we use the finite volume and finite difference discretization for space-fractional convection-diffusion equation and compare the results obtained by above-mentioned methods. Motivated by the preceding works above, this analysis examines the application of finite-difference method (FDM) and finite-volume method (FVM) to obtain numerical solutions of space fractional-order convection-diffusion equation in terms of Riemann-Liouville fractional derivative. More specifically, we focus on the space-fractional convection-diffusion equation of the underlying form:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = p \frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in [a,b] \times (0,T), \tag{1}$$

along with the underlying initial condition

$$u(x,0) = f(x), \tag{2}$$

where $t \geq 0$, $x \in [a,b]$, ϵ and p are positive parameters, β is a parameter describing the order of space fractional derivative in light of the Riemann-Liouville sense of order $0 < \beta \leq 1$, $f(x)$ is smooth given analytical function of x, and $u(x,t)$ is unknown analytic function to be determined afterwards. Hereinafter, we assume that the fractional models (1)-(2) fulfill the necessary and sufficient conditions of existence and unique solutions.

The basic motivation of the current work is to investigate and design a novel iterative algorithm for generating the numerical solutions of the space-fractional convection-diffusion equation by employing the Riemann-Liouville fractional operator. The solution methodology relies on constructing the FDM and FVM to gain solutions in a uniform form of a rapidly convergent series. Towards this end, it is necessary to construct a numerical foundation and equivalently numerical infrastructure. Simultaneously, stability analysis and error estimation are discussed. Finally, some numerical experiments are provided to illustrate the great flexibility and reliability of the developed algorithm. The remaining parts of this analysis are formulated as follows: In the second section, some basic and elementary definitions of fractional calculus are presented. Next, in Section 3, the solutions for the space-fractional convection-diffusion equation are presented as well using FDM and FVM. Some numerical experiments are provided in Section 4 to demonstrate the flexibility, accuracy, and plainness of the present methods. Section 5 is the end of this analysis with a brief conclusion.

2. PRELIMINARIES AND BASIC CONCEPTS

Fractional calculus arises in many branches of chemistry, physics, engineering, and applied sciences, including the oceanography, gravity, atmosphere, aerodynamics, fractals, electrodynamics, and rheology [49-59]. As effective tools to describe the hereditary properties of basic, substances, and processes, they are widely utilized to synthesize and formulate fractional evolution systems with priority given for providing a more comprehensive explanation of dynamics, chaos, and the pattern of state change over space. In this orientation, there are many fractional derivatives in literature, such as the Caputo derivative, Riemann-Liouville derivative, Riesz derivative, Grünwald-Letnikov derivative, Caputo-Fabrizio derivative, Feller derivative, Atangana-Baleanu-Caputo derivative, and conformable fractional derivative [51-54]. This section presents some important preliminaries and definitions, that we need in the rest of the analysis.

Definition 2.*1*. *[51] The Riemann-Liouville integral of fractional order* $\beta > 0$, $J_a^{\alpha} u(x)$ *is defined as:*

$$J_a^{\beta} u(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} u(t) dt, \text{ provided that } u \in L_1[a,b].$$

For $\beta = 0$, we have $J_a^0 u(x) = u(x)$ is the identity operator.

Definition 2.*2*. *[51] The Riemann-Liouville fractional derivative of order* $\beta > 0$ *is defined as:*

$$\mathcal{D}_a^{\beta} u(x) = \mathcal{D}^n J_a^{(n-\beta)} u(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d^n}{dx^n}\right) \left[\int_a^x \frac{u(t)}{(x-t)^{\beta+1-n}} dt\right]. \quad (3)$$

where n is the smallest integer that exceeds β.

For $\beta = 0$, we have $\mathcal{D}_a^0 u(x) = u(x)$ is the identity operator.

For $\beta \in \mathbb{N}$, $\mathcal{D}_a^{\beta} u(x) = \frac{d^{\beta} u(x)}{dx^{\beta}}$.

Definition 2.3. *[51] Let* $\beta > 0, u \in C^{[\beta]}[a,b]$. *Then,*

$$\widetilde{\mathcal{D}}_a^\beta u(x) = \lim_{h\to 0} \frac{1}{h^\beta} \sum_{k=0}^{\left[\frac{x-a}{h}\right]} (-1)^k \binom{\beta}{k} u(x-kh),\ a < x \le b, \tag{4}$$

with $h = \frac{x-a}{N}$ is called the Grünwald-Letnikov fractional derivative of order β of the function u, where N is number of uniformly spaced nodes.

The following theorems show the relation between the Riemann-Liouville fractional derivatives and Grünwald-Letnikov fractional derivative.

Theorem 2.1. ***[51]*** *Let* $\beta > 0, n = \lceil \beta \rceil$ *and* $u \in C^n[a,b]$. *Then,*

$$\widetilde{\mathcal{D}}_a^\beta u(x) = \mathcal{D}_a^\beta u(x),\ a < x \le b.$$

Theorem 2.2. ***[51]*** *Let* $\beta > 0$, *and* $u \in C[a,b]$. *Then, we have*

$$J_a^\beta u(x) = \lim_{h\to 0} h^\beta \sum_{k=0}^{\left[\frac{x-a}{h}\right]} (-1)^k \binom{-\beta}{k} u(x-kh), h = \frac{x-a}{N}, a < x \le b, \tag{5}$$

where $(-1)^k \binom{-\beta}{k} = \frac{\beta(\beta-1)(\beta-2)\dots(\beta+k-1)}{k!} = \frac{\Gamma(\beta+k)}{\Gamma(\beta)\Gamma(k+1)}$, N is number of uniformly spaced nodes, and the function $\Gamma(x)$ is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. If we define weights $w_0^\beta = 1$, $w_1^\beta = \beta$, and $w_k^\beta = \left(1 - \frac{(1-\beta)}{k}\right) w_{k-1}^\beta, k = 2,3,\dots$. Then, one can rewrite Eq. (5) as follows,

$$J_a^\beta u(x) = \lim_{h\to 0} h^\beta \sum_{k=0}^{\left[\frac{x-\beta}{h}\right]} w_k^\beta u(x-kh), h = \frac{x-a}{N}. \tag{6}$$

3. Numerical FV and FD Discretization Methods

FDM is the most direct method for discretizing the FPDEs. By thinking about a point in space where we take the continuous representation of models and replace them with a set of discrete equations, which are called finite difference equations. FDM is usually defined as a normal network or regular grid which leads to highly effective solution methods for the problems under study. FDM is specified for each dimension, making it easy to scale and increase the elements-order for a higher-order resolution. By installing the simulation in a limited domain using a uniform grid, efficient implementations are obtained as well even if the associated coefficient matrices are dense and hence fast computational methods are needed [38-41]. Unified grids are useful for very large-scale simulations on supercomputers typically employed in seismological, astrophysical, and meteorological simulations. With the help of FVM, the posed model is divided into very small but limited size elements with simple geometric shapes. FVM relies on the fact that many physical laws are conservation laws that enter into one cell on one side and need to leave the same cell on the other side. By following this idea, a formula is obtained which consists of flux conservation and fluid flow systems specified in the averaged sense on cells. In this section, the procedure used for the suggested methods is described to obtain solutions of the space-fractional convection-diffusion equation. It is worth noting here that for each $u^0 = [u_0^0, u_1^0, \dots, u_N^0] = [g(x_0), g(x_1), \dots, g(x_N)]$, there exist a unique vector equation depending by the initial condition $u(x,0) = g(x)$, where $g(x)$ is analytic smooth function.

3.1. Finite volume method

Consider the space-fractional *convection-diffusion equation of the underlying form:*

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^\beta u(x,t)}{\partial x^\beta} = p \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{7}$$

where $t \geq 0, x \in [a,b]$, ϵ and p are positive parameters, $\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ represents the space-fractional derivative in terms of Riemann-Liouville sense of order $0 < \beta \leq 1$. To begin with, using the definition of Riemann-Liouville fractional derivative when $\beta \in (0,1]$, we have the following:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial}{\partial x} J_a^{1-\beta} u(x,t) = p \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{8}$$

where $J_a^{1-\beta}$ is the Riemann-Liouville integral with respect to x. By letting $\alpha = 1 - \beta$, we have that $0 \leq \alpha < 1$. Consequently, the aforementioned equation can be written as follows:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[-\epsilon J_a^\alpha u(x,t) + p \frac{\partial u(x,t)}{\partial x}\right]. \tag{9}$$

Now, discretize the finite domain Ω = $[a,b]$ with $N+1$ uniformly spaced grid $x_i = a + ih$, $i = 0,1,..,N$, where $h = \frac{b-a}{N}$ be the space step. Thus, integrating Eq. (9) over the i^{th} control volume $[x_{i-1/2}, x_{i+1/2}]$ so that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial u(x,t)}{\partial t} dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial x} [-\epsilon J_a^\alpha u(x,t) + p \frac{\partial u(x,t)}{\partial x}] \, dx. \tag{10}$$

Divide each side by h, the standard finite volume discretization can be obtained as follows:

$$\frac{d\bar{u}_i(t)}{dt} = \frac{\epsilon}{h} [J_a^\alpha u(x_{i-1/2},t) - J_a^\alpha u(x_{i+1/2},t)] + \frac{p}{h} \left[\frac{\partial u(x_{i+1/2},t)}{\partial x} - \frac{\partial u(x_{i-1/2},t)}{\partial x} \right]. \tag{11}$$

where $\bar{u}_i(t) = \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx$ is the control volume averages of $u(x,t)$. Anyhow, we can use the fractionally-shift Grünwald formula to approximate $J_a^\alpha u(x,t)$ and the central difference formula to approximate the partial derivative $\frac{\partial u(x,t)}{\partial x}$. Subsequently, we require the fractional shift $p = \frac{1}{2}$,

$$J_a^\alpha u(x,t)|_{x=x_{i-1/2}} = J_a^\alpha u\left(x_{i-\frac{1}{2}}, t\right) \approx h^\alpha \sum_{j=0}^{i} w_j^\alpha u\left(x_{i-j}, t\right), \tag{12}$$

$$J_a^\alpha u(x,t)|_{x=x_{i+1/2}} = J_a^\alpha u\left(x_{i+\frac{1}{2}}, t\right) \approx h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u\left(x_{i-j+1}, t\right), \tag{13}$$

where $w_0^\alpha = 1$, $w_1^\alpha = \alpha$, and $w_j^\alpha = \left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha$, $j = 2,3, ...$

For $\frac{\partial u(x,t)}{\partial x}$, one can use

$$\frac{\partial u(x_i,t)}{\partial x} = \frac{u(x_{i+1},t) - u(x_{i-1},t)}{2h} + \mathcal{O}(h^2), \tag{14}$$

to get that

$$\frac{\partial u\left(x_{i-1/2},t\right)}{\partial x} \approx \frac{u(x_{i+1/2},t) - u(x_{i-3/2},\ t)}{2h}, \tag{15}$$

$$\frac{\partial u\left(x_{i+1/2},t\right)}{\partial x} \approx \frac{u(x_{i+3/2},t) - u(x_{i-1/2},t)}{2h}. \tag{16}$$

With the help of the standard averaging scheme $u\left(x_{i\pm1/2},t\right) \approx \frac{[u(x_i,t)+u\left(x_{i\pm1},t\right)]}{2}$ for the above Eqs. (15)-(16), we construct the approximations of first derivative in term of function values at the nodes x_j such that,

$$\frac{\partial u\left(x_{i-\frac{1}{2}},t\right)}{\partial x} \approx \frac{1}{4h}[u(x_i,t)+u(x_{i+1},t)-u(x_{i-2},t)-u(x_{i-1},t)], \tag{17}$$

$$\frac{\partial u\left(x_{i+\frac{1}{2}},t\right)}{\partial x} \approx \frac{1}{4h}[u(x_{i+1},t)+u(x_{i+2},t)-u(x_{i-1},t)-u(x_i,t)]. \tag{18}$$

Hence, Eq. (11) can be approximated by

$$\begin{aligned}\frac{d\bar{u}_i(t)}{dt} = \epsilon h^{\alpha-1}&\left[\sum_{j=0}^{i} w_j^{\alpha} u\big(x_{i-j},t\big) - \sum_{j=0}^{i+1} w_j^{\alpha} u\big(x_{i-j+1},t\big)\right] \\ &+ \frac{p}{4h^2}[\,u(x_{i-2},t)-2u(x_i,t)+u(x_{i+2},t)].\end{aligned} \tag{19}$$

Noting that if $u(x,t)$ is smooth function, then the value of control volume averages $\bar{u}_i(t)$ agrees with the value of $u(x,t)$ at the midpoint of the interval $[x_{i-1/2},x_{i+1/2}]$ to $\mathcal{O}(h^2)$. So, Eq. (19) can be rewritten as:

$$\begin{aligned}\frac{du(x_i,t)}{dt} = \epsilon h^{\alpha-1}&\left[\sum_{j=0}^{i} w_j^{\alpha} u\big(x_{i-j},t\big) - \sum_{j=0}^{i+1} w_j^{\alpha} u\big(x_{i-j+1},t\big)\right] \\ &+ \frac{p}{4h^2}[\,u(x_{i-2},t)-2u(x_i,t)+u(x_{i+2},t)].\end{aligned} \tag{20}$$

Letting $t_n = n\tau,\ n = 0,1,2,\ldots,$ whereas τ is the time step. Then, by using the standard backward difference to approximate the time derivative in Eq. (20), it yields that,

$$\left.\frac{du(x_i,t)}{dt}\right|_{t=t_{n+1}} = \frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\tau} + \mathcal{O}(\tau).$$

Now, letting $u_i^n \approx u(x_i,t_n)$, which denote the numerical solution. So, we have

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{\tau} = \epsilon h^{\alpha-1}&\left[\sum_{j=0}^{i} w_j^{\alpha} u_{i-j}^{n+1} - \sum_{j=0}^{i+1} w_j^{\alpha} u_{i-j+1}^{n+1}\right] \\ &+ \frac{p}{4h^2}[u_{i-2}^{n+1}-2u_i^{n+1}+u_{i+2}^{n+1}].\end{aligned}$$

By collecting like terms, we can rewrite Eq. (20) as follows,

$$\frac{u_i^{n+1}-u_i^n}{\tau}=\frac{1}{h}\sum_{j=0}^{N}k_{ij}u_j^{n+1},\quad i=0,1,2,\dots,N, \tag{21}$$

where K_{ij} given by

$$k_{ij}=\begin{cases}\epsilon h^{\alpha}\left[w_{i-j}^{\alpha}-w_{i-j+1}^{\alpha}\right], & j<i-2,\\ \epsilon h^{\alpha}\left[w_{2}^{\alpha}-w_{3}^{\alpha}\right]+\frac{p}{4h}, & j=i-2,\\ \epsilon h^{\alpha}\left[w_{1}^{\alpha}-w_{2}^{\alpha}\right], & j=i-1,\\ \epsilon h^{\alpha}\left[w_{0}^{\alpha}-w_{1}^{\alpha}\right]-\frac{p}{2h}, & j=i,\\ \epsilon h^{\alpha}\left[-w_{0}^{\alpha}\right], & j=i+1,\\ \frac{p}{4h}, & j=i+2,\\ 0, & j>i+2.\end{cases}$$

Utilizing the numerical solution vector $U^n=[u_0^n,u_1^n\dots,u_N^n]$, we have the following vector equation:

$$\left(I+\frac{\tau}{h}A\right)U^{n+1}=U^n,$$

where Matrix A has elements $a_{ij}=-K_{ij.}$

Theorem 3.1. For $i=0,1,\dots,N$, the numerical scheme $\frac{u_i^{n+1}-u_i^n}{\tau}=\frac{1}{h}\sum_{j=0}^{N}k_{ij}u_j^{n+1}$ is conditionally stable.

Proof: Substituting $u_i^n=\hat{u}^n\, exp(iI\xi)\,, I=\sqrt{-1},$ into

$$\frac{u_i^{n+1}-u_i^n}{\tau}=\frac{1}{h}\sum_{j=0}^{N}k_{ij}u_j^{n+1}, i=0,1,2,\dots,N,$$

gives $\hat{u}^{n+1}\, exp(iI\xi)-u^n\, exp(iI\xi)=r\sum_{j=0}^{N}k_{ij}\hat{u}^{n+1}\exp(jI\xi)\,, r=\frac{\tau}{h}$. $\hat{u}^{n+1}=\rho(\xi)\hat{u}^n$, where $\rho(\xi)=\frac{1}{\left[1-r\sum_{j=0}^{N}k_{ij}\exp((j-i)I\xi)\right]}$ which satisfies the von Neumann condition whenever $\left|1-r\sum_{j=0}^{N}k_{ij}exp\left((j-i)I\xi\right)\right|\geq 1$. By using the reverse triangle inequality, we have

$\left|1-r\sum_{j=0}^{N}k_{ij}exp\left((j-i)I\xi\right)\right|\geq\left|1-\left|r\sum_{j=0}^{N}k_{ij}exp\left((j-i)I\xi\right)\right|\right|$.

Thus, von Neumann condition holds for $\left|1-\left|r\sum_{j=0}^{N}k_{ij}exp\left((j-i)I\xi\right)\right|\right|\geq 1$, *i.e., either* $1-\left|r\sum_{j=0}^{N}k_{ij}exp\left((j-i)I\xi\right)\right|\geq 1$, *which is impossible to be hold, or* $1-\left|r\sum_{j=0}^{N}k_{ij}exp\left((j-\right.\right.$

$i)I\xi)\big| \leq -1$, which is equivalent to $\big|\sum_{j=0}^{N} k_{ij} exp\,((j-i)I\xi)\big| \geq \frac{2}{r}$. Therefore, $\rho(\xi)$ satisfies the von Neumann condition whenever $\big|\sum_{j=0}^{N} k_{ij} exp\,((j-i)I\xi)\big| \geq \frac{2}{r}$, for $i = 0,1,2,\dots,N$.

3.2. Finite difference method

Herein, we use a finite difference method for solving the space-fractional convection-diffusion equation with constant coefficients:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = p \frac{\partial^2 u(x,t)}{\partial x^2},$$

along with the following initial condition

$$u(x,0) = f(x),$$

where $x \in [a,b]$, $t \geq 0, 0 < \beta \leq 1$, ϵ and p are positive parameters, and $\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}$ is the space fractional, the space fractional derivative in the Riemann-Liouville sense. Using the definition of Riemann-Liouville fractional derivative of order $0 < \beta \leq 1$, we have

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial}{\partial x} J_a^{1-\beta} u(x,t) = p \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{22}$$

where $J_a^{1-\beta}$ is the Riemann-Liouville integral with respect to x. Take $\alpha = 1 - \beta$, we have $0 \leq \alpha < 1$.

Now discretize the finite domain $\Omega = [a,b]$ with $N + 1$ uniformly spaced nodes $x_i = a + ih$, $i = 0,1,..,N$, where $h = \frac{b-a}{N}$ be the space step. We approximate the α order fractional Riemann-Liouville integral with standard Grünwald formula and approximate the first and second derivative with central difference formula:

$$J_a^{\alpha} u(x,t) = h^{\alpha} \sum_{j=0}^{N} w_j^{\alpha} u(x - jh, t) + o(1), \tag{23}$$

$$\frac{\partial u(x_i,t)}{\partial x} = \frac{u(x_{i+1},t) - u(x_{i-1},t)}{2h} + \mathcal{O}(h^2), \tag{24}$$

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} = \frac{u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)}{h^2} + \mathcal{O}(h^2). \tag{25}$$

Applying the finite difference discretization method for evaluating Eq. (22) at $x = x_i$, and using the above Eqs (23)-(25) to get

$$\frac{du(x_i,t)}{dt} = -\frac{\epsilon}{2h}\left[h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u\left(x_{i-j+1},t\right) - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u\left(x_{i-j-1},t\right)\right] + p\left[\frac{u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)}{h^2}\right]. \quad (26)$$

Letting $t_n = n\tau,\ n = 0,1,2,\ldots,$ where τ is the time step, and using the standard backward difference to approximate the temporal derivative in Eq. (26) as follows,

$$\left.\frac{du(x_i,t)}{dt}\right|_{t=t_{n+1}} = \frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{\tau} + \mathcal{O}(\tau), \quad (27)$$

Now, letting $u_i^n \approx u(x_i,t_n)$ is the numerical solution, then we have

$$\frac{u_i^{n+1} - u_i^n}{\tau} = -\frac{\epsilon}{2h}\left[h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u_{i-j+1}^{n+1} - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u_{i-j-1}^{n+1}\right] + p\left[\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2}\right]. \quad (28)$$

Thus, by collecting like terms, we can rewrite the above equation as follows,

$$\frac{u_i^{n+1} - u_i^n}{\tau} = -\sum_{j=0}^{N} b_{ij} u_j^{n+1}, \quad (29)$$

where

$$b_{ij} = \begin{cases} \dfrac{\epsilon h^{\alpha-1}[w_{i-j+1}^\alpha - w_{i-j-1}^\alpha]}{2}, & j < i-1, \\ \dfrac{\epsilon h^{\alpha-1}[w_2^\alpha - w_0^\alpha]}{2} - \dfrac{p}{h^2}, & j = i-1, \\ \dfrac{\epsilon h^{\alpha-1} w_1^\alpha}{2} + \dfrac{2p}{h^2}, & j = i, \\ \dfrac{\epsilon h^{\alpha-1} w_0^\alpha}{2} - \dfrac{p}{h^2}, & j = i+1, \\ 0, & j > i+1. \end{cases}$$

Utilizing the numerical solution vector $U^n = [u_0^n, u_1^n, \ldots, u_N^n]$, we have the following vector equation:

$$(I+\tau A)U^{n+1}=U^{n}, \tag{30}$$

where the matrix A has elements $a_{ij}=b_{ij}$. we can re write the above equation as $U^{n+1}=MU^{n}$, where $M=(I+\tau A)^{-1}$ is the iteration matrix.

Theorem 3.2. Let $\epsilon>0, \rho>0$ and $0\leq\alpha<1$ $satisfy$ $\rho>\frac{\epsilon h^{\alpha+1}}{2}$, where h is the spatial step. Then, the coefficients a_{ij} satisfy

$$|a_{ii}|>\sum_{\substack{j=0\\ j\neq i}}^{N}|a_{ij}|, i=1,2,\dots,N.$$

Proof: Consider the following sum for a given i:

$$\sum_{\substack{j=0\\ j\neq i}}^{N}|a_{ij}|=\sum_{j=0}^{i-2}|a_{ij}|+|a_{i,i-1}|+|a_{i,i+1}|+\sum_{j=i+2}^{N}|a_{ij}|$$

$$=\sum_{j=0}^{i-2}\left|\frac{\epsilon h^{\alpha-1}[w_{i-j-1}^{\alpha}-w_{i-j+1}^{\alpha}]}{2}\right|+\left|\frac{\epsilon h^{\alpha-1}[w_{2}^{\alpha}-w_{0}^{\alpha}]}{2}-\frac{\rho}{h^{2}}\right|+\left|\frac{\epsilon h^{\alpha-1}w_{0}^{\alpha}}{2}-\frac{\rho}{h^{2}}\right|.$$

Now, by the hypotheses $\rho>\frac{\epsilon h^{\alpha+1}}{2}$, we include that each term is negative. So, we have:

$$\sum_{\substack{j=0\\ j\neq i}}^{N}|a_{ij}|=\sum_{j=0}^{i-2}\frac{\epsilon h^{\alpha-1}[w_{i-j-1}^{\alpha}-w_{i-j+1}^{\alpha}]}{2}+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}[w_{2}^{\alpha}-w_{0}^{\alpha}]}{2}\right)+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}w_{0}^{\alpha}}{2}\right).$$

Replacing the finite sum with infinite sum

$$\sum_{\substack{j=0\\ j\neq i}}^{N}|a_{ij}|<\sum_{j=-\infty}^{i-2}\frac{\epsilon h^{\alpha-1}[w_{i-j-1}^{\alpha}-w_{i-j+1}^{\alpha}]}{2}+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}[w_{2}^{\alpha}-w_{0}^{\alpha}]}{2}\right)+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}w_{0}^{\alpha}}{2}\right).$$

The telescoping sum have the form $(w_{1}^{\alpha}-w_{3}^{\alpha})+(w_{2}^{\alpha}-w_{4}^{\alpha})+(w_{3}^{\alpha}-w_{5}^{\alpha})+(w_{4}^{\alpha}-w_{6}^{\alpha})$. Hence, we have

$$\sum_{\substack{j=0\\ j\neq i}}^{N}|a_{ij}|<\frac{\epsilon h^{\alpha-1}[w_{1}^{\alpha}+w_{2}^{\alpha}]}{2}+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}[w_{2}^{\alpha}-w_{0}^{\alpha}]}{2}\right)+\left(\frac{\rho}{h^{2}}-\frac{\epsilon h^{\alpha-1}w_{0}^{\alpha}}{2}\right)$$

$$=\frac{\epsilon h^{\alpha-1}w_{1}^{\alpha}}{2}+\frac{2\rho}{h^{2}}\quad=|a_{ii}|.$$

Corollary 3.1. The iteration matrix M in the scheme (30) is convergent, and hence the scheme itself is conditionally stable.

Proof: With the aid of Theorem 3.2, we have that A is strictly diagonally dominant with positive diagonal elements. Hence, $I+\tau A$ is so and then the iteration matrix $M=(I+\tau A)^{-1}$ exists, in which its spectral radius satisfies

$$\rho(M)=\rho(I+\tau A)^{-1}=\left(1+\tau\rho(A)\right)^{-1}<1.$$

4. Numerical Experiments

Several physical applications that are formed utilizing FPDEs cannot be found their exact solutions. This section aims to use the proposed approaches, the FDM and FVM, to construct the solutions for space-fractional convection-diffusion equation. In this section, we demonstrate the validity of our analysis by giving two numerical examples. Eventually, numerical experiments are considered for comparing the FV and FD methods, the computations are performed by Mathematica Software 11.0.

Example 4.1*:* Consider the space-fractional convection-diffusion equation

$$\frac{\partial u(x,t)}{\partial t}+(0.01)\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=(0.002)\frac{\partial^{2}u(x,t)}{\partial x^{2}}, \tag{31}$$

subject to the following initial condition

$$u(x,0)=-\sin(\pi x), \tag{32}$$

where $x\in[1,1.5]$, $t\geq 0$ and $0<\beta\leq 1$.

In particular, the exact solution at $\beta=1$ is given by $u(x,t)=-\sin\left(\pi(x-(0.01)t)\right)\exp\left(-(0.002)(\pi^{2}t)\right)$. and when the $\beta\in(0,1)$ the exact solution is given by$\sum_{n=0}^{\infty}\frac{(-1)^{n}\pi^{2n}x^{2\beta}}{\Gamma(n\alpha+1)}\left[-\sin\left(\pi\left(x^{\beta}-(0.01)t\right)\right)\exp\left(-(0.002)(\pi^{2}t)\right)\right]$

In accordance with the proposed approaches, put $h=0.015625$ and $\tau=0.01$. In the following, some representative results of fractional convection-diffusion equation (31) and (32) are shown in Table 1. Whereas Table 1 displayed the absolute errors for the fractional models (31) and (32) at time $t=0.5$ using the FDM and FVM. From the numerical simulation, it can be seen how compatible the numerical solutions are with the exact solution at each selected node.

Table 1. Absolute error of Example 4.1 at $\beta = 1$ with time $t = 0.5$ using FVM and FDM.

x	Exact solution	FVM Absolute Error	FDM Absolute Error
1.00000	-0.015553	1.39255×10^{-7}	1.66307×10^{-7}
1.01563	0.033046	5.96755×10^{-7}	4.78730×10^{-7}
1.03125	0.081564	1.93938×10^{-6}	1.57099×10^{-6}
1.04688	0.129887	3.07424×10^{-6}	2.21967×10^{-6}
1.0625	0.177896	5.70262×10^{-6}	3.27646×10^{-6}
1.07813	0.225477	6.32957×10^{-6}	7.02039×10^{-6}
1.09375	0.272515	1.06647×10^{-5}	1.78096×10^{-5}
1.10938	0.318896	1.14916×10^{-5}	2.07548×10^{-5}
1.12500	0.364509	7.40457×10^{-5}	3.61227×10^{-5}

Example 4.*2:* Consider the space- fractional convection-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} + (0.1)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = (0.02)\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{30}$$

subject to the following initial condition:

$$u(x,0) = e^{\zeta x}, \tag{31}$$

where $\zeta = 1.17712434446770$, $x \in [-2,0.5]$, $t \geq 0$ and $0 < \beta \leq 1$.

In particular, the exact solution at $\beta = 1$ is given by $u(x,t) = e^{\zeta x - 0.09t}$ and when the $\beta \in (0,1)$ the exact solution is given by $(x,t,\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2\beta}}{\Gamma(n\alpha+1)} e^{\zeta x^{\beta} - 0.09t}$. In accordance with the proposed approaches, put $h = 0.0625$ and $\tau = 0.01$. In the following, some representative results of fractional convection-diffusion equation (33) and (34) are shown in Table 2. Table 2 displayed the absolute errors for the fractional models (33) and (34) at time $t = 0.5$ using the FDM and FVM. From the numerical simulation, it can be seen how compatible the numerical solutions are with the exact solution at each selected node.

Table 2. Absolute error of Example 4.2 at $\beta = 1$ with time $t = 0.5$ using FVM and FDM.

x	Exact solution	FVM Absolute Error	FDM Absolute error
-2.0000	0.090786	2.448740×10^{-7}	2.548004×10^{-7}
-1.9375	0.097717	1.853871×10^{-6}	5.950876×10^{-7}
-1.8750	0.105177	2.715357×10^{-6}	1.913549×10^{-6}
-1.8125	0.113207	4.084997×10^{-6}	2.269837×10^{-6}
-1.7500	0.121850	5.079988×10^{-6}	2.580286×10^{-6}
-1.6875	0.131152	7.852180×10^{-6}	2.925587×10^{-6}
-1.6250	0.141165	1.297671×10^{-5}	1.180799×10^{-5}
-1.5625	0.151942	1.620270×10^{-5}	3.043481×10^{-5}
-1.5000	0.163542	3.214643×10^{-5}	3.804975×10^{-5}
-1.4375	0.176027	3.273511×10^{-5}	4.173032×10^{-5}
-1.3750	0.189466	4.233925×10^{-5}	4.385183×10^{-5}

5. CONCLUSION

In this paper, finite difference and finite volume methods have been lucratively used for solving the convection-diffusion equation of fractional order with constant coefficient. We use finite volume method for solving the given equation, for this we take the integral over the i^{th}control volume, then Reimann-Liouville fractional integral is discretized using standard shift Grünwald formula. We got a conservative solution, whereas the solution obtained by the finite difference method is not conservative, because we require to use the central difference formula is first applied, then the Reimann –Liouville fractional integral is discretized using standard Grünwald formula. Numerical results are found using Mathematica software 11.0 and found that both the methods have approximately close. The obtained numerical results indicate that the finite volume method is more approximate to the exact solution than finite difference method in two different experiments. Thus, it can be concluded that the Finite volume method is better for solving the convection-diffusion equation of fractional order with constant coefficient

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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