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COMPLETELY SIMPLE SEMIGROUP WITH BASIS PROPERTY

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Abstract. An inverse semigroup (group) S is called an inverse semigroup (group) with basis property, if each two minimal (irreducible) generating sets (with respect to inclusion) of an arbitrary subsemigroup (group) H of S is equivalent (i.e. they have the same cardinality).

It is proved that every completely simple semigroup with basis property is either group with basis property or its sandwich matrix has at most two rows or two column and its maximal subgroup is either trivial group or a primary cyclic group.

Keywords: completely simple semigroup, basis property.

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1. Introduction

Let G be a finite group. we define $d(G)$ to be the least number of generators required to generate the group. This leads us to a simple definition of what it means for a generating set of a group to be minimal.

A generating set X is said to be minimal if it has no proper subset which is also form a generating set.

The subset X of a semigroup S is called independent, if for all $x \in S$, $x \notin \langle X \setminus \{x\} \rangle$.

An independent set X is called a basis of the subsemigroup $\langle X \rangle$.

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In 1978 Jones [5] introduced and initiated the study of semigroups with the basis property. Jones [6] states that if S is an inverse semigroup and $U \leq V \leq S$ then a U-basis for V is a subset X of V which is minimal such that $\langle U \cup X \rangle = S$. So a minimal generating set for V is a $\emptyset$-basis.

Definition1. A basis property of a universal algebra A means that every two minimal (with respect to inclusion) generating sets (bases) of an arbitrary subalgebra of A have the same cardinality [2].

Definition 2. A semigroup (group) S is called a semigroup (group) with basis property if there exists a basis minimal (irreducible) generating sets (with respect to inclusion) for every subsemigroup (group) H of S and every two bases are equivalent (i.e. they have the same cardinality) [6].

Let G be a p-group, then the minimal generating sets (that is, generating sets for which no proper subset also generates G) all have the same cardinality.

Notice that finitely generated vector spaces have the property that all minimal generating sets have the same cardinality.

Jones [6] introduced another concept which was stated for inverse semigroups.

Definition 3. An inverse semigroup S has the strong basis property if for any inverse subsemigroup V of S and inverse subsemigroup U of V, any two U-bases for V have the same cardinality.

Shiryaev [10] showed that the completely simple semigroup S is filtering if and only if either S signal band, group with two elements, or combinatorial Brandt semigroup with five elements.

Also Jones[6] showed that a free inverse semigroup, semilattices and finite combinatorial inverse semigroups posses the basis property.

Let $(\mathbb{Z}, +)$ be an additive abelian group, we see that $\mathbb{Z} = \langle 1 \rangle = \langle 2, 3 \rangle$ even though $2 \notin \langle 3 \rangle$ and $3 \notin \langle 2 \rangle$. Thus $\mathbb{Z}$ does not have the basis property, so complete semisimplicity alone is not sufficient to ensure the basis property. Hence even free groups do not have the basis property.

The first results on the basis property of groups was in [6] he showed that a group with basis property is solvable and all elements of such a group have prime power order. A classification of a group with the basis property was announced by Dickson to Jones in [6], but yet to be published. However a classification of finite groups with the basis property was announced by Al Khalaf [2] exploiting Higman's result, this classification requires a technical condition on the p-group and he proved the following theorem:

Theorem 1[2] Let a finite group G be a semidirect product of a p-group $P = Fit(G)$ (Fitting subgroup) of G by a cyclic q-group $\langle y \rangle$, of order q^b, where $p \neq q$ (p and q are primes), $b \in \mathbb{N}$. Then the group G has basis property if and only if for every element in $\langle y \rangle$, $y \neq e$ and for any invariant subgroup H of P the automorphism φ_u must define an isotopic representation on every quotient Frattini subgroup of H.

In [4] some results of Maschke, Clifford and Krull-Schmidt common to both group and module theory, are used to classify the group with basis property.

Finally Jones [7,8] studied basis property from the point of view exchange properties, then semigroups are determined for regular and periodic semigroup. The main method of proof is to eliminate undesirable types of semigroups by showing that each contains a certain special subsemigroup, for instance the bicyclic semigroup.

Definition 4. A semigroup S is a periodic if all cyclic subsemigroup $\langle a\rangle$, $a \in S$ are finite.

Proposition 1. Let S be a group, which is a semigroup with basis property. Then S is a periodic group with basis property.

Proof. Let S be a group, which is a semigroup with basis property and let H be a cyclic subgroup of S such that $H = \langle a\rangle$, $a \in S$. We prove that H is finite. Suppose that H is infinite. We see that a set $\left\{a, a^{-1}\right\}$ is a basis of H. Consider the set $X = \left\{a^{-6}, a^{10}, a^{15}\right\}$. We have $a = \left(a^{-6}\right)^4 a^{10} a^{15} \in \langle X\rangle$ and $a^{-1} = a^5 a^{-6} \in \langle X\rangle$, so $\langle X\rangle = H$. Therefore

$$\left\langle a^{-6}, a^{10}\right\rangle \subseteq \left\langle a^2, a^{-2}\right\rangle \neq H,$$

$$\left\langle a^{-6}, a^{15}\right\rangle \subseteq \left\langle a^{-3}, a^3\right\rangle \neq H,$$

$$\left\langle a^{10}, a^{15}\right\rangle \subseteq \left\langle a^5\right\rangle \neq H.$$

Thus X is a basis of H with three elements, so H has two bases with different cardinality. But this is contradicting with basis property for S. Hence S is periodic, we note that for every $Y \subseteq S$ the subsemigroup $\langle Y\rangle$ is a subgroup, therefore S is a group with basis property.

Theorem 2. [3] A semigroup S is completely simple if and only if it is isomorphic to a Rees matrix semigroup $M(\ G; I, \Lambda; P)$ for some group G, nonempty sets I, Λ, and regular sandwich matrix P.

Thus in the following propositions we suppose that S is a Rees matrix semigroup $M(\ G, I, \Lambda; P)$ with sandwich matrix $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$ over group G.

Proposition 2. Let S be a semigroup of type $M(\ G; I, \Lambda; P)$. If S is a semigroup with basis property and satisfies the following condition:

$$\exists i \in I, \exists\lambda \in \Lambda(\lambda \neq \mu \wedge p_{\lambda i} \neq 0 \wedge p_{\mu i} \neq 0). \quad (2-1)$$

Then either $|G| = 1$ or G is a primary cyclic group.

Proof Let S be a semigroup with basis property satisfying the condition of proposition, then consider that $S = I \times G \times \Lambda$. Putting $H_\lambda = \{i\} \times G \times \{\lambda\}$, $H_\mu = \{i\} \times G \times \{\mu\}$ and $H = H_\lambda \cup H_\mu$. Since $p_{\lambda i} \neq 0$ and $p_{\mu i} \neq 0$, so H is a subsemigroup of S isomorphic to right subgroup, which are isomorphic to G. Hence we will study the set $G \times \{1,2\}$ denoting it by K, then for $g_1, g_2 \in G$ and $i, j \in \{1,2\}$ the product in K given by

$$(g_1, i) = (g_2, j) = (g_1 g_2, j),$$

So we have the homomorphism $\varphi : K \to G$ is given by $\varphi(g_1, i) = g_1$. Since S is semigroup with basis property, and G is a subsemigroup of S, then G is a group, which is a semigroup with basis property, hence by proposition 1 G is a group with basis property and so it is a periodic group.

Suppose that $|G| > 1$, i.e. G does not primary cyclic group, then G has independent set $X = \{x_1, x_2\}$. Putting $G_1 = \langle X \rangle$, $K_1 = G_1 \times \{1,2\}$. Assume that $e \in G$ is an identity of G. Since G is a periodic group, then for all $i, j \in \{1,2\}$ there exists $s_1,\ s_2 \in \mathbb{N}$ such that $x_1^{s_1} = e$ and $x_2^{s_2} = e$. Let us take the subsemigroup $K_1 = \langle (x_1, 1),\ (x_2, 2)\ \rangle$ of semigroup K. Notice that the set $X_1 = \{(x_1\ , 1),\ (x_2\ , 2)\}$ is a basis of K_1. Now let $Y = \{(e,1), (x_1\ , 2),\ (x_2\ , 2)\}$ be a set of K_1. So we will prove that the Y is a basis of K_1. Since $(x_1\ , 1) = (x_1\ , 2)(e, 1)$ and $(x_2\ , 2) \in Y$ then we have $\langle Y \rangle = K_1$ on the other hand we will show that Y is an independent set . In fact we have $(e, 1) \notin \langle (x_1\ , 2),\ (x_2\ , 2) \rangle$. Suppose that $(x_1\ , 2) \in \langle Y \setminus \{(x_1\ , 2)\} \rangle$. Then there exists a word u such that

$$(x_1\ , 2) = u((e, 1),\ (x_2\ , 2)). \qquad (2-2)$$

Using the homomorphism φ for $(2-2)$ we get $\varphi(x_1\ , 2) = x_1\ = u(x_1^{s_1}\ , x_2\ ) \in \langle X \setminus \{x_1\ \} \rangle$, contradicting with X, which is an independent set.

Similarly $(x_2\ , 2) \in \langle Y \setminus \{(x_2\ , 2)\} \rangle$ and contradicting . Hence Y is a basis of K_1 with three elements.

Let now $V = \{(x_1, 1)(x_2, 2)\}$, since $(e, 1) = \left(x_1^{s_1}, 1\right) = (x_1, 1)^{s_1} \in \langle V \rangle$ and

$$(x_2, 2) = (x_1, 1)(e, 2) = (x_1, 1)\left(x_2^{s_2}, 2\right) = (x_1, 1)(x_2, 2)^{s_2} \in \langle V \rangle.$$

Then $Y \subseteq V$. And so

$$K_1 = \langle Y \rangle \subseteq \langle V \rangle \subseteq K_1.$$

Hence $K_1 = \langle V \rangle$ and V is a generating set of semigroup K_1, with two elements. So K_1 has a basis with cardinality 2. Thus K_1 has two different bases and we get a contradiction with basis property for S and K. Then G must be cyclic and periodic, so it is finite and using [9] we get that G either trivial i.e. $|G| = 1$ or primary cyclic group.

Proposition 3. Let S be a completely simple semigroup. If S does not form a group, then its maximal subgroup is either trivial or primary cyclic group.

Proof. If the matrix P has two non zero elements in some row or column, then the proposition holds by proposition 2. If it is not, then since P is regular and S is not a group, so in every row and every column there exists only one non zero element and by cancelation we can get e . Thus consider that $I = \Lambda$ and P is an identity matrix, i.e. S is a Brandt semigroup. Suppose, that G is not cyclic or primary group, then we have a contradiction. In fact in this case the group G has an independent set $X = \{x_1, x_2\}$ with two elements.

Taking $G_1 = \langle x_1, x_2 \rangle$, $K = \{i, j\} \times G_1 \times \{i, j\}$, and $i, j \in I,\ i \neq j$. Let $|x_1| = s_1\ ,\ |x_2| = s_2$. Since the matrix P is an identity, and the product in S given by:

$\forall\ i_1, i_2\ , i_3, i_4\ \in I,\ \ g_1, g_2 \in G_1,$

$$(i_1, g_1, i_2\ )(i_3, g_2, i_4\ ) = \begin{cases} (i_1, g_1 g_2, i_4\ ), & if\ i_2 = i_3 \\ 0, & if\ i_2 \neq i_3 \end{cases} . \qquad (2-3)$$

Consider the two sets;

$$X = \{(i,e,j)\,,\ (j,x_1,i),(j,x_2,j\,)\}\,,$$

$$Y = \{(i,e,j)\,,\ (j,e,i),(i,x_1,i\,),\ (j,x_2,j)\}\,.$$

From $(2-3)$ we have

$$(j,e,i) = (j,x_1,i)\,(i,e,j)^{s_1-1}\,(j,x_1,i) \in \langle X\rangle\,,$$

$$(i,x_1,i) = (i,e,j)\,(j,x_1,i) \in \langle X\rangle\,,$$

$$(j,x_1,i) = (j,e,i)\,(i,x_1,i)\,(i,e,j) \in \langle Y\rangle\,.$$

Hence we get $\langle X\rangle = \langle Y\rangle$. If we can prove that Y is an independent, then the semigroup S has a basis with four elements and also has another basis with smaller elements and we have a contradiction with the basis property for S. Hence the element

$$(i,e,j) \notin \ \{(j,e,i)\,,(i,x_1,i)\,,(j,x_2,j)\}\,.$$

Since the non zero product for these elements by the component i from the right may be only of the form $(i,x_1,i)^{\,m}$, i.e. the third component is i.

Similarly the element $(j,e,i) \notin \ \{(i,e,j)\,,(i,x_1,i)\,,(j,x_2,j)\}$ and so $(i,x_1,i) \notin \ \{(i,e,j)\,,(j,e,i)\,,(j,x_2,j)\}$ since the set $\{x_1,x_2\}$ is an independent in G. Similarly the element $(j,x_2,j) \notin \ \{(i,e,j)\,,(j,e,i)\,,(i,x_1,i)\}$. Hence we get a contradiction.

Corollary 1. From the propositions 1,2,3, we see that the group G either trivial or primary cyclic group with a single generating element, so we denote it by g. Then for $i \in I, \lambda \in \Lambda$ if $[H]$ is a class $H_{i\lambda} = \{i\} \times G \times \{\lambda\}$ is a subgroup of semigroup S, so it has a unique generated element $h_{i\lambda}$. Hence $h_{i\lambda} = (i, g_{i\lambda}, \lambda)$, for some $g_{i\lambda} \in G$.

Proposition 4. For some $i, j \in I$ and $\lambda, \mu \in \Lambda,\ \ \lambda \neq \mu$. If the following condition $p_{\lambda i}p_{\mu j}p_{\lambda j} \neq 0$ holds, then $p_{\mu i} \neq 0$.

Proof. Suppose that

$$p_{\lambda i}p_{\mu j}p_{\lambda j} \neq 0,\ i \neq j, \lambda \neq \mu,\ p_{\mu i} = 0. \qquad (2-4)$$

Then we get a contradiction. Multiplying the rows and the columns of matrix P by non zero elements, then we consider, that

$$p_{\lambda i} = e\ ,\ p_{\mu j} = e\ ,\ p_{\lambda j} = e. \qquad (2-5)$$

By proposition 3 the group G is cyclic. Let g be a generating element of the group. We study two sets:

$$X = \ \{(i,g,\lambda)\,,(j,g,\mu)\,,(j,g,\lambda)\}\,,$$

$$Y = \ \{(j,g,\lambda)\,,(i,e,\mu)\}\,.$$

If we prove that $\langle X\rangle = \langle Y\rangle$ and the independent of the set X , then from proposition 3, we get a contradiction with basis property for S. Assume that $|G| = m$. From $(2-4)$ and $(2-5)$ we have

$$(i,g,\lambda) = (i,e,\mu)(j,g,\lambda),$$

$$(j,g,\mu) = (i,g,\lambda)(i,e,\mu),$$

$$(i,e,\mu) = (i,g,\lambda)^{m-1}(j,g,\mu).$$

Then we get $\langle X\rangle \subseteq \langle Y\rangle \subseteq \langle X\rangle$ and $\langle X\rangle = \langle Y\rangle$. Since $p_{\mu i} = 0$ then the element $(j,g,\lambda) \neq (i,g,\lambda)(j,g,\mu)$. Thus

$$(i,g,\lambda) \notin \langle (j,g,\mu)(j,g,\lambda)\rangle,$$

$$(j,g,\mu) \notin \langle (i,g,\lambda)(j,g,\lambda)\rangle.$$

Thus X is an independent set.

Proposition 5. Let $i,j,k \in I$, $i \neq j \neq k$ and $\lambda,\mu,\gamma \in \Lambda$, $\lambda \neq \mu \neq \gamma$. Then for some $l \in \{i,j,k\}$ and $\chi \in \{\lambda,\mu,\gamma\}$ must be satisfies the equation $p_{\chi l} = 0$.

Proof. Putting $I_1 = \{i,j,k\}$ and $\Lambda_1 = \{\lambda,\mu,\gamma\}$. Suppose that

$$\forall l \in I_1, \forall \chi \in \Lambda_1, p_{\chi l} \neq 0, \quad (2-6)$$

Then every $[H]$ class $H = \{1\} \times G \times \{\chi\}$ is a group isomorphic to G and let $h_{l\chi}$ generating element of G. From lemma (3.2) [5] and using $(2-6)$ we have

$$\forall s,t \in I_1, \forall \chi,\xi \in \Lambda_1, H_{s\chi}H_{t\xi} = H_{s\xi}. \quad (2-7)$$

Consider the sets

$$X = \{h_{j\lambda}, h_{k\gamma}, h_{i\mu}, h_{j\gamma}\},$$

$$Y = \{h_{i\lambda}, h_{j\mu}, h_{k\gamma}\}.$$

We see that X is an independence and from $(2-7)$ we have $\langle X\rangle = \langle Y\rangle$, so we get a contradiction with basis property for S.

Proposition 6. Let $i,j,k \in I$, $i \neq j \neq k$ and $\lambda,\mu,\gamma \in \Lambda$, $\lambda \neq \mu \neq \gamma$ and
$p_{\lambda i} = p_{\lambda k} = p_{\mu j} = p_{\gamma j} = 0$ and $p_{\mu i} \neq 0, p_{\mu k} \neq 0$, $p_{\gamma i} \neq 0$, $p_{\gamma k} \neq 0$. $(2-8)$
Then $p_{\lambda j} = 0$.

Proof. Putting $I_1 = \{i,j,k\}$ and $\Lambda_1 = \{\lambda,\mu,\gamma\}$, $H = I_1 \times G \times \Lambda_1$. Suppose that $(2-8)$ holds and $p_{\lambda j} \neq 0$. Multiplying the rows and the columns of the matrix P by elements of G , we get,

$$p_{\lambda j} = p_{\mu i} = p_{\gamma i} = p_{\mu k} = e. \qquad (2-9)$$

By proposition 3 the group G is cyclic, so assume that $G=\langle g\rangle$, $|G|=m$, and also $p_{\gamma k}=g^{l}$, $l\in\mathbb{N}$. Consider the sets

$$X=\{(i,e,\lambda),(k,e,\lambda),(j,g,\mu),\ (j,g,\gamma)\},$$

$$Y=\{(k,e,\lambda),(j,g,\mu),(i,g,\gamma)\}.$$

Now using $(2-9)$ we have

$$(i,e,\lambda)=(i,g,\gamma)^{m-1}(k,e,\lambda)\in\langle Y\rangle$$

$$(j,g,\gamma)=((j,g,\mu)(k,e,\lambda))^{m-1}(j,g,\mu)(i,g,\gamma)\in\langle Y\rangle$$

$$(i,g,\gamma)=(i,e,\lambda)(j,g,\gamma)\in\langle X\rangle\ .$$

Thus $\langle X\rangle=\langle Y\rangle$. It is easy to see that X is an independence, so we have a contradiction with basis property for S. Hence $p_{\lambda j}=0$.

Proposition 7. Let $\{i,j,k\}\subseteq I$ and $\{\lambda,\mu,\gamma\}\subseteq\Lambda$, and

$p_{\lambda j}=p_{\lambda k}=p_{\mu i}=p_{\mu k}=p_{\gamma i}=p_{\gamma j}=0$ and $p_{\lambda i}\neq 0, p_{\mu j}\neq 0.$ $(2-9)$

Then $p_{\gamma k}=0$.

Proof. Suppose that $I_1=\{i,j,k\}$, $\Lambda_1=\{\lambda,\mu,\gamma\}$ and $p_{\gamma k}\neq 0$. Thus we can assume that $p_{\lambda i}=p_{\mu j}=p_{\gamma k}=e$, then the subsemigroup $H=\ I_1\times\{e\}\times\Lambda_1\cup\{0\}$ is a combinatorial semigroup Brandt, so we can consider H as the set $I_1\times I_1\cup\{0\}$ with operation:

$$(l,m)(n,q)=\begin{cases}(l,q)\ , & \text{if}\quad m=n\\ 0\ , & \text{if}\quad m\neq n,\end{cases}$$

and $0(l,m)=(l,m)0=0$, $\forall\ l,m,q\ \in I_1$, we want to prove that H is a semigroup with basis property.

Let the sets:

$$X=\{(i,j),(j,i),(k,j),(j,k)\},$$

$$Y=\{(k,i),(i,j),(j,k)\}.$$

Similarly, as in the proposition 6, we can prove that $\langle X\rangle=\langle Y\rangle$ and X is an independence. Thus we get a contradiction with basis property for H.

Proposition 8. Let $|I|\geq 2$ and $|\Lambda|\geq 2$. Then $\min\{|I|,|\Lambda|\}=2$.

Proof. Suppose that $|I|\geq 3$, and $|\Lambda|\geq 3$. Since P is a regular matrix, then there is a different indexes $\{i,j,k\}\subseteq I$, $\{\lambda,\mu,\gamma\}\subseteq\Lambda$ such that

$$p_{\lambda i}\neq 0, p_{\mu j}\neq 0, p_{\gamma k}\neq 0.\qquad (2-10)$$

Denote by P_1 for a submatrix P which is the intersection λ,μ,γ-rows and i,j-k-columns. Substitutions λ,μ,γ and correspondence i,j,k by $1,2,3$. Then from $(2-10)$ we have that the leading diagonal of matrix P_1 consist from nonzero elements. From proposition 4, we have that the matrix P_1 is symmetric, if we substitute every element by e.

From proposition 5 we have in out the leading diagonal exists a 0. From proposition 7 we have that in out the diagonal it is impossible to be all elements equal zero. Then we have a contradiction, since by proposition 6 and proposition 4 there is no two elements equal zero in the same row or in the same column. Then in every row there exists three non zeros , and in other there exists two nonzeros and two zeros are symmetric depend to the leading diagonal. Hence we get a contradiction with proposition 4, because does not exists submatrix of rank 2 has only one zero. Thus $\min\{|I|,|\Lambda|\}=2$.

Proposition 9 . Let S be a completely simple semigroup such that $|I|=2\leq|\Lambda|$ and $I=\{1,2\}$. Then for all $\lambda,\mu\in\Lambda$, $\lambda\neq\mu$, we have

$$\left\langle p_{\mu i}p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}\right\rangle = G. \qquad (2-11)$$

Proof. It is sufficient to consider the subsemigroup

$$H\cong M(G;\{i,j\},\{\lambda,\mu\};P_1)$$

such that
$P_1=\begin{pmatrix} p_{\lambda i} & p_{\lambda j}\\ p_{\mu i} & p_{\mu j}\end{pmatrix}\sim\begin{pmatrix} p'_{\lambda i} & p'_{\lambda j}\\ p'_{\mu i} & p'_{\mu j}\end{pmatrix}$, $p'_{\lambda i}=p'_{\lambda j}=p'_{\mu i}=e$, $p'_{\mu j}=h$, so
$P_1=\begin{pmatrix} e & e\\ e & h\end{pmatrix}$ and $\langle h\rangle\neq G=\langle g\rangle$.

Consider the sets

$$X=\{(i,e,\mu),(j,e,\lambda),(j,g,\mu)\},$$

$$Y=\{(j,g,\lambda),(j,g,\mu)\}.$$

Now again it is sufficiency to see that $\langle X\rangle=\langle Y\rangle$, and the set X is an independence, then we get a contradiction with a basis property of S. By proposition 2, we consider that the group G is a cyclic $p-$group, where p is prime and $|G|=p^{\alpha}=m$, $\alpha\geq 1$. Since $\langle h\rangle\neq G$ we have $h=g^k$, where $p\,|k$. Hence $(2k+1,\ p)=1$, then there exists $u\in\mathbb{N}$ such that $u(2k+1)\equiv 1(mod\ m)$. Now from

$$p_{\lambda i}=p_{\lambda j}=p_{\mu i}=e,\ p_{\mu j}=h,$$

we have

$$(i,e,\mu)=(i,g,\lambda)^{m-1}(j,g,\mu)\in\langle Y\rangle,$$

$$(j,e,\lambda)=(j,g,\mu)(i,g,\lambda)^{m-1}\in\langle Y\rangle$$

$$(i,g,\lambda)=\left(i,g^{(2k+1)u},\lambda\right)=\left(i,g^{2k+1},\lambda\right)^{u}=(i,p_{\mu j}gp_{\mu j},\lambda)^{u}=$$

$$((i,e,\mu)(j,g,\mu)(j,e,\lambda)\,)^{u}\in\langle X\rangle.$$

Thus $\langle X\rangle = \langle Y\rangle$. The independence of X get from that the first and the second elements of X under multiplication can't get the third from X the another elements from X have the same component in the first or third . A contradiction give us that $(2-11)$ holds for basis property of S.

Corollary 2. If S is a completely simple semigroup with a basis property and it is not a right, left group, or rectangular band, then S is finite and $|S| \leq 2p^{\alpha+1}$ such that p^{α} is the order of the maximal subgroup.

Proof. Without the loss of generality, consider that $|\Lambda| = 2 \leq |I|$ and the group G is an additive group of residue modulo p^{α}. Then from $(2-11)$ we find that the matrix P transforms into the form

$$\begin{pmatrix} 0 & s_1 & \dots & s_r & \dots \\ 0 & 0 & \dots & 0 & \dots \end{pmatrix},$$

Such that $s_i \not\equiv 0(\ mod\ p)$ and $s_j \not\equiv 0(\ mod\ p)$. Then the number of column smaller than P, so $|S| \leq 2\,|G|\,|I| \leq 2p^{\alpha+1}$.

Proposition 10. Let $|\Lambda| = 2$, a matrix P has in every column 0. Then the group G is trivial.

Proof. Let G be a group of order m and generating element g. By the condition of the proposition we must consider the case when $I = \Lambda = \{1,2\}$ and the matrix P is an identity. We study the sets

$$X = \{(1,e,2)\,,(2,e,1)\,,(2,g,2)\}\,,$$

$$Y = \{(1,e,2)\,,(2,g,1)\}\,.$$

Since

$$(2,e,1) = ((2,g,1)\ (1,e,2))^{m-1}\,(2,g,1) \in \langle Y\rangle\,,$$

$$(2,g,2) = (2,g,1)\,(1,e,2) \in \langle Y\rangle\,,$$

$$(2,g,1) = (2,g,2)\,(2,e,1) \in \langle X\rangle\,.$$

We get that $\langle X\rangle = \langle Y\rangle$. It is easy to see that X is an independence and we have a contradiction.

The Main Theorem

Theorem 3. Let S be a completely simple semigroup, then S is semigroup with basis property if and only if it is satisfying the following conditions:

(1) S is group with basis property,

(2) S is left or right group, which is the maximal subgroup either trivial or primary cyclic subgroup,

(3) S isomorphic to Rees matrix semigroup $M(G;I,\Lambda;P)$ with sandwich matrix P over group G, which is either trivial or primary cyclic, and then $\min\{|I|\,,\ |\Lambda|\} = 2$. Also sandwich matrix $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$ satisfying

$$\forall i,j \in I\ \forall\lambda,\mu \in \Lambda\ \left(i \neq j,\ \lambda \neq \mu \Rightarrow \left\langle p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1}\right\rangle = G\right)\,.$$

Proof. In fact, let S be a semigroup with basis property. If S a group then S is a group with basis property by proposition1. On the other hand if we assume that S is not a group, then S isomorphic to Rees matrix semigroup $M(G;I,\Lambda;P)$ with the sandwich matrix P over group G. Since S is not a group, then $|I| \geq 1$ or $|\Lambda| \geq 1$. If we take $|I| = 1$ or $|\Lambda| \geq 1$, then by proposition 2 the group G either trivial or a primary cyclic group (also if $|\Lambda| = 1$, S is either rectangular band, or rectangular band primary cyclic group. Therefore let $|I| \geq 2$, $|\Lambda| \geq 2$. Then by proposition 8 we have $\min\{|I|, |\Lambda|\} = 2$. If S is a completely simple semigroup, then by proposition 9 the condition (2-11) holds, a group G either trivial or primary cyclic group by proposition 3.

The suffice condition proves in the same investigation .

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