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ROOK POLYNOMIALS OF YOUNG DIAGRAMS AND THEIR COMPLEMENTS ASSOCIATED WITH NUMERICAL SEMIGROUPS

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Abstract. In this study, using the bijective correspondence between numerical sets and Young diagrams, we express the rook polynomials of the Young diagrams corresponding to certain families of numerical semigroups and the complements of these Young diagrams.

Keywords: numerical semigroups; numerical sets; rook polynomials; Young diagrams

2020 AMS Subject Classification: 20M14; 20M20; 05A05; 05A10

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. For a given subset $A \subseteq \mathbb{N}_0$, the set A is called a numerical set if $0 \in A$ and the set $\mathbb{N}_0 \setminus A$ is finite. Moreover, if the numerical set A is closed under addition, that is, if $a + b \in A$ for all $a, b \in A$, then A is called a numerical semigroup. Numerical semigroups are used in the study of problems that arise in many areas of mathematics. In particular, they play an important role in understanding topics such as algebraic geometry, commutative algebra, combinatorics, and number theory, as well as the Frobenius problem and the analysis of linear Diophantine equations.

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A Young diagram is a left-aligned arrangement of boxes in columns, in which the number of boxes in each column is not less than that in the adjacent column to its right. Young diagrams are combinatorial objects with many applications in representation theory. There is a bijective correspondence between Young diagrams and numerical sets. This bijective correspondence was presented in [1]. Subsequently, many studies have been carried out using this bijective correspondence. For example, thanks to this bijective correspondence, various algebraic properties of numerical semigroups have been studied in detail and characterized in different works [2–9].

Let B be a chessboard consisting of squares arranged in rows and columns. Each square is located at the intersection of a row and a column. A rook is a chess piece that can move along rows and columns. The rook placement problem involves placing t rooks on the board B such that no two rooks share the same row or the same column. In other words, there is no more than one rook in any given row or column. We can also express this intuitive definition in the following form, which we will use throughout this article. For $e, f \in \mathbb{N}$, let $M_{e,f}[\mathbb{F}_2]$ denote the set of $e \times f$ matrices whose entries are 0 and 1. Let B be an element of $M_{e,f}[\mathbb{F}_2]$, and write $B = (b_{i,j})$. Under this interpretation, a placement of t rooks on B corresponds to a selection of t independent 1's in B . By “independent,” we mean that no two or more of the selected 1's are in the same row or the same column. The rook polynomial of a board B is defined as given in Equation 1.

$$(1) \quad R_B(x) = \sum_{t=0}^{\infty} r_t(B) x^t.$$

Here, the coefficient $r_t(B)$ represents the number of placements of t rooks on B . For any board, we have $r_0(B) = 1$, and $r_1(B)$ is equal to the number of cells in B . Rook polynomials, one of the important tools of combinatorics, are used effectively in the theory of restricted permutations. In addition, they have been extensively studied in various areas of mathematics, such as group representation theory, enumeration of matrices over finite fields, and hypergeometric series.

Süer and Sezgin investigated the rook polynomials of Young diagrams associated with almost symmetric Arf numerical semigroups and the complements of these Young diagrams [6]. Süer determined the rook polynomials of Young diagrams associated with certain numerical semigroups generated by three elements [10]. In this study, we investigate the rook polynomials of Young diagrams associated with certain families of numerical semigroups and the complements

of these Young diagrams. Young diagrams are a useful tool for visualizing the structure of numerical semigroups and examining their various properties. Rook polynomials, on the other hand, are important combinatorial objects that allow us to count specific arrangements. The study of numerical semigroups through Young diagrams and rook polynomials contributes to a better understanding of the algebraic and combinatorial properties of these structures. This approach allows us to examine the structural properties of numerical semigroups from a different perspective and to uncover new relationships.

The structure of the article is as follows: In Section 2, we present the definitions and theorems necessary to arrive at our main results. In Section 3, we present the rook polynomials of Young diagrams associated with certain families of numerical semigroups and the complements of these Young diagrams (see Theorem 3.1, Theorem 3.3, Theorem 3.4, 3.6, and Theorem 3.7).

2. PRELIMINARIES

Let A be a numerical semigroup. Every numerical semigroup is finitely generated. Hence, there exist finitely many elements $a_1, \dots, a_n \in A$ such that

$$(2) \quad A = \left\{ \sum_{i=1}^n \mu_i a_i \mid a_i \in A, \mu_i \in \mathbb{N}_0, n \in \mathbb{N} \right\}.$$

The elements $a_1, \dots, a_n$ are called generators of A , and A is denoted by $A = \langle a_1, \dots, a_n \rangle$. If no proper subset of $a_1, \dots, a_n$ generates A , then A is said to be minimally generated by $a_1, \dots, a_n$.

Let A be a numerical set. The elements of $\mathbb{N}_0$ that do not belong to A are called the gaps of A , and the set of all gaps of A is denoted by $G(A)$. The number of elements in $G(A)$ is called the genus of A and is denoted by $g(A)$. If $G(A) = \emptyset$, then $\mathbb{N}_0$ itself is considered a numerical set. Otherwise, if $G(A) \neq \emptyset$, A is referred to as a proper numerical set. Additionally, the largest element of $G(A)$ is called the Frobenius number of A and is denoted by $F(A)$. The number $C(A) = F(A) + 1$ is called the conductor of A . Elements smaller than $C(A)$ are called the small elements of A . If A has n small elements, $0 = a_0 < a_1 < \dots < a_{n-1}$ and is expressed as $A = \{0, a_1, \dots, a_{n-1}, a_n = C(A), \rightarrow\}$. In this notation, the arrow indicates that all subsequent integers are elements of A .

Let A be a numerical set. We can draw a Young diagram associated with A . This Young diagram is obtained by drawing a continuous, polygonal path starting from the origin in $\mathbb{Z}^2$. Starting with $a = 0$, the following steps are performed:

- (1) if $a \in A$, a unit-length line segment is drawn to the right.
- (2) if $a \notin A$, a unit-length line segment is drawn upward.

These operations are repeated until $a = F(A)$. The polygonal path obtained at the end of this process, the region bounded by the y -axis and the line $y = g(A)$ is a Young diagram formed by unit squares. This Young diagram is called the Young diagram associated with A , and is represented by Y_A . It is clear that each Young diagram corresponds to a unique proper numerical set A . Therefore, the mapping $A \rightarrow Y_A$ defines a bijection between the set of proper numerical sets and the set of Young diagrams.

Example 2.1. Let $A = \{0, 3, 6, 8, \rightarrow\}$ be a numerical semigroup. The set of gaps of A is $G(A) = \{1, 2, 4, 5, 7\}$, the genus of A is $g(A) = 5$, and the Frobenius number of A is $F(A) = 7$. The Young diagram associated with A is shown in Figure 1.

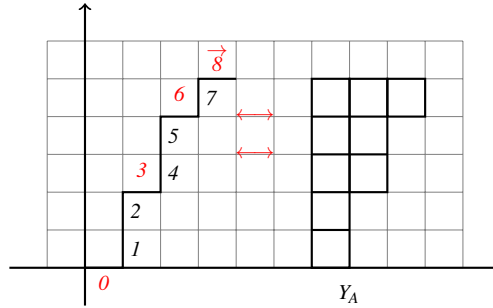


FIGURE 1. The Young diagram Y_A associated with the numerical semigroup $A = \{0, 3, 6, 8, \rightarrow\}$

Theorem 2.2. [11, In Chapter 7] Let $R_{e,f}(x)$ denote the rook polynomial of a rectangular $e \times f$ board. Hence,

$$(3) \quad R_{e,f}(x) = \sum_{t=0}^{\min(e,f)} \binom{e}{t} \binom{f}{t} t! x^t.$$

Definition 2.3. [12] Let B be a board. A subboard N of B has the property $n_{i,j} = b_{\varepsilon_1(i), \varepsilon_2(j)}$ with injective mappings ε_1 and ε_2 from the row and column indices of N to the row and column

indices of B . The board N with this property is represented by a matrix $(n_{i,j})$ over $\mathbb{F}_2$. Furthermore, if i is in the image of ε_1 and j is in the image of ε_2 , N contains a row i and a column j of B .

Definition 2.4. [12] Let B be a board and N be a block of B . If the following conditions are satisfied, then we say that N is a subboard of B .

- If N contains any rows i, i' of B and does not contain any column j of B , then $b_{i,j} = b_{i',j}$.
- If N does not contain any row i of B and contains any columns j, j' of B , then $b_{i,j} = b_{i,j'}$.

Formally, an $s \times t$ block N is a subboard of B consisting of the intersection of a set of s rows and a set of t columns. All cells outside this block are the same. Keep in mind that each cell is actually a 1×1 block.

Definition 2.5. [12] Let B be a $e \times f$ board and N an $s \times t$ block of B . For $0 \leq j \leq \min(s, t)$, board $B_{N,j}$ is expressed as follows.

- Remove j of the s rows contained in N from B .
- Remove j of the t columns contained in N from B .
- Remove all the cells of N from B .

$B_{N,j}$ is termed the j^{th} inclusion board of B relative to N . $B_{N,j}$ is well-defined because the relevant rows and columns remain unchanged except for the removed block N .

Example 2.6. Let the gray region shown in Figure 2 be a subboard of the Young diagram Y_A given in Example 2.1, and denote this board by N .

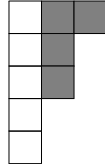
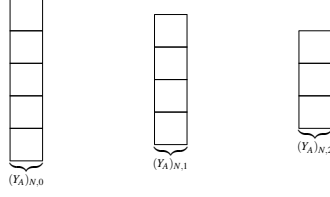


FIGURE 2. The subboard of the Young diagram $Y_{A=\{0,3,6,8,\rightarrow\}}$ (shown in gray)

For $0 \leq j \leq 2$, $(Y_A)_{N,j}$ is given as follows in Definition 2.5.



Theorem 2.7. [12, Theorem 3.5] Let B be a $e \times f$ board and N an $s \times t$ block of B . Let $r_k(N)$ be the coefficient of the x^k term in the rook polynomial of N . Moreover, let $B_{N,j}$ be the j^{th} inclusion board of B relative to N , for $0 \leq j \leq \min(s, t)$. In this case,

$$(4) \quad R_B(x) = \sum_{j=0}^{\min(s,t)} r_j(N) x^j R_{B_{N,j}}(x).$$

Example 2.8. Let's find the rook polynomial of the Young diagram Y_A given in Example 2.1. As stated in Example 2.6, let us consider the board N corresponding to the gray region, which is a subboard of the Young diagram Y_A . From Theorem 2.2, the rook polynomials of rectangles $(Y_A)_{N,0}$, $(Y_A)_{N,1}$, and $(Y_A)_{N,2}$ are found as follows.

$$R_{(Y_A)_{N,0}}(x) = 1 + 5x$$

$$R_{(Y_A)_{N,1}}(x) = 1 + 4x$$

$$R_{(Y_A)_{N,2}}(x) = 1 + 3x.$$

It can be easily calculated that the rook polynomial of N is $R_N(x) = 1 + 4x + 2x^2$.

From Theorem 2.7, the rook polynomial of Y_A is obtained as follows.

$$\begin{aligned} R_{Y_A}(x) &= \sum_{j=0}^2 r_j(N) x^j R_{(Y_A)_{N,j}}(x) \\ &= r_0(N) x^0 R_{(Y_A)_{N,0}}(x) + r_1(N) x^1 R_{(Y_A)_{N,1}}(x) + r_2(N) x^2 R_{(Y_A)_{N,2}}(x) \\ &= 1(1 + 5x) + 4x(1 + 4x) + 2x^2(1 + 3x) \\ &= 1 + 9x + 18x^2 + 6x^3. \end{aligned}$$

Notice that the rook polynomial of the Young diagram Y_A remains unchanged when the diagram is rotated by 90, 180, or 270 degrees clockwise or counterclockwise, or when a symmetry is applied with respect to the x or y axes.

Definition 2.9. [13] *The complement of a Young diagram is found by completing a rectangular grid with a width equal to the number of boxes in the first row and a height equal to the number of boxes in the first column.*

Throughout this paper, the complement of the Young diagram Y_A corresponding to a numerical set A will be denoted by $Y_{A'}$.

Example 2.10. *The complement of the Young diagram Y_A given in Example 2.1 is the gray region in Figure 3.*

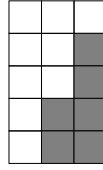


FIGURE 3. The complement of $Y_{A=\{0,3,6,8,\rightarrow\}}$ (shown in gray)

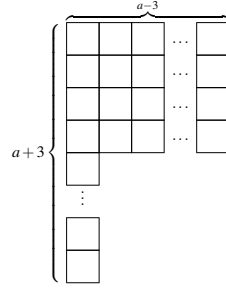
3. MAIN RESULTS

In this section, we determine the rook polynomials of several families of numerical semigroups that have been previously studied in the literature. For the first two families, it was shown that they are not λ -minimal with respect to a given integer partition λ [14]. The last family consists of Sally type numerical semigroups, for which the Gorenstein and symmetry properties were studied in [15].

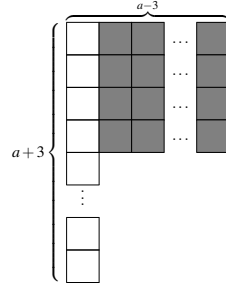
Theorem 3.1. *Let $a \geq 9$ and $A_a = \langle a, a+1, \dots, 2a-5 \rangle$. Then*

$$\begin{aligned}
 (1) \quad R_{Y_{A_a}}(x) &= 1 + (5a - 13)x + (10a^2 - 62a + 88)x^2 + (10a^3 - 108a^2 + 362a - 360)x^3 \\
 &\quad + (5a^4 - 82a^3 + 475a^2 - 1118a + 840)x^4 \\
 &\quad + (a^5 - 23a^4 + 201a^3 - 817a^2 + 1478a - 840)x^5 \\
 (2) \quad R_{Y_{A'_a}}(x) &= \sum_{t=0}^{a-4} \binom{a-1}{t} \binom{a-4}{t} t! x^t.
 \end{aligned}$$

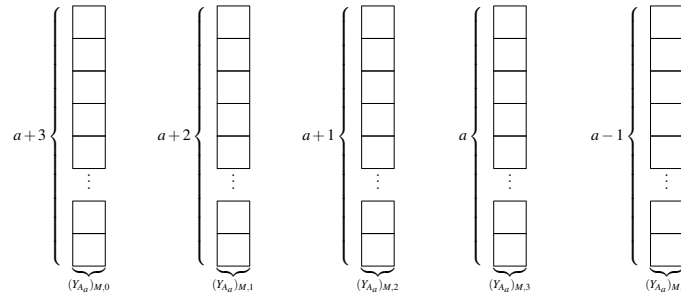
Proof. Let $a \geq 9$ and $A_a = \langle a, a+1, \dots, 2a-5 \rangle$. In this case, it is clear that $A_a = \langle a, a+1, \dots, 2a-5 \rangle = \{0, a, a+1, \dots, 2a-5, 2a, \rightarrow\}$. When we draw the Young diagram Y_{A_a} of the numerical semigroup A_a , we obtain the shape shown in Figure 4.

FIGURE 4. The Young diagram $Y_{A_a} = \langle a, a+1, \dots, 2a-5 \rangle$

As shown in Figure 5, let the subboard of the Young diagram Y_{A_a} be the gray colored region, and denote this region by the board M .

FIGURE 5. The subboard of the Young diagram $Y_{A_a} = \langle a, a+1, \dots, 2a-5 \rangle$ (shown in gray)

The subboard M is a block of size $4 \times (a-4)$, and for $0 \leq j \leq 4$, the boards $(Y_{A_a})_{M,j}$ are obtained as follows according to Definition 2.5.



Applying Theorem 2.2, the rook polynomials of rectangles $(Y_{A_a})_{M,0}$, $(Y_{A_a})_{M,1}$, $(Y_{A_a})_{M,2}$, $(Y_{A_a})_{M,3}$, $(Y_{A_a})_{M,4}$ and M are obtained as follows.

$$(5) \quad R_{(Y_{A_a})_{M,0}}(x) = 1 + (a+3)x.$$

$$(6) \quad R_{(Y_{A_a})_{M,1}}(x) = 1 + (a+2)x.$$

$$(7) \quad R_{(Y_{A_a})_{M,2}}(x) = 1 + (a+1)x.$$

$$(8) \quad R_{(Y_{A_a})_{M,3}}(x) = 1 + ax.$$

$$(9) \quad R_{(Y_{A_a})_{M,4}}(x) = 1 + (a-1)x.$$

$$(10) \quad \begin{aligned} R_M(x) &= 1 + (4a-16)x + (6a^2-54a+120)x^2 + (4a^3-60a^2+296a-480)x^3 \\ &+ (a^4-22a^3+179a^2-638a+840)x^4. \end{aligned}$$

Using Theorem 2.7, the rook polynomial of the Young diagram Y_{A_a} is as in Equation 11.

$$(11) \quad \begin{aligned} R_{Y_{A_a}}(x) &= \sum_{j=0}^4 r_j(M) x^j R_{(Y_{A_a})_{M,j}}(x) \\ &= r_0(M) x^0 R_{(Y_{A_a})_{M,0}}(x) + r_1(M) x^1 R_{(Y_{A_a})_{M,1}}(x) + r_2(M) x^2 R_{(Y_{A_a})_{M,2}}(x) \\ &+ r_3(M) x^3 R_{(Y_{A_a})_{M,3}}(x) + r_4(M) x^4 R_{(Y_{A_a})_{M,4}}(x). \end{aligned}$$

Let's substitute Equations 5, 6, 7, 8, 9, and 10 into Equation 11. Then, $R_{Y_{A_a}}(x)$ becomes as shown in Equation 12.

$$(12) \quad \begin{aligned} R_{Y_{A_a}}(x) &= \sum_{j=0}^4 r_j(M) x^j R_{(Y_{A_a})_{M,j}}(x) \\ &= r_0(M) x^0 R_{(Y_{A_a})_{M,0}}(x) + r_1(M) x^1 R_{(Y_{A_a})_{M,1}}(x) + r_2(M) x^2 R_{(Y_{A_a})_{M,2}}(x) \\ &+ r_3(M) x^3 R_{(Y_{A_a})_{M,3}}(x) + r_4(M) x^4 R_{(Y_{A_a})_{M,4}}(x) \\ &= 1x^0 [1 + (a+3)x] + (4a-16)x [1 + (a+2)x] \\ &+ (6a^2-54a+120)x^2 [1 + (a+1)x] + (4a^3-60a^2+296a-480)x^3 [1 + ax] \\ &+ (a^4-22a^3+179a^2-638a+840)x^4 [1 + (a-1)x] \\ &= 1 + (5a-13)x + (10a^2-62a+88)x^2 + (10a^3-108a^2+362a-360)x^3 \\ &+ (5a^4-82a^3+475a^2-1118a+840)x^4 \\ &+ (a^5-23a^4+201a^3-817a^2+1478a-840)x^5. \end{aligned}$$

Now let's find the rook polynomial of the complement of Y_{A_a} . The complement of Y_{A_a} is the region shown in gray in Figure 6.

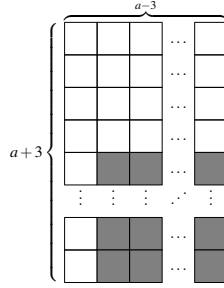


FIGURE 6. The complement of $Y_{A_a} = \langle a, a+1, \dots, 2a-5 \rangle$ (shown in gray)

From Theorem 2.2,

$$R_{Y_{A'_a}}(x) = \sum_{t=0}^{a-4} \binom{a-1}{t} \binom{a-4}{t} t! x^t.$$

□

Example 3.2. Let $a = 9$ and $A_9 = \langle 9, 10, 11, 12, 13 \rangle = \{0, 9, 10, 11, 12, 13, 18, \rightarrow\}$. The Young diagram Y_{A_9} corresponding to the numerical semigroup A_9 is as shown in Figure 7.

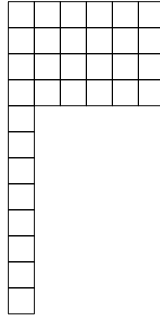


FIGURE 7. The Young diagram $Y_{A_9} = \langle 9, 10, 11, 12, 13 \rangle$

From Theorem 3.1, $R_{Y_{A_9}}(x) = 1 + 32x + 340x^2 + 1440x^3 + 2280x^4 + 960x^5$ and $R_{Y_{A'_9}}(x) = 1 + 40x + 560x^2 + 3360x^3 + 8400x^4 + 6720x^5$.

Theorem 3.3. Let $a \geq 4$ and $A = \langle 2a + 1, 2a + 2, \dots, 3a + 1 \rangle$. Let $R_T(x) = t_0(T) + t_1(T)x + t_2(T)x^2 + \dots + t_{a+1}(T)x^{a+1}$ be the rook polynomial of Y_A . Then, the coefficients of $R_{Y_A}(x)$ are given as follows:

$t_{a+1}(T) = 2a(a+1)!$ and for $0 \leq j \leq a$

$$t_j(T) = \sum_{k=0}^j m_k(M) \binom{a+2-k}{j-k} \binom{a}{j-k} (j-k)!.$$

Where $m_k(M) = \binom{1}{k} \binom{2a}{k} k!$ for $k = 0, 1$, and $m_k(M) = 0$ for $k > 1$.

Proof. Let $a \geq 4$ and $A = \langle 2a+1, 2a+2, \dots, 3a+1 \rangle$. It can easily be seen that $A = \langle 2a+1, 2a+2, \dots, 3a+1 \rangle = \{0, 2a+1, 2a+2, \dots, 3a+1, 4a+2, \rightarrow\}$. In this case, the Young diagram of A , denoted by Y_A , is as shown in Figure 8.

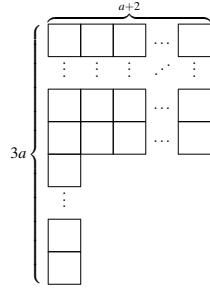


FIGURE 8. The Young diagram $Y_{A=\langle 2a+1, 2a+2, \dots, 3a+1 \rangle}$

The Young diagram Y_A in Figure 8 is first rotated 90 degrees counterclockwise; then, when we take its symmetry with respect to the x -axis, we obtain the board shown in Figure 9. Let us denote this board by T . Note that the rook polynomials of Y_A and T are identical.

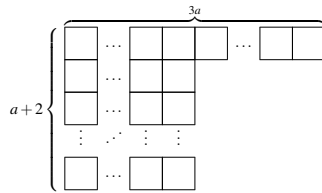
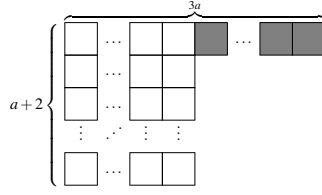
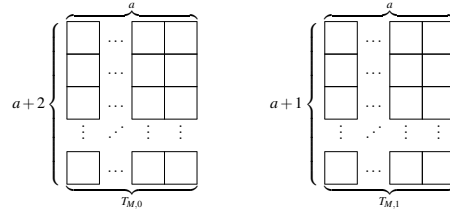


FIGURE 9. The board T

Let $R_T(x) = t_0(T) + t_1(T)x + t_2(T)x^2 + \dots + t_{a+1}(T)x^{a+1}$ be the rook polynomial of T .

Let the gray colored region shown in Figure 10 be a subboard of the board T . Denote this subboard by M .

The subboard M is a block of size $1 \times 2a$, and for $0 \leq j \leq 1$, the boards $T_{M,j}$ are obtained as follows according to Definition 2.5.

FIGURE 10. A subboard of the board T (shown in gray)

From Theorem 2.2, the rook polynomials of rectangles $T_{M,0}$, $T_{M,1}$, and M are found as follows.

$$(13) \quad R_{T_{M,0}}(x) = \sum_{t=0}^a \binom{a+2}{t} \binom{a}{t} t! x^t.$$

$$(14) \quad R_{T_{M,1}}(x) = \sum_{t=0}^a \binom{a+1}{t} \binom{a}{t} t! x^t.$$

$$(15) \quad R_M(x) = m_0(M) + m_1(M)x.$$

Where $m_k(M) = \binom{1}{k} \binom{2a}{k} k!$ for $k = 0, 1$.

From Theorem 2.7, the rook polynomial of T is as in Equation 16.

$$(16) \quad \begin{aligned} R_{Y_A}(x) = R_T(x) &= \sum_{j=0}^1 m_j(M) x^j R_{T_{M,j}}(x) \\ &= m_0(M) x^0 R_{T_{M,0}}(x) + m_1(M) x^1 R_{T_{M,1}}(x). \end{aligned}$$

Let's substitute Equations 13, 14, and 15 into Equation 16. Thus, $R_T(x)$ is given by Equation 17.

$$\begin{aligned} R_T(x) &= \sum_{j=0}^1 m_j(M) x^j R_{T_{M,j}}(x) \\ &= m_0(M) x^0 R_{T_{M,0}}(x) + m_1(M) x^1 R_{T_{M,1}}(x) \end{aligned}$$

$$\begin{aligned}
&= m_0(M) x^0 \left[\sum_{t=0}^a \binom{a+2}{t} \binom{a}{t} t! x^t \right] + m_1(M) x \left[\sum_{t=0}^a \binom{a+1}{t} \binom{a}{t} t! x^t \right] \\
&= \left[m_0(M) \binom{a+2}{0} \binom{a}{0} 0! \right] x^0 \\
&+ \left[m_0(M) \binom{a+2}{1} \binom{a}{1} 1! + m_1(M) \binom{a+1}{0} \binom{a}{0} 0! \right] x \\
&+ \left[m_0(M) \binom{a+2}{2} \binom{a}{2} 2! + m_1(M) \binom{a+1}{1} \binom{a}{1} 1! \right] x^2 + \dots \\
&\dots + \left[m_0(M) \binom{a+2}{a-1} \binom{a}{a-1} (a-1)! + m_1(M) \binom{a+1}{a-2} \binom{a}{a-2} (a-2)! \right] x^{a-1} \\
&+ \left[m_0(M) \binom{a+2}{a} \binom{a}{a} a! + m_1(M) \binom{a+1}{a-1} \binom{a}{a-1} (a-1)! \right] x^a \\
(17) \quad &+ \left[m_1(M) \binom{a+1}{a} \binom{a}{a} a! \right] x^{a+1}.
\end{aligned}$$

From Equation 17, we can easily write the coefficients of $R_T(x)$ as follows.

- $t_{a+1}(T) = 2a(a+1)!$
- $t_j(T) = \sum_{k=0}^j m_k(M) \binom{a+2-k}{j-k} \binom{a}{j-k} (j-k)!$ for $0 \leq j \leq a$.

Note that $m_k(M) = 0$ for $k > 1$, since it is not possible to place 2 or more rooks on the board M . □

Theorem 3.4. *Let $a \geq 4$ and $A = \langle 2a+1, 2a+2, \dots, 3a+1 \rangle$. Then*

$$R_{Y_{A_a'}}(x) = \sum_{t=0}^{a+1} \binom{2a}{t} \binom{a+1}{t} t! x^t.$$

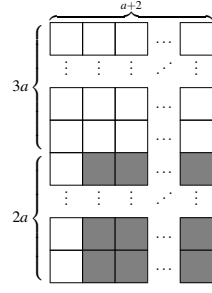
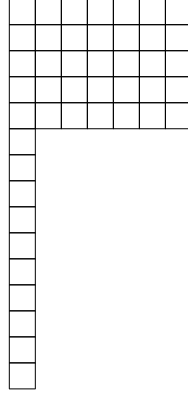
Proof. Let $a \geq 4$ and $A = \langle 2a+1, 2a+2, \dots, 3a+1 \rangle$. We know that the Young diagram Y_A corresponding to $A = \langle 2a+1, 2a+2, \dots, 3a+1 \rangle$ is shown in Figure 8. The complement of Y_A is the region shown in gray in Figure 11.

By Theorem 2.2,

$$R_{Y_{A_a'}}(x) = \sum_{t=0}^{a+1} \binom{2a}{t} \binom{a+1}{t} t! x^t.$$

□

Example 3.5. *Let $a = 5$ and $A = \langle 11, 12, 13, 14, 15, 16, \rangle = \{0, 11, 12, 13, 14, 15, 16, 22, \rightarrow\}$. In this case, the Young diagram Y_A corresponding to A is shown in Figure 12.*

FIGURE 11. The complement of $Y_{A=\langle 2a+1, 2a+2, \dots, 3a+1 \rangle}$ (shown in gray)FIGURE 12. The Young diagram $Y_{A=\langle 11, 12, 13, 14, 15, 16 \rangle}$

By Theorem 3.3, the rook polynomial of Y_A is obtained as follows.

For $k = 0, 1$, $m_0(M) = 1$ and $m_1(M) = 10$.

$$j = 0 \Rightarrow t_0(T) = 1$$

$$j = 1 \Rightarrow t_1(T) = \sum_{k=0}^1 m_k(M) \binom{7-k}{1-k} \binom{5}{1-k} (1-k)! = m_0(M) \binom{7}{1} \binom{5}{1} 1! + m_1(M) \binom{6}{0} \binom{5}{0} 0! = 1.35 + 10.1 = 45$$

$$j = 2 \Rightarrow t_2(T) = \sum_{k=0}^2 m_k(M) \binom{7-k}{2-k} \binom{5}{2-k} (2-k)! = m_0(M) \binom{7}{2} \binom{5}{2} 2! + m_1(M) \binom{6}{1} \binom{5}{1} 1! + m_2(M) \binom{5}{0} \binom{5}{0} 0! = 1.420 + 10.30 + 0.1 = 720$$

$$j = 3 \Rightarrow t_3(T) = \sum_{k=0}^3 m_k(M) \binom{7-k}{3-k} \binom{5}{3-k} (3-k)! = m_0(M) \binom{7}{3} \binom{5}{3} 3! + m_1(M) \binom{6}{2} \binom{5}{2} 2! + m_2(M) \binom{5}{1} \binom{5}{1} 1! + m_3(M) \binom{4}{0} \binom{5}{0} 0! = 1.2100 + 10.300 + 0.25 + 0.1 = 5100$$

$$j = 4 \Rightarrow t_4(T) = \sum_{k=0}^4 m_k(M) \binom{7-k}{4-k} \binom{5}{4-k} (4-k)! = m_0(M) \binom{7}{4} \binom{5}{4} 4! + m_1(M) \binom{6}{3} \binom{5}{3} 3! + m_2(M) \binom{5}{2} \binom{5}{2} 2! + m_3(M) \binom{4}{1} \binom{5}{1} 1! + m_4(M) \binom{3}{0} \binom{5}{0} 0! = 1.4200 + 10.1200 + 0.200 + 0.20 + 0.1 = 16200$$

$$j = 5 \Rightarrow t_5(T) = \sum_{k=0}^5 m_k(M) \binom{7-k}{5-k} \binom{5}{5-k} (5-k)! = m_0(M) \binom{7}{5} \binom{5}{5} 5! + m_1(M) \binom{6}{4} \binom{5}{4} 4! + m_2(M) \binom{5}{3} \binom{5}{3} 3! + m_3(M) \binom{4}{2} \binom{5}{2} 2! + m_4(M) \binom{3}{1} \binom{5}{1} 1! + m_5(M) \binom{2}{0} \binom{5}{0} 0! = 1.2520 + 10.1800 + 0.600 + 0.120 + 0.15 + 0.1 = 20520$$

$$t_6(T) = 10.6! = 7200.$$

Therefore, $R_{Y_A}(x) = 1 + 45x + 720x^2 + 5100x^3 + 16200x^4 + 20520x^5 + 7200x^6$. Furthermore, by Theorem 3.4, $R_{Y_{A'}}(x) = 1 + 60x + 1350x^2 + 14400x^3 + 75600x^4 + 181440x^5 + 151200x^6$.

Theorem 3.6. Let $a \geq 4$ and $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle$. Then

$$R_{Y_{A_a}}(x) = 1 + (2a+7)x + (a^2 + 10a + 7)x^2 + (4a^2 + 4a - 2)x^3 + (2a^2 - 4a + 2)x^4.$$

Proof. Let $a \geq 4$ and $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle$. In this case, it is clear that $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle = \{0, a, a+1, a+4, \dots, 2a+2, 2a+4, 2a+5, \rightarrow\}$. The Young diagram of A_a is shown in Figure 13.

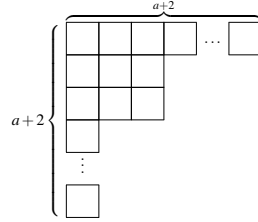


FIGURE 13. The Young diagram $Y_{A_a=\langle a, a+1, a+4, \dots, 2a-1 \rangle}$

As shown in Figure 14, let the subboard of the Young diagram Y_{A_a} be the gray colored region, and denote this region by the board M .

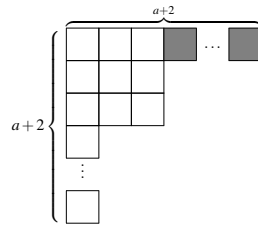
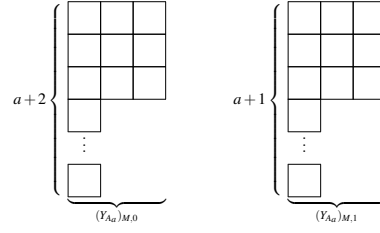


FIGURE 14. The subboard of the Young diagram $Y_{A_a=\langle a, a+1, a+4, \dots, 2a-1 \rangle}$ (shown in gray)

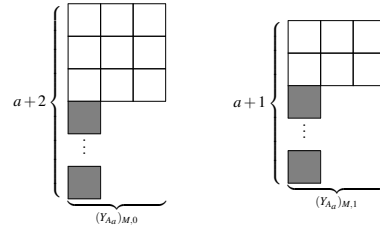
The subboard M is a block of size $1 \times (a-1)$, and for $0 \leq j \leq 1$, the boards $(Y_{A_a})_{M,j}$ are obtained as follows according to Definition 2.5.



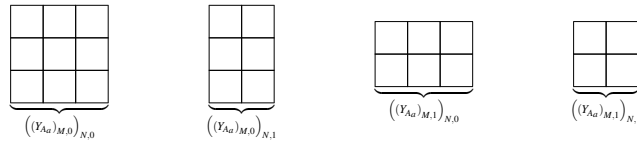
From Theorem 2.7, the rook polynomial of the Young diagram Y_{A_a} is as in Equation 18.

$$\begin{aligned}
 R_{Y_{A_a}}(x) &= \sum_{j=0}^1 r_j(M) x^j R_{(Y_{A_a})_{M,j}}(x) \\
 (18) \quad &= r_0(M) x^0 R_{(Y_{A_a})_{M,0}}(x) + r_1(M) x^1 R_{(Y_{A_a})_{M,1}}(x).
 \end{aligned}$$

Let the gray-shaded region below be a subboard of the boards $(Y_{A_a})_{M,0}$ and $(Y_{A_a})_{M,1}$, and denote it by N . Note that the rook polynomials of N and M are the same.



The subboard N is a block of size $(a-1) \times 1$, and for $0 \leq j \leq 1$, the boards $((Y_{A_a})_{M,0})_{N,j}$ and $((Y_{A_a})_{M,1})_{N,j}$ are obtained as follows according to Definition 2.5.



From Theorem 2.2, the rook polynomials of rectangles $((Y_{A_a})_{M,0})_{N,0}$, $((Y_{A_a})_{M,0})_{N,1}$, $((Y_{A_a})_{M,1})_{N,0}$, $((Y_{A_a})_{M,1})_{N,1}$, M and N are obtained as follows.

$$(19) \quad R_{((Y_{A_a})_{M,0})_{N,0}}(x) = 1 + 9x + 18x^2 + 6x^3.$$

$$(20) \quad R_{((Y_{A_a})_{M,0})_{N,1}}(x) = 1 + 6x + 6x^2.$$

$$(21) \quad R_{\left((Y_{A_a})_{M,1}\right)_{N,0}}(x) = 1 + 6x + 6x^2.$$

$$(22) \quad R_{\left((Y_{A_a})_{M,1}\right)_{N,1}}(x) = 1 + 4x + 2x^2.$$

$$(23) \quad R_M(x) = R_N(x) = 1 + (a-1)x.$$

From Theorem 2.7, the rook polynomials of the boards $(Y_{A_a})_{M,0}$ and $(Y_{A_a})_{M,1}$ are as given in Equations 24 and 25, respectively.

$$(24) \quad \begin{aligned} R_{(Y_{A_a})_{M,0}}(x) &= \sum_{j=0}^1 r_j(N) x^j R_{\left((Y_{A_a})_{M,0}\right)_{N,j}}(x) \\ &= r_0(N) x^0 R_{\left((Y_{A_a})_{M,0}\right)_{N,0}}(x) + r_1(N) x^1 R_{\left((Y_{A_a})_{M,0}\right)_{N,1}}(x). \end{aligned}$$

$$(25) \quad \begin{aligned} R_{(Y_{A_a})_{M,1}}(x) &= \sum_{j=0}^1 r_j(N) x^j R_{\left((Y_{A_a})_{M,1}\right)_{N,j}}(x) \\ &= r_0(N) x^0 R_{\left((Y_{A_a})_{M,1}\right)_{N,0}}(x) + r_1(N) x^1 R_{\left((Y_{A_a})_{M,1}\right)_{N,1}}(x). \end{aligned}$$

Substitute Equations 19, 20, 21, 22, and 23 into Equation 24 and 25, we obtain Equations 26 and 27.

$$(26) \quad \begin{aligned} R_{(Y_{A_a})_{M,0}}(x) &= \sum_{j=0}^1 r_j(N) x^j R_{\left((Y_{A_a})_{M,0}\right)_{N,j}}(x) \\ &= r_0(N) x^0 R_{\left((Y_{A_a})_{M,0}\right)_{N,0}}(x) + r_1(N) x^1 R_{\left((Y_{A_a})_{M,0}\right)_{N,1}}(x) \\ &= 1x^0 (1 + 9x + 18x^2 + 6x^3) + (a-1)x (1 + 6x + 6x^2) \\ &= 1 + (a+8)x + (6a+12)x^2 + 6ax^3. \end{aligned}$$

$$(27) \quad \begin{aligned} R_{(Y_{A_a})_{M,1}}(x) &= \sum_{j=0}^1 r_j(N) x^j R_{\left((Y_{A_a})_{M,1}\right)_{N,j}}(x) \\ &= r_0(N) x^0 R_{\left((Y_{A_a})_{M,1}\right)_{N,0}}(x) + r_1(N) x^1 R_{\left((Y_{A_a})_{M,1}\right)_{N,1}}(x) \\ &= 1x^0 (1 + 6x + 6x^2) + (a-1)x (1 + 4x + 2x^2) \\ &= 1 + (a+5)x + (4a+2)x^2 + (2a-2)x^3. \end{aligned}$$

Substituting Equations 23, 26, and 27 into Equation 18, the rook polynomial of Y_{A_a} is obtained as given in Equation 28.

$$\begin{aligned}
 R_{Y_{A_a}}(x) &= \sum_{j=0}^1 r_j(M) x^j R_{(Y_{A_a})_{M,j}}(x) \\
 &= r_0(M) x^0 R_{(Y_{A_a})_{M,0}}(x) + r_1(M) x^1 R_{(Y_{A_a})_{M,1}}(x) \\
 &= 1x^0 (1 + (a+8)x + (6a+12)x^2 + 6ax^3) \\
 &\quad + (a-1)x (1 + (a+5)x + (4a+2)x^2 + (2a-2)x^3) \\
 &= 1 + (2a+7)x + (a^2 + 10a + 7)x^2 + (4a^2 + 4a - 2)x^3 \\
 &\quad + (2a^2 - 4a + 2)x^4.
 \end{aligned}
 \tag{28}$$

□

Theorem 3.7. Let $a \geq 4$ and $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle$. Let $R_{Y_{A'}}(x) = m_0(M) + m_1(M)x + m_2(M)x^2 + \dots + m_{a+1}(M)x^{a+1}$ be the rook polynomial of $Y_{A'}$. Then, the coefficients of $R_{Y_{A'}}(x)$ are given as follows:

$m_a(M) = (a-1)!(a^3 - 2a^2 + 3a - 2)$, $m_{a+1}(M) = (a-1)!(a^2 - 3a + 2)$ and for $0 \leq j \leq a-1$

$$m_j(M) = \sum_{k=0}^j n_k(N) \binom{a+1-k}{j-k} \binom{a-1}{j-k} (j-k)!.$$

Where $n_k(N) = \binom{a-1}{k} \binom{2}{k} k!$ for $k = 0, 1, 2$, and $n_k(N) = 0$ for $k > 2$.

Proof. Let $a \geq 4$ and $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle$. We know that the Young diagram Y_{A_a} corresponding to $A_a = \langle a, a+1, a+4, \dots, 2a-1 \rangle$ is shown in Figure 13.

The complement of Y_{A_a} is the gray-shaded region shown in Figure 15.

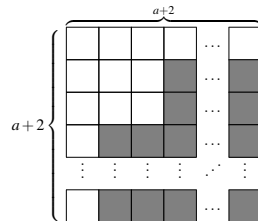


FIGURE 15. The complement of $Y_{A_a} = \langle a, a+1, a+4, \dots, 2a-1 \rangle$ (shown in gray)

By rotating the board $Y_{A'_a}$ by 90 degrees clockwise and then taking its symmetry with respect to the x -axis, we obtain the board shown in Figure 16. Let us denote this board by M . Furthermore, let N denote the gray-shaded region, which is a subboard of M . Note that the rook polynomials of $Y_{A'_a}$ and M are identical.

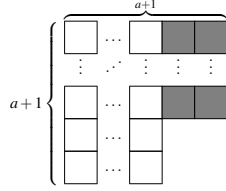
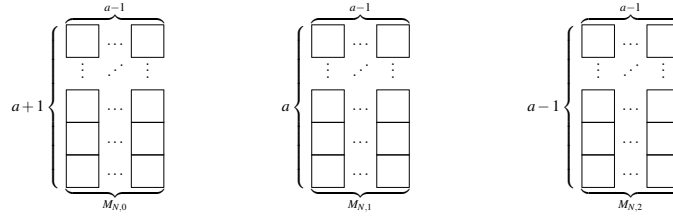


FIGURE 16. The subboard of M (shown in gray)

Let $R_M(x) = m_0(M) + m_1(M)x + m_2(M)x^2 + \cdots + m_{a+1}(M)x^{a+1}$ be the rook polynomial of M .

The subboard N is a block of size $(a-1) \times 2$, and for $0 \leq j \leq 2$, the boards $M_{N,j}$ are obtained as follows according to Definition 2.5.



From Theorem 2.2, the rook polynomials of rectangles $M_{N,0}$, $M_{N,1}$, $M_{N,2}$, and N are found as follows.

$$(29) \quad R_{M_{N,0}}(x) = \sum_{t=0}^{a-1} \binom{a+1}{t} \binom{a-1}{t} t! x^t.$$

$$(30) \quad R_{M_{N,1}}(x) = \sum_{t=0}^{a-1} \binom{a}{t} \binom{a-1}{t} t! x^t.$$

$$(31) \quad R_{M_{N,2}}(x) = \sum_{t=0}^{a-1} \binom{a-1}{t} \binom{a-1}{t} t! x^t.$$

$$(32) \quad R_N(x) = n_0(N) + n_1(N)x + n_2(N)x^2.$$

Where $n_k(N) = \binom{a-1}{k} \binom{2}{k} k!$ for $k = 0, 1, 2$.

From Theorem 2.7, the rook polynomial of M is as in Equation 33.

$$\begin{aligned}
 R_{Y_{A'}}(x) = R_M(x) &= \sum_{j=0}^2 n_j(N) x^j R_{M_{N,j}}(x) \\
 &= n_0(N) x^0 R_{M_{N,0}}(x) + n_1(N) x^1 R_{M_{N,1}}(x) \\
 (33) \quad &+ n_2(N) x^2 R_{M_{N,2}}(x).
 \end{aligned}$$

Substituting Equations 29, 30, 31, and 32 into Equation 33, the rook polynomial of M is obtained as given in Equation 34.

$$\begin{aligned}
 R_M(x) &= \sum_{j=0}^2 n_j(N) x^j R_{M_{N,j}}(x) \\
 &= n_0(N) x^0 R_{M_{N,0}}(x) + n_1(N) x^1 R_{M_{N,1}}(x) + n_2(N) x^2 R_{M_{N,2}}(x) \\
 &= n_0(N) x^0 \left[\sum_{t=0}^{a-1} \binom{a+1}{t} \binom{a-1}{t} t! x^t \right] + n_1(N) x \left[\sum_{t=0}^{a-1} \binom{a}{t} \binom{a-1}{t} t! x^t \right] \\
 &+ n_2(N) x^2 \left[\sum_{t=0}^{a-1} \binom{a-1}{t} \binom{a-1}{t} t! x^t \right] \\
 &= \left[n_0(N) \binom{a+1}{0} \binom{a-1}{0} 0! \right] x^0 \\
 &+ \left[n_0(N) \binom{a+1}{1} \binom{a-1}{1} 1! + n_1(N) \binom{a}{0} \binom{a-1}{0} 0! \right] x \\
 &+ \left[n_0(N) \binom{a+1}{2} \binom{a-1}{2} 2! + n_1(N) \binom{a}{1} \binom{a-1}{1} 1! \right. \\
 &+ \left. n_2(N) \binom{a-1}{0} \binom{a-1}{0} 0! \right] x^2 + \dots \\
 &\dots + \left[n_0(N) \binom{a+1}{a-1} \binom{a-1}{a-1} (a-1)! + n_1(N) \binom{a}{a-2} \binom{a-1}{a-2} (a-2)! \right. \\
 &+ \left. n_2(N) \binom{a-1}{a-3} \binom{a-1}{a-3} (a-3)! \right] x^{a-1} \\
 &+ \left[n_1(N) \binom{a}{a-1} \binom{a-1}{a-1} (a-1)! + n_2(N) \binom{a-1}{a-2} \binom{a-1}{a-2} (a-2)! \right] x^a \\
 (34) \quad &+ \left[n_2(N) \binom{a-1}{a-1} \binom{a-1}{a-1} (a-1)! \right] x^{a+1}.
 \end{aligned}$$

From Equation 34, we can easily write the coefficients of $R_M(x)$ as follows.

- $m_a(M) = (a-1)! (a^3 - 2a^2 + 3a - 2)$
- $m_{a+1}(M) = (a-1)! (a^2 - 3a + 2)$
- $m_j(M) = \sum_{k=0}^j n_k(N) \binom{a+1-k}{j-k} \binom{a-1}{j-k} (j-k)! \text{ for } 0 \leq j \leq a-1.$

Note that $n_k(N) = 0$ for $k > 2$, since it is not possible to place 3 or more rooks on the board N . □

Example 3.8. Let $a = 5$ and $A_5 = \langle 5, 6, 9 \rangle = \{0, 5, 6, 9, 10, 11, 12, 14, \rightarrow\}$. In this case, the Young diagram corresponding to A_5 is shown in Figure 17.

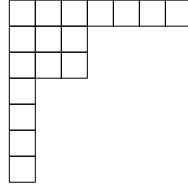


FIGURE 17. The Young diagram $Y_{A_5} = \langle 5, 6, 9 \rangle$

From Theorem 3.6, we have $R_{Y_{A_5}}(x) = 1 + 17x + 82x^2 + 118x^3 + 32x^4$, and from Theorem 3.7, $R_{Y_{A'_5}}(x)$ is given as follows.

For $k = 0, 1, 2$, $n_0(N) = 1$, $n_1(N) = 8$, and $n_2(N) = 12$.

$$j = 0 \Rightarrow m_0(M) = 1$$

$$j = 1 \Rightarrow m_1(M) = \sum_{k=0}^1 n_k(N) \binom{6-k}{1-k} \binom{4}{1-k} (1-k)! = n_0(N) \binom{6}{1} \binom{4}{1} 1! + n_1(N) \binom{5}{0} \binom{4}{0} 0! = 1 \cdot 24 + 8 \cdot 1 = 32$$

$$j = 2 \Rightarrow m_2(M) = \sum_{k=0}^2 n_k(N) \binom{6-k}{2-k} \binom{4}{2-k} (2-k)! = n_0(N) \binom{6}{2} \binom{4}{2} 2! + n_1(N) \binom{5}{1} \binom{4}{1} 1! + n_2(N) \binom{4}{0} \binom{4}{0} 0! = 1 \cdot 180 + 8 \cdot 20 + 12 \cdot 1 = 352$$

$$j = 3 \Rightarrow m_3(M) = \sum_{k=0}^3 n_k(N) \binom{6-k}{3-k} \binom{4}{3-k} (3-k)! = n_0(N) \binom{6}{3} \binom{4}{3} 3! + n_1(N) \binom{5}{2} \binom{4}{2} 2! + n_2(N) \binom{4}{1} \binom{4}{1} 1! + n_3(N) \binom{3}{0} \binom{4}{0} 0! = 1 \cdot 480 + 8 \cdot 120 + 12 \cdot 16 + 0 \cdot 1 = 1632$$

$$j = 4 \Rightarrow m_4(M) = \sum_{k=0}^4 n_k(N) \binom{6-k}{4-k} \binom{4}{4-k} (4-k)! = n_0(N) \binom{6}{4} \binom{4}{4} 4! + n_1(N) \binom{5}{3} \binom{4}{3} 3! + n_2(N) \binom{4}{2} \binom{4}{2} 2! + n_3(N) \binom{3}{1} \binom{4}{1} 1! + n_4(N) \binom{2}{0} \binom{4}{0} 0! = 1 \cdot 360 + 8 \cdot 240 + 12 \cdot 72 + 0 \cdot 12 + 0 \cdot 1 = 3144$$

$$m_5(M) = 2112$$

$$m_6(M) = 288.$$

$$\text{Therefore, } R_{Y_{A'_5}}(x) = 1 + 32x + 352x^2 + 1632x^3 + 3144x^4 + 2112x^5 + 288x^6.$$

4. CONCLUSION

In this study, the rook polynomials of Young diagrams associated with certain families of numerical semigroups and the complements of these Young diagrams have been presented. This study may shed light on further investigations into the relationship between numerical semigroups and rook polynomials.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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