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EXISTENCE AND UNIQUENESS OF COMMON FIXED POINTS INVOLVING $(\varphi,\ \psi)$ –CONTRACTIVE CONDITION OF FOUR SELF MAPPINGS IN CONE METRIC SPACES

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Abstract. The object of this paper is to prove the existence and the uniqueness of common fixed points of four mappings, compatible in pairs, involving (φ, ψ) - contractive condition in cone metric spaces. These results generalize and extend some existing fixed point results in the literature.

Keywords: Cone metric space, weak compatible, complete space, Cauchy sequence, regular cone.

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1. Introduction

Huang and Zhang [8] recently generalized the concept of metric spaces replacing the set of real numbers by an ordered Banach space defining in this way a cone metric space. These authors defined convergent and Cauchy sequences in the terms of interior points of the underlying cone. They later proved some fixed point theorems for different contractive mappings. Their results have been generalized and extended by different authors see [1],[2], [3],[4],[5],[7],[8]. Alberd and Guerre-Delabriere [10] introduced the notion of the

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weak contraction. They proved the existence of fixed points for single-valued maps satisfying weak contractive condition on Hilbert spaces.

2. Preliminaries

The weak contraction was defined as follows

Definition 2.1 A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$

Definition 2.2 ([8]) Let E be a real Banach space and P be a subset of E. Then P is called a cone if

(i) P is closed, nonempty and $P \neq \{0\}$

(ii) $a, b \in \mathbb{R}, a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone $P \subset E$, we define a partial ordering " $\leq$ in E by $x \leq y$ if $y - x \in P$.

We write $x < y$ to denote $x \leq y$ and $x \neq y$ and we write $x \ll y$ to denote $y - x \in$ int P, where int P stands for the interior of P.

In the following we suppose that E is a Banach space with cone P with int $P \neq \phi$ and $\leq$ is the partial ordering with respect to P.

Definition 2.3 ([8]) A cone P is called regular if every increasing sequence which is bounded from above is convergent. A cone P is called regular if and only if every decreasing sequence which is bounded from below is convergent.

Proposition 2.1 ([9]) Let P be a cone in a real Banach space E. If $a \in P$ and $a \leq ka$ for some $k \in [0, 1)$, then $a = 0$

Proposition 2.2 ([9]) Let P be a cone in a real Banach space E.If $a \in E$ and $a \ll c$ for all $c \in$ int$P$, then $a = 0$.

Remark 2.1 (see [11]) We have λ int $P \subset$ int P for $\lambda > 0$ and int P+int $P \subset$ int P

Definition 2.4 ([8]) Let X be a nonempty set. Suppose that we are given a mapping $d : X \times X \to E$ that satisfies:

(a) $0 \leq d(x,y)$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$

(c) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a metric on X and (X,d) is called a cone metric space.

For examples of cone metric spaces, we refer the reader to [8].

Definition 2.5 ([8]) Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$ there is a positive integer N_c such that for all $n > N_c$ we have $d(x_n, x) \ll c$, then the sequence $\{x_n\}$ is said to be converge to x and x is called the limit of $\{x_n\}$. We write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.6 ([7]) Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$ there is a positive integer N_c such that for all $n, m > N_c$ we have $d(x_n, x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X.

Remark 2.2 It follows from the above definition that if $\{x_{2n}\}$ is a subsequence of a Cauchy sequence $\{x_n\}$ in a cone metric space (X,d) and $x_{2n} \to z$ then $x_n \to z$.

Definition 2.6 ([8]) Let (X,d) be a cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

Proposition 2.3 Let (X,d) be a cone metric space and P be a cone in a real Banach space E. If $u \leq v$ and $v \ll w$ then $u \ll w$. **Definition 2.7** ([2]) Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then w is called a point of coincidence of f and g.

Definition 2.8 ([13]) Let X be any set. A pair of self maps (f,g) is said to be weakly compatible if $u \in X$ and $fu = gu$ imply $gfu = fgu$.

Definition 2.9 Let $\psi : P \to P$ be a function.

(i) We say that ψ is strongly monotone increasing if for $x, y \in P$, $x \leq y \iff \psi(x) \leq \psi(y)$

(ii) ψ is said to be continuous at $x_0 \in P$ if for any sequence $\{x_n\}$ in P, $x_n \to x_0 \implies \psi(x_n) \to \psi(x_0)$.

Proposition 2.3 ([2]) Let (f, g) be a pair of weakly compatible self maps of a set X. if f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Recently Choudhury and Metiya [4] have proved the following results

Lemma 2.1 ([4]): Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$, with $x \neq y$. Let $\varphi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ be a function with the following properties:

(i) $\varphi(t) = 0$ if and only if $t = 0$

(ii) $\varphi(t) \ll t$, for $t \in intP$

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in \operatorname{int} P \cup \{0\}$ and $x, y \in X$.

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n, x_{n+1})\}$ is convergent to either $r = 0$ or $r \in intP$.

Theorem 2.1 ([5]) Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$, with $x \neq y$. Let $f, g : X \to X$ be a mapping satisfying the inequality

$$\psi(d(fx, fy)) \leq \psi(d(fx, gx)) - \varphi(d(fy, gy)) \text{ for all } x, y \in X.$$

where $\psi : P \to P$ and $\varphi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ are continuous functions satisfying the following properties:

(i) $\varphi(t) = 0 = \psi(t)$ if and only if $t = 0$

(ii) $\varphi(t) \ll t$, for $t \in intP$

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in \operatorname{int} P \cup \{0\}$ and $x, y \in X$.

If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, Then f and g have a unique point of coincidence in X. Moreover , if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Also they proved the following theorem:

Theorem 2.2 ([5]) : Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$, with $x \neq y$. Let $f, g : X \to X$ be a mapping satisfying the inequality

$$\psi\left(d\left(fx,fy\right)\right)\leq\psi\left(\frac{d\left(fx,gx\right)+d\left(fy,gy\right)}{2}\right)-\varphi\left(d\left(gx,gy\right)\right)\ \text{for all}\ x,y\in X$$

where $\psi : P \to P$ and $\varphi : \text{int } P \cup \{0\} \to \text{int } P \cup \{0\}$ are continuous functions with the following properties:

(i) $\varphi(t) = 0 = \psi(t)$ if and only if $t = 0$

(ii) $\varphi(t) \ll t$, for $t \in intP$

(iii) either $\varphi(t) \leq d(x,y)$ or $d(x,y) \leq \varphi(t)$, for $t \in \text{int } P \cup \{0\}$ and $x, y \in X$. Then T has a unique fixed point in X.

If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

3. Main Results

Theorem 3.1 Let (X, d) be a cone metric space with regular cone P, such that $d(x, y) \in$ *int* P for all $x, y \in X$, with $x \neq y$. Let A, B, S and $T : X \to X$ be four maps satisfying

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X) \tag{1}$$

and the inequality

$$\psi(d(Ax, By)) \leq \psi(\theta(x, y)) - \varphi(\theta(x, y)) \text{ for all } x, y \in X \tag{2}$$

where $\psi : P \to P$ and $\varphi : \text{int } P \cup \{0\} \to \text{int } P \cup \{0\}$ are continuous functions with the following properties:

(i) ψ is strongly monotonic increasing

(ii) $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$

(iii) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ for all $t_1, t_2 \in P$

and

$$\theta(x,y) \quad = \quad \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty)$$

for some $\lambda, \mu, \delta \in]0,1[$ and $\lambda+\mu+\delta<1$. If one of $A(X)$, $B(X)$, $T(X)$ or $S(X)$ is a complete subspace of X and the pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Given $x_0 \in X$, from (1), we define the sequence $\{x_n\}$ recursively as follows:

$$\left\{\begin{array}{lllll} Ax_{2n} & = & Tx_{2n+1} & = & y_{2n} \\ Bx_{2n+1} & = & Sx_{2n+2} & = & y_{2n+1} \end{array}\right. \quad \text{for all } n \in \mathbb{N} \tag{3}$$

First, we claim that if there exists $n_0 \in \mathbb{N}$ such that

$$\theta(x_{2n}, x_{2n-1}) = 0 \quad \text{then} \quad y_{2n} = y_{2n-1} \tag{4}$$

By taking $x = x_{2n}$ and $y = x_{2n-1}$ in (3), we get

$$\begin{aligned} 0 &= \theta(x_{2n}, x_{2n-1}) = \lambda d(Ax_{2n}, Sx_{2n}) \\ &\quad +\mu d(Bx_{2n-1}, Tx_{2n-1}) + \delta d(Sx_{2n}, Tx_{2n-1}) \end{aligned}$$

Using (3) again, we obtain

$$\begin{aligned} 0 &= \theta(x_{2n}, x_{2n-1}) = \lambda d(y_{2n}, y_{2n-1}) \\ &\quad +\mu d(y_{2n-1}, y_{2n-2}) + \delta d(y_{2n-1}, y_{2n-2}) \end{aligned}$$

then since $\lambda > 0$, we have

$$d(y_{2n}, y_{2n-1}) = -\frac{1}{\lambda}\left(\mu+\delta\right) d(y_{2n-1}, y_{2n-2})) \in P \cap (-P) = \{0\}$$

thus

$$y_{2n} = y_{2n-1} \tag{5}$$

Now, we claim that if (5) is true, then $y_{2n} = y_{2n+1}$. By taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3), we get

$$\begin{aligned} \theta(x_{2n}, x_{2n+1}) &= \lambda d(Ax_{2n}, Sx_{2n}) \\ &\quad +\mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n}, Tx_{2n+1}) \end{aligned}$$

Using (3), we obtain

$$\begin{aligned}\theta(x_{2n},x_{2n+1}) &= \lambda d(y_{2n},y_{2n-1}) \\ &\quad +\mu d(y_{2n+1},y_{2n})+\delta d(y_{2n-1},y_{2n})\end{aligned} \tag{6}$$

it follows from (6) that

$$\theta(x_{2n},x_{2n+1})=\mu d(y_{2n},y_{2n+1}) \tag{7}$$

and from (7), since $0<\mu<1$, we have

$$\theta(x_{2n},x_{2n+1})<d(y_{2n},y_{2n+1}) \tag{8}$$

Since ψ is monotone nondecreasing, therefore from (2) , we have

$$\begin{aligned}\psi(\theta(x_{2n},x_{2n+1})) &\leq \psi(d(y_{2n},y_{2n+1})) \\ &= \psi(d(Ax_{2n},Bx_{2n+1})) \\ &\leq \psi(d(y_{2n},y_{2n+1}))-\varphi(\mu d(y_{2n},y_{2n+1}))\end{aligned}$$

Hence by, we get

$$\varphi(\mu d(y_{2n},y_{2n+1}))=0$$

then from the property of φ we get $\mu d(y_{2n},y_{2n+1})=0$ and we have $y_{2n}=y_{2n+1}$. Continuing in this way, we can conclude that

$$y_n=y_{n+r} \quad \text{for} \quad \text{all } r\in\mathbb{N} \tag{9}$$

Thus the sequence $\{y_n\}$ is a Cauchy sequence.

From now on, we suppose that

$$\theta(x_{2n},x_{2n+1})>0 \quad \text{all } n\in\mathbb{N} \tag{10}$$

Setting

$$d_n=d\left(y_n,y_{n+1}\right) \tag{11}$$

By taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3) again, we get

$$\begin{aligned}\theta(x_{2n}, x_{2n+1}) &= \lambda d(Ax_{2n}, Sx_{2n}) \\ &\quad +\mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n}, Tx_{2n+1})\end{aligned}$$

therefore by (3) , we obtain

$$\begin{aligned}\theta(x_{2n}, x_{2n+1}) &= \lambda d(y_{2n}, y_{2n-1}) \\ &\quad +\mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n-1}, y_{2n}) \\ &= \lambda d_{2n-1} + \mu d_{2n} + \delta d_{2n-1}\end{aligned} \tag{12}$$

Hence, we have

$$\begin{aligned}\theta(x_{2n}, x_{2n+1}) &\leq \lambda d_{2n-1} + \mu d_{2n} + \delta d_{2n-1} \\ &\leq (\lambda + \delta)\, d_{2n-1} + \mu d_{2n}\end{aligned} \tag{13}$$

Now, we claim that

$$d_{2n} \leq d_{2n-1} \tag{14}$$

Suppose (14) is not true, that is, there exists $l \in \mathbb{N}$ such that

$$d_{2l-1} > d_{2l} \tag{15}$$

then from (15) and from (13) , we have

$$\theta(x_{2l}, x_{2l+1}) \leq (\lambda + \delta + \mu)\, d_{2l-1} < d_{2l-1} \tag{16}$$

then, from (13) , (15) , (16) and the property of ψ, it follows

$$\begin{aligned}\psi\left(d_{2l}\right) &\leq \psi\left(d_{2l-1}\right) \\ &\leq \psi\left(\theta(x_{2l}, x_{2l-1})\right) - \varphi\left(\theta(x_{2l}, x_{2l-1})\right) \\ &\leq \psi\left(d_{2l-1}\right) - \varphi\left(\theta(x_{2l}, x_{2l-1})\right)\end{aligned}$$

Using the property of φ, we obtain $\varphi\left(\theta(x_{2l}, x_{2l-1})\right) = 0$, which implies that

$$\theta(x_{2l}, x_{2l-1}) = 0$$

which is a contradiction with (10) . Continuing in this way, we can conclude that

$$d_{n+1} \leq d_n \quad \text{for all} \quad n \in \mathbb{N} \tag{17}$$

Hence the sequence $\{d_n\}$ is a nondecreasing sequence. Since the cone P is regular and $d_n \geq 0$, there exists $d \in P$ such that $d_n \to d$ as $n \to \infty$. We shall prove that $d = 0$. It follows from (13) and (14) that

$$\theta(x_{2n}, x_{2n+1}) \leq (\lambda + \delta + \mu)\, d_{2n-1} < d_{2n-1} \tag{18}$$

and

$$d\left(Ax_{2n}, Bx_{2n-1}\right) = d_{2n-1} \tag{19}$$

Using (2), (18), (19) and the property of ψ, we get

$$\psi\left(d_{2n-1}\right) \leq \psi\left(d_{2n-1}\right) - \varphi\left(\theta(x_{2n}, x_{2n-1})\right) \tag{20}$$

Letting $n \to \infty$ in (20), using (12) and the continuity of φ and ψ we obtain

$$\psi\left(d\right) \leq \psi\left(d\right) - \varphi\left(\left(\lambda + \delta + \mu\right) d\right)$$

using the property of φ, we get $(\lambda + \delta + \mu)\, d = 0$. Therefore $d = 0$ and we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = 0 \tag{21}$$

We shall prove that the sequence $\{y_n\}$ is a Cauchy sequence. It is enough to show that $\{y_{2n}\}$ is a Cauchy sequence. On contrary, suppose that $\{y_{2n}\}$ is not a Cauchy sequence, then there exists $c \in E, c \gg 0$ and monotone increasing sequences of natural numbers $\{_{2m_k}\}$ and $\{_{2n_k}\}$ such that $n_k > m_k$

$$d\left(y_{2m_k}, y_{2n_k}\right) \geq c \tag{22}$$

and

$$d\left(y_{2m_k}, y_{2n_k - 2}\right) \leq c \tag{23}$$

Combining (22) and (23) and using triangular inequality, we obtain

$$\begin{aligned} c &\leq d\left(y_{2m_k}, y_{2n_k}\right) \\ &\leq d\left(y_{2m_k}, y_{2n_k-2}\right) + d\left(y_{2n_k}, \ y_{2n_k-1}\right) + d\left(y_{2n_k-1}, y_{2n_k}\right) \\ &< c + d_{2n_k-2} + d_{2n_k-1} \end{aligned} \tag{24}$$

Letting $k \to \infty$ in (24) and using (21) we get

$$\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k}\right) = c \tag{25}$$

Now,

$$d\left(y_{2m_k}, y_{2n_k+1}\right) \leq d\left(y_{2n_k+1}, y_{2n_k}\right) + d\left(y_{2n_k}, \ y_{2m_k}\right) \tag{26}$$

and

$$-d\left(y_{2n_k+1}, y_{2n_k}\right) + d\left(y_{2n_k}, \ y_{2m_k}\right) \leq d\left(y_{2m_k}, y_{2n_k+1}\right) \tag{27}$$

hence, combining (26) and (27) , we obtain

$$\begin{aligned} -d_{2n_k} + d\left(y_{2n_k}, \ y_{2m_k}\right) &\leq d\left(y_{2m_k}, y_{2n_k+1}\right) \\ &\leq d_{2n_k} + d\left(y_{2n_k}, \ y_{2m_k}\right) \end{aligned} \tag{28}$$

Letting $k \to \infty$ in (28) and using (25) and (22) we get

$$c \leq \lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k+1}\right) \leq c$$

Thus

$$\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k+1}\right) = c \tag{29}$$

With the same way as in (26) , (27) and (28) we get

$$\lim_{k\to\infty} d\left(y_{2n_k}, y_{2m_k+1}\right) = c \tag{30}$$

and

$$\lim_{k\to\infty} d\left(y_{2n_k+1}, y_{2m_k+1}\right) = c \tag{31}$$

Now by (2) and (3) we have

$$\begin{aligned}(32)\qquad \psi(d(Ax_{2m_k}, Bx_{2n_k+1})) &= \psi(d(y_{2m_k}, y_{2n_k+1}))\\ &\leq \psi(\theta(x_{2m_k}, x_{2n_k+1})) - \varphi(\theta(x_{2m_k}, x_{2n_k+1}))\end{aligned}$$

$$\begin{aligned}(33)\qquad \theta(x_{2m_k}, x_{2n_k+1}) &= \lambda d\left(y_{2m_k}, y_{2m_k-1}\right) + \mu d\left(y_{2n_k}, y_{2n_k-1}\right)\\ (34)\qquad &\quad +\delta d\left(y_{2n_k}, y_{2m_k-1}\right)\\ &= \lambda d_{2m_{k-1}} + \mu d_{2n_{k-1}} + \delta d\left(y_{2n_k}, y_{2m_{k-1}}\right)\end{aligned}$$

Letting $k \to \infty$ in (33) and using (29), (30) and (31) we obtain

$$\lim_{k\to\infty} \theta(x_{2m_k}, x_{2n_k+1}) = \delta c$$

Combining (32) , (33) and the property of φ and ψ we obtain

$$\begin{aligned}\psi\left(c\right) &\leq \psi\left(\delta c\right) - \varphi\left(\delta c\right)\\ &\leq \psi\left(c\right) - \varphi\left(\delta c\right)\end{aligned}$$

thus $\varphi\left(\delta c\right) = 0$, since $\delta > 0$, then from the property of φ , it follows that $c = 0$, which is a contradiction with the fact that $c \gg 0$. Hence $\{y_{2n}\}$ is a Cauchy sequence. Letting $n \to \infty$ and $m \to \infty$ in

$$\left\|d\left(y_{2n+1}, y_{2m+1}\right) - d\left(y_{2n}, y_{2m}\right)\right\| \leq \|d_{2n}\| + \|d_{2m}\|$$

we get $\lim\limits_{n,m\to\infty} d\left(y_{2n+1}, y_{2m+1}\right) = 0$. Hence $\{y_{2n+1}\}$ is a Cauchy sequence. Thus $\{y_n\}$ is a Cauchy sequence and we have

$$(35)\qquad \lim_{n,m\to\infty} d\left(y_n, y_m\right) = 0.$$

Case 1. $S\left(X\right)$ is complete. Since $\{y_{2n+1}\}$ is a Cauchy sequence in a complete metric space $\left(S\left(X\right), d\right)$, therefore there exists $u \in X$ such that $z = S(u) \in S\left(X\right)$. Since $\{y_n\}$ is a Cauchy sequence in (X,d) and $y_{2n+1} \to z$, it follows that $y_{2n} \to z$. Now,

$$d(Au,Su) \leq d(Au,Bx_{2n+1}) + d(Bx_{2n+1},Su) = d(Au,Bx_{2n+1}) + d(y_{2n+1},Su) \tag{36}$$

using (3) with $x = u$ and $y = x_{2n+1}$. we have

$$\theta(u,x_{2n+1}) = \lambda d(Au,Su) + \mu d(Bx_{2n+1},Tx_{2n+1}) + \delta d(Su,Tx_{2n+1}) \tag{37}$$

$$= \lambda d(Au,Su) + \mu d(y_{2n+1},y_{2n}) + \delta d(Su,y_{2n}) \tag{38}$$

using (36), (37) and the property of φ, we get

$$\begin{aligned} \psi(d(Au,Su)) &\leq \psi(d(Au,Bx_{2n+1}) + d(y_{2n+1},Su)) \\ &\leq \psi(\theta(u,x_{2n+1})) + \psi(d(y_{2n+1},Su)) - \varphi(\theta(u,x_{2n+1})) \\ &= \psi(\lambda d(Au,Su) + \mu d(y_{2n+1},y_{2n}) + \delta d(Su,y_{2n}) \\ &\quad \psi(d(y_{2n+1},Su)) - \varphi(\theta(u,x_{2n+1})). \end{aligned}$$

As $y_{2n} \to Su = z$ and $y_{2n+1} \to Su = z$, letting $n \to \infty$ and since φ and ψ are continuous, we get

$$\psi(d(Au,Su)) \leq \psi(d(Au,Su)) - \varphi(\lambda d(Au,Su))$$

thus $\varphi(\lambda d(Au,Su)) = 0$ and we get $Au = Su$. Thus $Au = Su = z$. Therefore z is a point of coincidence of the pair (A,S). Since (A,S) is weakly compatible, $Az = Sz$.

Step 2. As $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Au = Tv$. So

$$z = Au = Su = Tv \tag{39}$$

Taking $x = u$ and $y = v$ in (3) we have

$$\begin{aligned} \theta\,(u,v)) &= \lambda d(Au,Su) + \mu d(Bv,Tv) \\ &\quad + \delta d(Su,Tv) \end{aligned}$$

using (39) we have

$$\begin{aligned} \theta(z,v) &= \mu d(z,Bv) \\ &< d(z,Bv) \end{aligned}$$

using (2) and the property of ψ we get

$$\psi\left(d(z,Bv)\right) \leq \psi\left(d(z,Bv)\right) - \varphi\left(\mu d(z,Bv)\right)$$

Hence $\varphi\left(\mu d(z,Bv)\right) = 0$ and $d(z,Bv) = 0$, thus $z = Bv$. As the pair (B,T) is weak compatible we get $Bz = Tz$. Taking $x = z$, $y = z$ in (3) and using $Az = Sz, Bz = Tz$ we get

$$\begin{aligned} \theta(z,z) &= \lambda d(Az,Sz) + \mu d(Bz,Tz) + \delta d(Sz,Tz) \\ &\leq \delta d(Az,Bz) \\ &< d(Az,Bz) \end{aligned}$$

using (2), we obtain

$$\psi\left(d(Az,Bz)\right) \leq \psi\left(d(Az,Bz)\right) - \varphi\left(\delta d(Az,Bz)\right)$$

Hence $\varphi\left(\delta d(Az,Bz)\right) = 0$ and $d(Az,Bz) = 0$, thus $Az = Bz$ and we have $Az = Bv$ $= z = Tz$. Thus z is a point of coincidence of the four self maps A, B, S and T

Case 2. $T\left(X\right)$ is complete. The proof of this case is similar to case 1.

Case 3. $A\left(X\right)$ is complete. $\{y_{2n} = Ax_{2n}\}$ is a Cauchy sequence in $A\left(X\right)$ which is complete. Hence $y_{2n} \to z = Aw$ for some $w \in X$. As $A\left(X\right) \subset T(X)$ there exists $p \in X$ such that $z = Aw = Tp$. It follows from case 2 that $Az = Bz = Tz = Sz$. Thus also in this case, the maps A, B, S and T have a common point of coincidence.

Case 4. $B\left(X\right)$ is complete. The proof of this case is similar to case 3.

Step 3. We have $z = Bz = Sz$. Let $Au = Su$ be another point of coincidence of the pair $\left(A,S\right)$. Now

$$\begin{aligned} d(z,Au) &\leq d(z,Bx_{2n+1}) + d(Bx_{2n+1},Au) \\ &\leq d(z,y_{2n+1}) + d(y_{2n+1},Au) \end{aligned}$$

Taking $x = u$ and $y = x_{2n+1}$ in (3) we get

$$\theta\left(u, x_{2n+1}\right) = \lambda d(Au, Su) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Su, Tx_{2n+1})$$

$$\begin{aligned} d(z, Au) &\leq d(z, y_{2n+1}) + \lambda d(Au, Su) + \mu d(Bx_{2n+1}, Tx_{2n+1}) \\ &\quad + \delta d(Su, Tx_{2n+1}) \\ &\leq d(z, y_{2n+1}) + \mu d_{2n} + \delta d(Au, y_{2n+1}) \end{aligned}$$

letting $n \to \infty$, we get

$$\begin{aligned} d(z, Au) &\leq \delta d(Au, z) \\ &< d(Au, z) \end{aligned}$$

using (2) and the property of ψ we get

$$\psi\left(d(z, Au)\right) \leq \psi\left(d(z, Au)\right) - \varphi\left(\left(\delta\right) d(z, Av)\right)$$

which implies that $\varphi\left(\delta d(z, Au)\right) = 0$ and $d(z, Au) = 0$, so $z = Au$. Hence the point of coincidence of (A, S) is unique. As the pair (A, S) is weakly compatible by proposition 14, z is the unique common fixed point of A, S. Hence $z = Az = Bz = Tz = Sz$ is the unique common fixed point of A, B, T and S.

Taking $T = S$ in Theorem 20 we have the following corollary for three self mappings.

Corollary 3.1 Let (X, d) be a cone metric space with regular cone P, such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$, with $x \neq y$. Let A, B and $T : X \to X$ be self mappings on X satisfying

$$A\left(X\right) \cup B\left(X\right) \subset S\left(X\right) \tag{40}$$

and the inequality

$$\psi(d(Ax, By)) \leq \psi(\theta(x, y)) - \varphi(\theta(x, y)) \text{ for all } x, y \in X \tag{41}$$

where $\varphi, \psi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ are continuous and monotone increasing functions with:

(i) $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$

(ii) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$

where

$$\theta(x, y) = \lambda d(Ax, Sx) + \mu d(By, Sy) + \delta d(Sx, Sy)$$

for some $\lambda, \mu, \delta \in]0, 1[$ with $\lambda + \mu + \delta < 1$. If one of $S(X)$ or $A(X) \cup B(X)$ is a complete subspace of X and the pairs (A, S) and (B, S) are weakly compatible, then A, B and S have a unique common fixed point in X.

Proof. The case in which $A(X) \cup B(X)$ is complete follows from the cases in which $A(X)$ or $B(X)$ is complete. For these we have $y_{2n} = Ax_{2n} \to z \in A(X)$ and $y_{2n} = Ax_{2n} \to z \in B(X)$ are discussed above in Cases 3 and 4 respectively.

Taking $B = A$ and $T = S$ in Theorem 20 we obtain

Corollary 3.2 Let (X, d) be a cone metric space with regular cone P, such that $d(x, y) \in \text{int}\, P$ for all $x, y \in X$, with $x \neq y$. Let A and $S : X \to X$ be self mappings on X satisfying

$$A(X) \subset S(X) \tag{42}$$

and the inequality

$$\psi(d(Ax, Ay)) \leq \psi(\theta(x, y)) - \varphi(\theta(x, y)) \text{ for all } x, y \in X \tag{43}$$

where $\varphi, \psi : \text{int}\, P \cup \{0\} \to \text{int}\, P \cup \{0\}$ are continuous and monotone increasing functions with

(i) $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$

(ii) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ where

$$\theta(x, y) = \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \delta d(Sx, Sy)$$

for some $\lambda, \mu, \delta \in]0, 1[$ with $\lambda + \mu + \delta < 1$. If one of $S(X)$ or $A(X)$ is a complete subspace of X and the pairs (A, S) is weakly compatible, then A and S have a unique common fixed point in X.

4 Application.

In this section we define a concept of integral with respect to a cone see ([12]) and introduce the Branciari's result in cone metric spaces.

Definition 4.1 ([12]) Suppose that P is a normal cone in E. We define

$$
\begin{aligned}
[a,b] &= \{x \in E : x = tb + (1-t)a, \text{ for some } a, b \in E \text{ and } t \in [0,1] \qquad (44)\\
[a,b[&= \{x \in E : x = tb + (1-t)a, \text{ for some } a, b \in E \text{ and } t \in 0,1[
\end{aligned}
$$

Definition 4.2 ([12]) The set $\{a = x_0, x_1, x_2, \cdots x_n = b\}$ is called a partition for $[a,b]$ if and only if the sets $\{[x_{i-1,}x_i]\}_{i=1}^n$ are pairwise disjoint and $[a,b] = \{[x_{i-1,}x_i]\}_{i=1}^n \cup \{b\}$.

Definition 4.3 ([12]) For each partition Q of $[a,b]$ and each increasing function $\phi : [a,b] \to P$, we define cone lower summation and cone upper summation as

$$
\begin{aligned}
L_n^{Con}(\phi, Q) &= \sum_{i=0}^{n-1} \phi(x_i) \|x_i - x_{i+1}\| \qquad (45)\\
U_n^{Con}(\phi, Q) &= \sum_{i=0}^{n-1} \phi(x_{i+1}) \|x_i - x_{i+1}\|
\end{aligned}
$$

respectively.

Definition 4.4 ([12]) Suppose that P is a normal cone in $E.\phi : [a,b] \to P$ is called an integrable function on $[a,b]$ with respect to cone P or to simplicity, Cone integrable function, if and only if for all partition Q of $[a,b]$

$$
\lim_{n\to\infty} L_n^{Con}(\phi, Q) = S^{Com} = \lim_{n\to\infty} U_n^{Con}(\phi, Q) \qquad (46)
$$

where S^{Com} must be unique.

We denote the common value S^{Com} by

$$
\int_a^b \phi(x) d_p(x) \text{ or to simplicity by } \int_a^b \phi d_p. \qquad (47)
$$

We denote the set of all cone integrable function $\phi : [a,b] \to P$ by $L^1(X,P)$.

Lemma 4.1 ([12]) $1-$ If $[a,b] \subset [a,c]$, then $\int_a^b f dp \leq \int_a^c f dp$, for $f \in L^1(X,P)$.

$2-$ $\int_a^b (\alpha f + \beta g) dp = \alpha \int_a^b f d_p + \beta \int_a^b g d_p$, for $f, g \in L^1(X,P)$ and $\alpha, \beta \in \mathbb{R}$.

Definition 4.5 ([12]) The function $\phi : P \to E$ is called sub-additive cone integrable function if and only if for $a, b \in P$

$$\int_O^{a+b} \phi d_p \leq \int_0^a \phi d_p + \int_0^b \phi d_p$$

Example 4.1 Let $E = X = \mathbb{R}, d(x,y) = |x-y|, P = [0,\infty)$, and $\phi(t) = \frac{1}{1+t}$ for all $t > 0$. Then for all $a, b \in P$, $\int_0^{a+b} \frac{1}{1+t} dt = \ln(a+b+1)$, $\int_0^a \frac{1}{1+t} dt = \ln(a+1)$, $\int_0^b \frac{1}{1+t} dt = \ln(b+1)$. Since $ab \geq 0$, then $a+b+1 \leq a+b+1+ab$. Therefore

$$\ln(a+b+1) \leq \ln(a+1) + \ln(b+1)$$

This shows that ϕ is an example of sub-additive cone integrable function.

Set $Y = \{f : P \to P,\ f$ is a non-vanishing map and a sub-additive cone integrable in each $[a,b] \subset P$ such that for each $\forall\ \epsilon \gg 0, \int_0^\epsilon f d_p \gg 0\}$.

Theorem 4.1 Let (X,d) be a cone metric space with regular cone P, such that $d(x,y) \in$ *int* P for all $x, y \in X$, with $x \neq y$. Let A, B, S and $T : X \to X$ be four maps satisfying

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X) \tag{48}$$

and the inequality

$$\int_0^{\psi(d(Ax,By))} \chi d_p \leq \int_0^{\psi(\theta(x,y))} \chi d_p - \int_0^{\varphi(\theta(x,y))} \chi d_p \text{ for all } x, y \in X \text{ and for } \chi \in Y \tag{49}$$

where $\psi : P \to P$ and $\varphi : \text{int } P \cup \{0\} \to \text{int } P \cup \{0\}$ are continuous functions with the following properties:

(i) ψ increasing

(ii) $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$

(iii) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ for all $t_1, t_2 \in P$

and

$$\theta(x,y) \quad = \quad \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty)$$

for some $\lambda, \mu, \delta \in]0,1[$ and $\lambda + \mu + \delta < 1$. If one of $A(X)$, $B(X)$, $T(X)$ or $S(X)$ is a complete subspace of X and the pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Setting $\Lambda(x) = \int_0^x \chi d_p$, then (49) gives

$$\Lambda(\psi(d(Ax, By))) \leq \Lambda(\psi(\theta(x, y))) - \Lambda(\varphi(\theta(x, y)))$$

which further can be written

$$\Gamma_1(d(Ax, By) \leq \Gamma_1(\theta(x, y)) - \Gamma_2((\theta(x, y))$$

where $\Gamma_1 = \Lambda \circ \psi$ and $\Gamma_2 = \Lambda \circ \varphi$. Hence by Theorem 20, we have the desired result.

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