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NOTIONS OF AMENABILITY ON SEMIGROUP ALGEBRAS

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Abstract. We give a survey of results and problems concerning notions of amenability in semigroup algebras. We also provide the relationship that exits between these notions of amenability and some important structures of the semigroups.

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1. Introduction

This article is a survey of the notions of amenability results around the semigroup algebras which are known to the author. The groundwork for amenability of Banach algebras was laid by Johnson in [28], and ever since its introduction, the concept of amenability has occupied an important place in research in Banach algebras. A Banach algebra A is amenable if every continuous derivation $D : A \to X'$ is inner for every Banach A-bimodule X. In particular, if G is a locally compact group then $L^1(G)$ is amenable (as a Banach algebra) if and only if G is amenable as a topological group [28].

It has been realized that in many instances amenability is too restrictive and so does not allow for the development of a rich general theory and does not include a variety of interesting examples. For this reason by relaxing some of the constraints in the definition of amenability, different notions of amenability have been introduced in the recent years. Most notable that we shall consider in this survey article are; the notions of

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(i) weak amenability ([29], [31]);

(ii) approximate amenability ([16],[19],[8], [33]);

(iii) pseudo-amenability ([18]);

(iv) character amenability ([37], [27]);

(v) approximate character amenability ([35]);

(vi) character pseudo - amenability ([38], [36]).

For details on various notions of amenability on general Banach algebras see [32].

Different notions of amenability of the semigroup algebra $\ell^1(S)$ have been widely studied in the recent years. Some structural implications of these notions of amenability of $\ell^1(S)$ for an arbitrary semigroup S have also been investigated by various authors. For most of these notions of amenability, it is not known in general when $\ell^1(S)$ possess these notions of amenability, only partial results were known in the literature, see ([1], [2], [7], [10], [11], [13], [14], [15], [22], [21], [41], [40]).

The purpose of this note is to give an overview of what has been done so far on amenability and different notions of amenability on semigroup algebra $\ell^1(S)$ of semigroup S, and raise some problems related to the characterization of the semigroup S for which $\ell^1(S)$ possesses these notions of amenability.

2. Preliminaries

First, we recall some standard notions; for further details, see [6] and [32].

Let A be an algebra and let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, for $x \in X$, $\delta_x : a \to a \cdot x - x \cdot a \quad (a \in A)$ is a derivation; derivations of this form are called the *inner derivations*.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \quad \|x \cdot a\| \leq k\|a\|\|x\| \quad (a \in A, x \in X).$$

By renorming X, we may suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations defined by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X. Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathcal{N}$. We take $X^{(0)} = X$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A,X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A,X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A,X) = \mathcal{Z}^1(A,X)/\mathcal{N}^1(A,X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A,X') = \{0\}$ for each Banach A-bimodule X and *weakly amenable* if $\mathcal{H}^1(A,A') = \{0\}$. For instance, the group algebra $L^1(G)$ of a locally compact group G is always weakly amenable [29], and is amenable if and only if G is amenable in the classical sense [28]. Also, a C^*-algebra is always weakly amenable [25] and is amenable if and only if it is nuclear ([5],[25]). For example, $C_0(\Omega)$ for any locally compact space Ω and $K(H)$ are amenable. But $B(H)$ is not nuclear, so it is not amenable.

Let A be a Banach algebra and let X be a Banach A-bimodule. A derivation $D : A \to X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = lim_\alpha(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. The Banach algebra A is

(i) *approximately contractible* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is approximately inner;

(ii) *approximately weakly amenable* if every continuous derivation $D : A \to A'$ is approximately inner;

(iii) *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.

Certainly every amenable Banach algebra is approximately amenable.

We also recall from [16] that a Banach algebra A is

(i) *pseudo-amenable* if there is a net $(m_\alpha) \subset A\hat{\otimes}A$, called an approximate diagonal for A, such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \textit{ and } \pi_A(m_\alpha) \cdot a \to a \quad (a \in A);$$

(ii) *pseudo-contractible* if it has a central approximate diagonal, i.e. an approxmate diagonal (m_α) satisfying $a \cdot m_\alpha = m_\alpha \cdot a$ for all $a \in A$ and all m_α.

Let A be a Banach algebra over $\mathcal{C}$ and $\varphi : A \to \mathcal{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathcal{C}$, and let Φ_A denote the character space of A (that is, the set of all characters on A).

3. Semigroups and semigroup algebras

In this section, we shall recall certain basic definitions and some properties of semigroups and semigroup algebras. For an introductory account of semigroup theory, see [26]. The standard reference for semigroup algebras is [7].

Definition 3.1. *A non-empty set S with an associative binary operation, denoted by*

$$(s,t) \to st, \quad S \times S \to S$$

is called a semigroup.

In the case where S is unital, the identity of S is denoted by e_S. Suppose S is non-unital, then the semigroup formed by adjoining an identity to S is denoted by $S^\sharp$. The semigroup S is abelian if $st = ts \quad (s,t \in S)$. A non-empty subset T of S is a subsemigroup if T is a semigroup for the the same binary operation on S.

Definition 3.2. *Let S be a semigroup, let $0 \in S$ such that*

$$s0 = 0s = 0 \quad (s \in S),$$

then 0 is a zero for the semigroup S.

Let S be a semigroup, and let 0 be an element not in S. Set $T = S \cup \{0\}$, with $s0 = 0s = 0 \quad (s \in S)$ and $0^2 = 0$. Then T is semigroup containing S as a subsemigroup. We say that T is formed by adjoining a zero to S and write $T = S^0$.

Definition 3.3. *Let S be a semigroup.*

(i) Let $s \in S$. An element $s^* \in S$ is called an inverse of s if

$$ss^*s = s \quad \textit{and} \quad s^*ss^* = s^*.$$

(ii) An element $s \in S$ is called regular if there exists $t \in S$ with $sts = s$.

(iii) An element $s \in S$ is called completely regular if there exists $t \in S$ with $sts = s$ and $ts = st$.

(iv) S is called regular if each $s \in S$ is a regular element.

(v) S is called completely regular if each $s \in S$ is a completely regular element.

(vi) S is called an inverse semigroup if S is regular and every element in S has a unique inverse.

(vii) An element $p \in S$ is called an idempotent if $p^2 = p$; the set of idempotents of S is denoted by $E(S)$.

Remark 3.4. (i) In general an inverse element will not be unique. We shall denote the inverse of an element s in an inverse semigroup by s^{-1}.

(ii) If $s \in S$ has an inverse, then s is regular. The converse is also true: Suppose there exists $t \in S$ such that $sts = s$. Set $u = tst$. Then

$$sus = s(tst)s = (sts)ts = sts = s \quad and \quad usu = (tst)s(tst) = t(sts)tst = t(sts)t = u.$$

Proposition 3.5. *A semigroup S is an inverse semigroup if and only if S is regular and the idempotents commutes.*

Proof.

This is Proposition 5.2.1 of [26].

Definition 3.6. *A semigroup S is called semilattice if S is commutative and $E(S) = S$.*

Definition 3.7. *Let S be a semigroup. For each $s \in S$, define $L_s(t) = st, R_s(t) = ts \quad (t \in S)$. An element $s \in S$ is left (resp. right) cancellable if L_s (resp. R_s) is injective on S, and s is cancellable if it is both left cancellable and right cancellable. The semigroup S is left (resp. right) cancellative if each element in S is left (resp. right) cancellable, and cancellative if each element is cancellable.*

Let S be a semigroup, we shall use the following notations and definitions from [39]. For $s, t \in S$, we define the sets

$$[st^{-1}] = \{u \in S : ut = s\}, \quad [t^{-1}s] = \{u \in S : tu = s\}.$$

For $s, t \in S$, we define a relation $\mathbb{D}$ on S by $s\mathbb{D}t$ if and only if there exists $x \in S$ with $Ss \cup \{s\} = Sx \cup \{x\}$ and $xS \cup \{x\} = tS \cup \{t\}$. This is an equivalence relation, see [39].

The following characterization of $\mathbb{D}-$ classes in an inverse semigroup is Proposition 2.11 of [39].

Proposition 3.8. *Let S be an inverse semigroup, and let $s, t \in S$. Then $s\mathbb{D}t$ if and only if there exists $x \in S$ with $s^{-1}s = xx^{-1}$ and $t^{-1}t = x^{-1}x$.*

Let S be an inverse semigroup, and $p \in E(S)$. We set

$$G_p = \{s \in S : ss^{-1} = s^{-1}s = p\},$$

where s^{-1} denote the inverse of s. Then G_p is a group with identity p and G_p contains any other subgroup of S with identity p. Thus G_p is called the maximal subgroup of S at p.

Let P be a partially ordered set. For $p \in P$, we define

$$(p] = \{x : x \leq p\} \text{ and } [p) = \{x : p \leq x\}.$$

Then P is locally finite if $(p]$ is finite for each $p \in P$, and is locally $C-$ finite for some constant $C > 1$ if $\mid (p] \mid < C$ for each $p \in P$. A partially ordered set that is $C-$ finite for some C is uniformly locally finite.

Let S be an inverse semigroup. Then S is [locally finite/ C-locally finite/ uniformly locally finite] if the partially ordered set $(E(S), \leq)$ has the corresponding property.

The following result is Proposition 2.14 of [39].

Proposition 3.9. *Let S be an inverse semigroup. Suppose that $(E(S), \leq)$ is [uniformly] locally finite. Then $(S, \leq)$ is [uniformly] locally finite.*

We next give the definitions of some important semigroups that we shall consider in this survey.

Definition 3.10. *Let S be a semigroup.*

(i) S is called a band semigroup if $S = E(S)$.

(ii) S is called a rectangular band semigroup if it is a band semigroup and for each $x, y \in S, xyx = x$.

Definition 3.11. *An inverse semigroup S is called a Clifford semigroup if $ss^{-1} = s^{-1}s$ for each $s \in S$.*

Let S be a Clifford semigroup, and let $s \in S$. Then $s \in G_{ss^{-1}}$, and hence S is a disjoint union of the groups $G_p \quad (p \in E(S))$, that is

$$S = \bigcup_{p \in E(S)} G_p,$$

where G_p's are the maximal subgroups of S.

Definition 3.12. *A semigroup S with a zero element 0 is called a Brandt semigroup if it satisfies the following axioms:*

(i) To each non-zero element s of S there corresponds unique elements t, u and s' in S such that

$$ts = s, \quad su = s \quad \text{and} \quad s's = u.$$

(ii) If $t, u \in E(S)$ and are non-zero, then $tSu \neq \{0\}$.

Definition 3.13. *A Brandt semigroup S over a group G with index set J consists of all canonical $J \times J$ matrix units over $G \bigcup \{0\}$ and a zero matrix 0. It is an inverse semigroup over G with index set J given by*

$$S = \{(g)_{ij} : g \in G, i, j \in J\} \bigcup \{0\},$$

where $(g)_{ij}$ is the $J \times J$-matrix with (k, l)-entry equal to g if $(k, l) = (i, j)$ and 0 if $(k, l) \neq (i, j)$ and multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{matrix} (gh)_{il} & (j = k) \\ 0 & (j \neq k) \end{matrix}$$

Definition 3.14. *A bicyclic semigroup is a semigroup $S = \{e, p, q : pq = e\}$ generated by an identity element e and two more elements p and q such that $pq = e$.*

The bicyclic semigroup S is regular and has played a very crucial role in both the semigroup and semigroup algebra theory.

Rees semigroups are described in [26, §3.2] and [7, Chapter 3]. Indeed, let G be a group, and let $m, n \in \mathcal{N}$; the zero adjoined to G is 0. A *Rees semigroup* has the form $S = \mathcal{M}(G, P, m, n)$; here $P = (a_{ij}) \in \mathbb{M}_{n,m}(G)$, the collection of $n \times m$ matrices with components in G. For $x \in G$, $i \in \mathcal{N}_m$, and

$j \in \mathcal{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{m,n}(G^0)$ with x in the $(i,j)^{\text{th.}}$ place and 0 elsewhere. As a set, S consists of the collection of all these matrices $(x)_{ij}$. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{k\ell} = (xa_{jk}y)_{i\ell} \quad (x, y \in G,\ i, k \in \mathcal{N}_m,\ j, \ell \in \mathcal{N}_n)\,;$$

it is shown in [26, Lemma 3.2.2] that S is a semigroup.

Similarly, we have the semigroup $\mathcal{M}^0(G, P, m, n)$, where the elements of this semigroup are those of $\mathcal{M}(G, P, m, n)$, together with the element 0, identified with the matrix that has 0 in each place (so that 0 is the zero of $\mathcal{M}^0(G, P, m, n)$), and the components of P are now allowed to belong to G^0. The matrix P is called the *sandwich matrix* in each case. The semigroup $\mathcal{M}^0(G, P, m, n)$ is a *Rees matrix semigroup with a zero over* G.

We write $\mathcal{M}^0(G, P, n)$ for $\mathcal{M}^0(G, P, n, n)$ in the case where $m = n$.

The above sandwich matrix P is *regular* if every row and column contains at least one entry in G; the semigroup $\mathcal{M}^0(G, P, m, n)$ is regular as a semigroup if and only if the sandwich matrix is regular.

Definition 3.15. *Let S be a semigroup. A principal series of ideals for S is a chain*

$$S = I_1 \supset I_2 \supset \cdots \supset I_{m-1} \supset I_m = K(S)$$

where $I_1, I_2, ..., I_m$ are ideals in S and there is no ideals of S strictly between I_j and I_{j+1} for each $j \in \mathcal{N}_{m-1}$ and $K(S)$ is the minimum ideal of S.

Let S be a regular semigroup with finitely many idempotents. Then by Theorem 3.12, and Theorem 3.13 of [7], $K(S)$ exits and S has a principal series. In this case each quotient I_j/I_{j+1} is a Rees matrix semigroup of the form $\mathcal{M}^0(G, P, n)$, where $n \in \mathcal{N}$, and the sandwish matrix P is invertible in $\mathcal{M}_n(l^1(G)$.

Finally in this section, we give the definitions and some basic properties of semigroup algebras.

Let S be a non-empty set. Then

$$\ell^1(S) = \left\{ f \in \mathcal{C}^S : \sum_{s \in S} |f(s)| < \infty \right\},$$

with the norm $\|.\|_1$ given by $\|f\|_1 = \sum_{s \in S} |f(s)|$ for $f \in \ell^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s \in S$.

Now suppose that S is a semigroup. For $f, g \in \ell^1(S)$, we set

$$(f \star g)(t) = \left\{ \sum f(r)g(s) : r, s \in S,\ rs = t \right\} \quad (t \in S)$$

so that $f \star g \in \ell^1(S)$. It is standard that $(\ell^1(S), \star)$ is a Banach algebra, called the *semigroup algebra on* S. For a further discussion of this algebra, see [6, 7], for example. In particular, with $A = \ell^1(S)$, we identify A' with $C(\beta S)$, where βS is the Stone-Čech compactification of S, and $(A'', \Box)$ with $(M(\beta S), \Box)$, where $M(\beta S)$ is the space of regular Borel measures on βS of S; in this way, $(\beta S, \Box)$ is a compact, right topological semigroup that is a subsemigroup of $(M(\beta S), \Box)$ after the identification of $u \in \beta S$ with $\delta_u \in M(\beta S)$.

Let S be a semigroup. The semigroup algebra $\ell^1(S)$ may have an identity even if S is non-unital. The necessary and sufficient conditions for $\ell^1(S)$ to have a bounded approximate identity for an inverse semigroup were given in [10] and these were generalized to other semigroups in [24]. For a general semigroup S, the following is [7, Proposition 4.3].

Proposition 3.16. *Let S be a semigroup for which there is a finite subset $F \subset E(S)$ such that*

$$S = \bigcup \{pSq : p, q \in F\}.$$

Suppose that $\ell^1(S)$ has a bounded approximate identity. Then $\ell^1(S)$ has an identity.

The following theorem from [39] gives the structure of semigroup algebra $\ell^1(S)$ for uniformly locally finite inverse semigroup S.

Proposition 3.17. *Let S be a uniformly locally finite inverse semigroup with $\mathbb{D}-$ classes $\{D_\lambda : \lambda \in J\}$. For each λ take an idempotent $p_\lambda \in D_\lambda$. Then there is an isomorphism of Banach algebras*

$$\ell^1(S) \cong l^1 - \bigoplus \{\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})) : \lambda \in J\}.$$

4. Amenability and weak amenability of semigroup algebras

We first recall that a discrete semigroup S is left amenable if the space $l^\infty(S)$ admits a functional m called a mean such that $m(1) = 1 = \|m\|$ and the mean is left invariant, i.e. $m(l_x f) = m(f)$, where $(l_x f)(y) = f(xy) \quad (x, y \in S, f \in l^\infty(S))$. Similarly for right amenable. If S is both left and right amenable, it is amenable. In the case of a group, or even an inverse semigroup, left (or right) amenability implies amenability. For example, the bicyclic semigroup S is an amenable semigroup [10].

The result below on the amenability of semigroup is well-known.

Proposition 4.1. *Let S be a semigroup such that S is regular and amenable. Suppose S is right cancellative. Then S is an amenable group.*

It is also a well-known result that the amenability of the semigroup algebra $\ell^1(S)$ implies that S is an amenable semigroup [10, lemma 3]. An example of a semigroup S for which $\ell^1(S)$ is not amenable is the bicyclic semigroup S, see [10].

The characterization of the semigroup S for which $\ell^1(S)$ is amenable is somehow complicated. Such a characterization is given in [7]. The following theorem determined exactly when $\ell^1(S)$ is amenable.

Theorem 4.2. *Let S be a semigroup. Then the semigroup algebra $\ell^1(S)$ is amenable if and only if the minimum ideal $K(S)$ exists, $K(S)$ is an amenable group, and S has a principal series*

$$S = I_1 \supset I_2 \supset \cdots \supset I_{m-1} \supset I_m = K(S)$$

such that each quotient I_j/I_{j+1} is a regular Rees matrix semigroup of the form $\mathcal{M}^0(G, P, n)$, where $n \in \mathcal{N}$, G is an amenable group and the sandwish matrix P is invertible in $\mathcal{M}_n(l^1(G)$.

Proof. This is [7, Theorem 10.12].

The next result summarizes some known structural implications of amenability of the semigroup algebra $\ell^1(S)$.

Theorem 4.3. *Let S be a semigroup. Suppose $\ell^1(S)$ is amenable. Then*

(i) S is amenable;

(ii) S is regular;

(iii) $E(S)$ is finite;

(iv) $\ell^1(S)$ has an identity.

Proof. (i) This is [10, Lemma 3]

(ii) This is [11, Theorem 2]

(iii) This is [11, Theorem 2]

(iv) This is [7, Corollary 10.6]

For a finite semigroup, we have the following result from [12].

Proposition 4.4. *Suppose S is a finite semigroup. Then the following statements are equivalent:*

(i) $\ell^1(S)$ is amenable.

(ii) S is regular and $\ell^1(S)$ is unital.

(iii) S is regular and $\ell^1(S)$ is semisimple.

For inverse and regular semigroups, we have the next result.

Theorem 4.5. *Let S be a semigroup.*

(i) Suppose S is inverse and $E(S)$ is finite. Then $\ell^1(S)$ is amenable if and only if each maximal subgroup of S is amenable.

(ii) Suppose S is regular and admits a principal series. Then $\ell^1(S)$ is amenable if and only if every maximal subgroup of S is amenable.

Proof. (i) This is [10, Theorem 8].

(ii) This is [12, Corollary 5.2].

For a Brandt semigroup S over a group G with index set I. It was shown in [2] that $\ell^1(S)$ is amenable if and only if G is amenable and I is a finite set.

It is not known in general when $\ell^1(S)$ is weakly amenable. For partial results, see [1] and [22].

$\ell^1(S)$ is weakly amenable for a large class of inverse semigroups, including the clifford semigroup, Rees matrix semigroup, and inverse semigroups with a finite set of idempotents. An example of a semigroup S for which $\ell^1(S)$ is not weakly amenable is the bicyclic semigroup S, see [3].

For the hereditary properties of weakly amenable semigroup algebras, we have the following result.

Theorem 4.6. *(i) Suppose S is commutative semigroup and T is a homomorphic image of S. If $\ell^1(S)$ is weakly amenable, then $\ell^1(T)$ is weakly amenable.*

(ii) Let S and T be semigroups with identity elements. Then $S \times T$ is weakly amenable if and only if S and T are weakly amenable.

Proof. (i) This follows from [23, Proposition 2.1].

(ii) This is [3, Theorem 1.7].

The following result is well-known about weak amenability.

Proposition 4.7. *Let A be a Banach algebra such that $\overline{A^2} \neq A$, then A is not weakly amenable. In particular, for a semigroup S, if $S^2 \neq S$, then $\ell^1(S)$ is not weakly amenable.*

For regular and inverse semigroups, we have the following results.

Theorem 4.8 *Let S be a semigroup.*

(i) If S is completely regular. Then $\ell^1(S)$ is weakly amenable.

(ii) If S is inverse whose semilattice of idempotents is finite. Then $\ell^1(S)$ is weakly amenable.

Proof. *(i) This is* [1, Theorem 3.6].

(ii) This is [3, Theorem 2.12].

In [1]*, Blackmore posed an open question on whether the weak amenability of the semigroup algebra $\ell^1(S)$ implies complete regularity of the semigroup S. This question was answer by Ghlaio and Read in* [22]*, where they gave counter examples for the problem. In particular, they gave examples of irregular semigroups S such that $\ell^1(S)$ is weakly amenable.*

5. Approximate and Pseudo amenability for semigroup algebras

In [16]*, Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nonetheless had a form of amenability. All known approximate amenable Banach algebras have bounded approximate identities until recently when Ghahramani and Read in* [22] *give examples of Banach algebras which are boundedly approximately amenable but which do not have bounded approximate identities. This answers a question open since the year 2004 when Ghahramani and Loy founded the notion of approximate amenability.*

Approximate amenability of $\ell^1(S)$ *have been widely investigated in* [2], [4], [9], [19], [21],[33], [41] *for different semigroups* S. *It is not known in general when the semigroup algebra* $\ell^1(S)$ *is approximately amenable; partial result is given in* [19, Theorem 9.2]. *Thus we cannot determine when* $\ell^1(S)$ *is approximately amenable. It was shown in* [21] *that* $\ell^1(S)$ *is not approximately amenable for the bicyclic semigroup* S.

Some known structural implications of approximate amenability of $\ell^1(S)$ *for an arbitrary semigroup* S *are given below.*

Theorem 5.1. *Let* S *be a semigroup.*

(1) Suppose $\ell^1(S)$ *is approximate amenable. Then*

(i) S *is regular*

(ii) S *is amenable*

(2) Suppose $\ell^1(S)$ *is approximate amenable and* $E(S)$ *is finite. Then* $\ell^1(S)$ *has an identity.*

Proof. *(1) This is* [19, Theorem 9.2].

(2) This is [9, Corollary 2.2.2].

How close the amenability of S *is to a sufficient condition for the approximate amenability of* $\ell^1(S)$ *is not clear.*

The next result gives an hereditary property of approximately amenable semigroup algebra.

Theorem 5.2. *Let* S *be a semigroup such that* $E(S)$ *is finite, and let* T *be an ideal in* S. *Suppose* $\ell^1(S)$ *is approximately amenable. Then* $\ell^1(T)$ *is approximately amenable.*

Proof. *This is* [9, Proposition 2.2.3].

In [2], *Bami and Samea investigated the approximate amenability of the semigroup algebra* $\ell^1(S)$ *for the cancellative semigroup* S *in terms of the amenability of* S. *They obtained the following results:*

Theorem 5.3. *(i) Let* S *be a left cancellative (right respectively) semigroup such that* $\ell^1(S)$ *is approximately amenable. Then* S *is left (right respectively) amenable.*

(ii) Let S *be a cancellative semigroup such that* $\ell^1(S)$ *is approximately amenable. Then* S *is amenable.*

In [18], *Ghahramani and Zhang introduced two notions of amenability for Banach algebras based on the existence of an approximate diagonal (not necessarily bounded). We recall the definitions of these two notions of amenability from* [18].

Definition 5.4. *A Banach algebra* A *is pseudo-amenable if there is a net* $(u_\alpha) \subset (A\hat{\otimes}A)$, *called an approximate diagonal for* A, *such that* $au_\alpha - u_\alpha a \to a$ *and* $\pi(u_\alpha)a \to a$ *for each* $a \in A$, *where* π *is the product morphism from* $A\hat{\otimes}A$ *into* A *defined by* $\pi(a \otimes b) = ab \quad (a, b \in A)$.

Definition 5.5. *A Banach algebra* A *is pseudo-contractible if it has a central approximate diagonal, i.e. an approximate diagonal* (u_α) *satisfying* $au_\alpha = u_\alpha a$ *for all* $a \in A$ *and all* u_α.

The notion of pseudo-amenability for Brandt semigroup algebra was studied in [40] *by Sadr. Afterward in* [14] *and* [13]*, where the authors considered the pseudo-amenability of semigroup algebras for certain classes of inverse, left cancellative, Band and Brandt semigroups. Indeed, it is shown that for an inverse semigroup with uniformly locally finite idempotent set S, the semigroup algebra $\ell^1(S)$ is pseudo-amenable if and only if each maximal subgroup of S is amenable* [14, Theorem 3.7]*. The following results were also obtained:*

Theorem 5.6. *(i) Let S be an inverse semigroup. Suppose $\ell^1(S)$ is pseudo-amenable, then S is an amenable semigroup.*
(ii) Let G be a group, I be a non-empty set and let $S = M^0(G, I)$ be the Brandt semigroup over G with index set I. Then $\ell^1(S)$ is pseudo-amenable if and only if G is amenable.
(iii) Let S be a band semigroup. Suppose $\ell^1(S)$ is pseudo-amenable, then S is semilattice and so amenable.
(iv) Let S be a uniformly locally finite band semigroup. Then $\ell^1(S)$ is pseudo-amenable if and only if S is semilattice.
(v) Let $S = \bigcup_{p\in E(S)} G_p$ be the Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is pseudo-amenable if and only if G_p is amenable for every $p \in E(S)$.

Proof. *(i) This is* [13, Theorem 3.1].
(ii) This follows from [40, Theorem 3] *and* [13, Corollary 3.2]*, see also* [14, Corollary 3.8].
(iii) This is [13, Theorem 3.4].
(iv This is [13, Corollary 3.5].
(v) This is [14, Corollary 3.9].

For the pseudo-contractibility of $\ell^1(S)$, it is not also known in general the semigroup S for which $\ell^1(S)$ is pseudo-contractible. Partial results are given in [13]*, where the authors gave the structural implication of the pseudo-contractibility of $\ell^1(S)$, and also characterized the pseudo-contractibility of $\ell^1(S)$ for uniformly locally finite inverse semigroup S. They obtained the following results:*

Theorem 5.7. *(1)Let S be a semigroup such that $\ell^1(S)$ is pseudo-contractible. Then*
(i) S is amenable, and
(ii) if S has a left or right identity, then S is finite.
(2) Let S be a uniformly locally finite inverse semigroup. Then the following are equivalent
(i) $\ell^1(S)$ is pseudo-contractible.
(ii) Each maximal subgroup of S is finite and each $\mathcal{D}$-class has finitely many idempotents.
(3) Let S be a uniformly locally finite semilattice. Then $\ell^1(S)$ is pseudo-contractible.

Proof. *(1) This is* [13, Corollary 2.10].

(2) This is [13, Theorem 2.4].

(3) This is [13, Corollary 2.7].

AS a consequence of Theorem 5.7 (2), we have the following characterizations of the pseudo-contractibility of $\ell^1(S)$ for the Brandt and Clifford semigroups S.

Theorem 5.8. *(i) Let G be a group, I be a non-empty set and $S = M^0(G, I)$ be the Brandt semigroup over G with index set I. Then $\ell^1(S)$ is pseudo-contractible if and only G and I are finite.*

(ii) Let $S = \bigcup_{p\in E(S)} G_p$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is pseudo-contractible if and only if for each $p \in E(S)$, G_p is a finite group.

Proof. *(i) This is* [13, Corollary 2.5].

(ii) This is [13, Corollary 2.6].

6. Character amenability for semigroup algebras

In [37]*, Monfared introduced the notion of character amenable Banach algebras. His definition of this notion requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in* [28]*, so all amenable Banach algebras are character amenable.*

We let $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach A- bimodule X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [27]*, see also* [37] *that*

(i) A is left φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_r}$;

(ii) A is right φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_l}$;

(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$;

(iv) A is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_A$;

(v) A is character amenable if it is both left and right character amenable.

We also recall from [27] *that, for* $\varphi \in \Phi_A$*, a left (right)* φ*-approximate diagonal for* A *is a net* (m_α) *in* $A\hat{\otimes}A$ *such that*

$$(i)\ \|m_\alpha \cdot a - \varphi(a)m_\alpha\| \to 0 \quad (\|a \cdot m_\alpha - \varphi(a)m_\alpha\| \to 0) \quad (a \in A);$$

$$(ii)\ \langle \varphi \otimes \varphi, m_\alpha \rangle = \varphi(\pi(m_\alpha)) \to 1,$$

where $\pi : A\hat{\otimes}A \to A$ *defined by* $\pi(a \otimes b) = ab \quad (a, b \in A)$ *is the product map.*

Let A *be a Banach algebra and* $\varphi \in \Phi_A$*. We recall from* [38] *that*

(i) A *is left (right)* φ*-pseudo - amenable if it has a left (right)* φ*-approximate diagonal;*

(i) A *is left (right) character pseudo - amenable if it has a left (right)* φ*-approximate diagonal for every* $\varphi \in \Phi_A$*;*

(iii) A *is character pseudo - amenable if it is both left and right character pseudo - amenable.*

No work has been done on character and approximate character amenability of $\ell^1(S)$*. It will be interesting to investigate these.*

In [36]*, the authors studied the character pseudo - amenability of semigroup algebras* $\ell^1(S)$*. They focused on certain semigroups such as inverse semigroup with uniformly locally finite idempotent set and Brandt semigroup and study the character pseudo - amenability of semigroup algebra* $\ell^1(S)$ *in relation to the semigroup* S*. In particular, they show that for a unital cancellative semigroup* S, *the character pseudo-amenability of* $\ell^1(S)$ *is equivalent to its amenability, this is in turn equivalent to* S *being an amenable group* [36, Corollary 4.6]*. For Brandt semigroup, they also obtained the following result, see* [36, Corollary 4.8].

Theorem 6.1. *Let* $S = M^0(G, I)$ *be the Brandt semigroup over the group* G *with index set* I*. Suppose* $\ell^1(S)$ *is left or right character pseudo - amenable. Then* G *is amenable.*

7. Relations between these notions of Amenability for semigroup algebras

In this section we give some relations between amenability, weak amenability, approximate amenability, pseudo-amenability and pseudo-contractibility for semigroup algebras.

For finite and commutative semigroups, we have following results due to [1, Proposition 4.12] *and* [2, Corollary 1.3] *respectively.*

Proposition 7.1. *(i) Let* S *be a commutative finite semigroup. Then* $\ell^1(S)$ *is amenable if and only if it is weakly amenable.*

(ii) Let S *be a finite semigroup. Then the approximately amenability and amenability of* $\ell^1(S)$ *are equivalent.*

For a general semigroup S with $E(S)$ finite, we have following relation due to [9, Theorem 2.2.8].

Theorem 7.2. *Let S be a semigroup such that $E(S)$ is finite. Then $\ell^1(S)$ is approximately amenable if and only if it is amenable.*

The following relations are for left (right) cancellative semigroup.

Theorem 7.3. *(1) Let S be a left cancellative semigroup. Then the following are equivalent:*

(i) $\ell^1(S)$ is pseudo-amenable.

(ii) S is an amenable group.

(iii) $\ell^1(S)$ is amenable.

(2) let S be a semigroup such that $\ell^1(S)$ is approximately amenable. Suppose S is right cancellative. Then S is an amenable group and $\ell^1(S)$ is amenable.

Proof.

(1) This is [13, Theorem 3.6].

(2) Follows from [19, Theorem 9.2] *and* [24, Theorem 2.3].

For Brandt semigroup, we have the following relations due to [41, Theorem 4.5] *and* [13, Corollary 2.5].

Theorem 7.4. *(1) Let $S = M^0(G, I)$ be the Brandt semigroup over the group G with index set I. Then the following are equivalent:*

(i) $\ell^1(S)$ is amenable.

(ii) $\ell^1(S)$ is approximately amenable.

(iii) I is finite and G is amenable.

(2) Let $S = M^0(G, I)$ be the Brandt semigroup over the group G with index set I. Then the following are equivalent:

(i) $\ell^1(S)$ is pseudo-contractible.

(ii) $\ell^1(S)$ is contractible.

(iii) I and G are finite.

Lastly, Mewomo and Akinbo in [33]*, showed that for $S = \mathcal{M}^o(G, P, I)$, a Rees matrix semigroup with zero over an amenable group G with finite index I, the approximate amenability of $\ell^1(S)$ is equivalent to its amenability.*

8. Open Questions

A fruitful area of research in amenability of semigroup algebras $\ell^1(S)$ has been to describe amenability (or some versions of amenability) of $\ell^1(S)$ in terms of the semigroup S.

Questions 8.1. *What are structural implications of the approximate amenability, pseudo-amenability, pseudo-contractibility and character pseudo-amenability of $\ell^1(S)$?*

It is not known in general when the semigroup algebra $\ell^1(S)$ is approximately amenable, pseudo-amenable and character pseudo-amenable. We only have partial results in literature.

Questions 8.2. *How do we characterize the approximate amenability, pseudo-amenability and character pseudo-amenability of the semigroup algebras $\ell^1(S)$?*

No work has been done on character and approximate character amenability of the semigroup algebras $\ell^1(S)$. It would be very interesting to investigate the character and approximate character amenability of $\ell^1(S)$ for different classes of semigroups such as inverse, Brandt, Clifford, Rees matrix semigroups. In particular, we ask the following questions:

Questions 8.3. *For what semigroup S is the semigroup algebra $\ell^1(S)$*

(i) character amenable

(ii) approximately character amenable

(iii) character pseudo-amenable

9. Conclusion

The above survey of results and problems to some extent aims at producing a catalogue listing which of our known semigroup algebras have the property of amenability in one of its version. It also provide different characterizations of these notions of amenability for semigroup algebras. This survey serves as a reference point for future research in the area of amenability for semigroup algebras.

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