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CAYLEY GRAPHS OF DIRECT PRODUCTS AND OF 0-DIRECT UNION OF SEMIGROUPS

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Abstract. In this paper, we first prove that Cayley graphs of direct products of semigroups equal direct products of Cayley graphs under certain conditions and then describe Cayley graphs of 0-direct union of semigroups.

Keywords: direct product of simigroups; generalized Cayley graphs; 0-direct union.

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1. Introduction and preliminaries

The investigation and characterization of digraphs that are Cayley graphs of certain algebraic structures have a long history. There are rich results of research on the Cayley graphs of groups. In recent years, stimulated by the interesting results, the Cayley graphs of semigroups have also been considered by various authors, see, for example, [1, 3, 5, 7, 10, 11, 12]. The Cayley graphs of semigroups are closely related to the finite state automata and have many valuable applications, see [4, 9].

If S is a semigroup, and T is a nonempty subset of S , the so-called *connection set*, then the *Cayley graph* $Cay(S,T)$ of S relative to T is usually defined as the graph with vertex set S and

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edge set $E(Cay(S,T))$ consisting of those ordered pairs (x,y), where $xt = y$ for some $t \in T$, see [1, 5].

In 2012, Zhu [14] first introduced the concept of generalized Cayley graphs of semigroups and discussed their fundamental properties, and then studied a special case, the universal Cayley graphs of semigroups.

Following [14], Zhu [15] continued discussion of generalized Cayley graphs of semigroups. The Cayley D-saturated property of generalized Cayley graphs of semigroups was considerd. In addition, for some basic graphs and their complete fission graphs, the author described all semigroups whose universal Cayley graphs are isomorphic to these graphs.

In this paper, we continue the research of Cayley graphs of semigroups. First, we consider the relation between Cayley graphs of direct products of semigroups and then describe Cayley graphs of 0-direct union of semigroups.

Now we are in a position to present the main definitions and notations of this paper.

Throughout the paper, for a graph Γ, denote by $V(\Gamma)$ and $E(\Gamma)$ its vertex set and edge set, respectively.

Recall that if S is an ideal of a semigroup T, then we call T an *ideal extension* of S. For any semigroup T, let

$$T^1 = \begin{cases} T & \text{if } T \text{ has an identify,} \\ T \cup \{1\} & \text{otherwise,} \end{cases}$$

where 1 is an extra identity element, see [2].

Definition 1.1 Let $\Gamma = (V,E)$, $\Gamma_i = (V_i,E_i)$, $i = 1,2$ be three graphs. We call Γ the direct product of Γ_1 and Γ_2 if the following two conditions hold:

(a) $V = V_1 \times V_2 = \{(v_1,v_2) \mid v_i \in V_i,\ i = 1,2\}$;

(b) $((a_1,b_1),(a_2,b_2)) \in E$ if and only if $(a_1,a_2) \in V_1$ and $(b_1,b_2) \in V_2$.

Definition 1.2 Let T be an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$. The *Cayley graph* $\mathrm{Cay}(S,\rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set $E(Cay(S,\rho))$ consisting of those ordered pairs (a,b), where $xay = b$ for some $(x,y) \in \rho$. We also call the Cayley graphs defined in this way the *generalized Cayley graphs*, in order to distinguish them from the usual ones.

Definition 1.3 Let I be an index set. Assume that $\{S_\alpha\}_{\alpha\in I}$ is a class of semigroups such that S_α has a zero element 0 for any $\alpha \in I$, and $S_\alpha \cap S_\beta = \{0\}$ for any $\alpha \neq \beta \in I$. Let $S = \cup_{\alpha\in I} S_\alpha$. For any $a, b \in S$, we define

$$ab = \begin{cases} ab & \text{if there exists } \alpha \in I \text{ such that } a, b \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then S is a semigroup with the above operation. And S is called *a* 0*-direct union of* $\{S_\alpha\}_{\alpha\in I}$.

Definition 1.4 Let T_i be an ideal extension of a semigroup S_i, $\rho_i \subseteq T_i^1 \times T_i^1$, $i = 1, 2$ and $\rho \subseteq (T_1 \times T_2)^1 \times (T_1 \times T_2)^1$. For any $x_1, y_1 \in T_1^1$, $x_2, y_2 \in T_2^1$, if the following four conditions hold:

$$((x_1,x_2),(y_1,y_2)) \in \rho \Leftrightarrow (x_1,y_1) \in \rho_1, (x_2,y_2) \in \rho_2, \tag{0.1}$$

$$(1,(y_1,y_2)) \in \rho \Leftrightarrow (1,y_1) \in \rho_1, (1,y_2) \in \rho_2, \tag{0.2}$$

$$((x_1,x_2),1) \in \rho \Leftrightarrow (x_1,1) \in \rho_1, (x_2,1) \in \rho_2, \tag{0.3}$$

$$(1,1) \in \rho \Leftrightarrow (1,1) \in \rho_1, (1,1) \in \rho_2, \tag{0.4}$$

then we call ρ connecting ρ_1 to ρ_2 by the means of direct product.

Let A be a set. If $\rho \subseteq A \times A$, then ρ is called a relation on set A. We define the *domain* $\mathrm{dom}(\rho)$ of ρ by

$$\mathrm{Dom}\rho = \{x \in A \mid \exists y \in A, (x,y) \in \rho\},$$

the *image* $\mathrm{Im}(\rho)$ of ρ by

$$\mathrm{Im}\rho = \{y \in A \mid \exists x \in A, (x,y) \in \rho\}$$

and let

$$\mathrm{ID}\rho = \mathrm{Im}\rho \cup \mathrm{Dom}\rho.$$

2. Cayley graphs of direct products

In this section, we consider the relation between Cayley graphs of direct products of semigroups and direct products of Cayley graphs of semigroups, establish the following result.

Theorem 2.1 *Let S_1, S_2 be semigroups, $S = S_1 \times S_2$ and T an ideal extension of S. Let T_i be an ideal extension of S_i, $\rho_i \subseteq T_i^1 \times T_i^1$, $i = 1, 2$ and $\rho \subseteq (T_1 \times T_1)^1 \times (T_1 \times T_1)^1$. Then*

$$Cay(S,\rho) = Cay(S_1,\rho_1) \times Cay(S_2,\rho_2).$$

Proof. For simplicity, we will assume that $\Gamma = Cay(S,\rho)$, $\Gamma_i = Cay(S_i,\rho_i)$, $i = 1,2$. Obviously, $V(\Gamma) = V(\Gamma_1 \times \Gamma_2)$.

By using Definition 1.2 and Definition 1.4, we have

$$((a_1,a_2),(b_1,b_2)) \in E(\Gamma) \text{ if and only if one of } (0.5),(0.7),(0.9),(0.11) \text{ holds.}$$

(i)

$$\exists((x_1,x_2),(y_1,y_2)) \in \rho, (x_1,x_2)(a_1,a_2)(y_1,y_2) = (b_1,b_2) \tag{0.5}$$

$$\Leftrightarrow \quad (x_1a_1y_1, x_2a_2y_2) = (b_1,b_2)$$

$$\Leftrightarrow \quad x_1a_1y_1 = b_1, x_2a_2y_2 = b_2,\ \exists(x_1,y_1) \in \rho_1, (x_2,y_2) \in \rho_2 \tag{0.6}$$

$$\Rightarrow \quad (a_1,b_1) \in E(\Gamma_1), (a_2,b_2) \in E(\Gamma_2)$$

$$\Leftrightarrow \quad ((a_1,a_2),(b_1,b_2)) \in E(\Gamma_1 \times \Gamma_2).$$

(ii)

$$\exists(1,(y_1,y_2)) \in \rho, 1(a_1,a_2)(y_1,y_2) = (b_1,b_2) \tag{0.7}$$

$$\Leftrightarrow \quad (1a_1y_1, 1a_2y_2) = (b_1,b_2)$$

$$\Leftrightarrow \quad 1a_1y_1 = b_1, 1a_2y_2 = b_2, \exists(1,y_1) \in \rho_1, \exists(1,y_2) \in \rho_2 \tag{0.8}$$

$$\Rightarrow \quad (a_1,b_1) \in E(\Gamma_1), (a_2,b_2) \in E(\Gamma_2)$$

$$\Leftrightarrow \quad ((a_1,a_2),(b_1,b_2)) \in E(\Gamma_1 \times \Gamma_2).$$

(iii)

$$\exists((x_1,x_2),1)\in\rho,(x_1,x_2)(a_1,a_2)1=(b_1,b_2) \tag{0.9}$$

$$\Leftrightarrow\quad (x_1a_11,x_2a_21)=(b_1,b_2)$$

$$\Leftrightarrow\quad x_1a_11=b_1,x_2a_21=b_2,\exists(x_1,1)\in\rho_1,\exists(x_2,1)\in\rho_2 \tag{0.10}$$

$$\Rightarrow\quad (a_1,b_1)\in E(\Gamma_1),(a_2,b_2)\in E(\Gamma_2)$$

$$\Leftrightarrow\quad ((a_1,a_2),(b_1,b_2))\in E(\Gamma_1\times\Gamma_2).$$

(iv)

$$\exists(1,1)\in\rho,1(a_1,a_2)1=(b_1,b_2) \tag{0.11}$$

$$\Leftrightarrow\quad (1a_11,1a_21)=(b_1,b_2)$$

$$\Leftrightarrow\quad 1a_11=b_1,1a_21=b_2,\exists(1,1)\in\rho_1,\exists(1,1)\in\rho_2 \tag{0.12}$$

$$\Rightarrow\quad (a_1,b_1)\in E(\Gamma_1),(a_2,b_2)\in E(\Gamma_2)$$

$$\Leftrightarrow\quad ((a_1,a_2),(b_1,b_2))\in E(\Gamma_1\times\Gamma_2).$$

Conversely, if $((a_1,a_2),(b_1,b_2))\in E(\Gamma_1\times\Gamma_2)$, then one of (0.6), (0.8), (0.10), (0.12) holds. Therefore, we find that (0.5) and (0.6), (0.7) and (0.8), (0.9) and (0.10), (0.11) and (0.12) are equivalent to each other. Thus one of (0.5), (0.7), (0.9), (0.11) holds. Hence we have $((a_1,a_2),(b_1,b_2))\in E(\Gamma)$. This completes the proof of Theorem 2.1.

3. Cayley graphs of 0-direct union of semigroups

Let us begin by introducing the notion of *concentric graph*.

Definition 3.1. Assume that $\Gamma_0=(V_0,E_0)$ is a graph and $a\in V_0$. Let $E_1=\{(v_0,a)\mid v_0\in V_0\}$, $E=E_0\cup E_1$ and $V=V_0$. The new graph $\Gamma=(V,E)$, denoted by $C(\Gamma_0;a)$, is called *a concentric graph* converging to the point a.

Now we give the main results of this section. In what follows, the symbol I will always mean an index set with $|I|>1$ (unless otherwise specified).

Theorem 3.2. *Assume that* $\{S_\alpha\}_{\alpha\in I}$ *is a class of semigroups such that* S_α *has a zero element* 0 *for any* $\alpha\in I$*, and* $S_\alpha\cap S_\beta=\{0\}$ *for any* $\alpha\neq\beta\in I$*. Let* T_α *be an ideal extension of* S_α *and* S*,* T *the* 0*-direct union of* $\{S_\alpha\}_{\alpha\in I}$*,* $\{T_\alpha\}_{\alpha\in I}$*, respectively. Set* $\rho\subseteq T^1\times T^1$*. If there exist at least two elements* α_1*,* $\alpha_2\in I$ *such that* $ID\rho\cap S_{\alpha_i}\neq\emptyset$*,* $i=1,\,2$*, then*

$$Cay(S,\rho)=\bigcup_{\alpha\in I}C(Cay(S_\alpha,\rho_\alpha);0),$$

where $\rho_\alpha=\rho\cap(T^1_\alpha\times T^1_\alpha)$*.*

Proof. First of all, it is clear that $V(Cay(S,\rho))=V(\bigcup\limits_{\alpha\in I}C(Cay(S_\alpha,\rho_\alpha);0))$.

Note that for any $a,b\in S$, we have

$$(a,b)\in E(Cay(S,\rho)) \text{ if and only if there exists } (x,y)\in\rho \text{ such that } xay=b,$$

where $\rho\subseteq T^1\times T^1$.

Suppose that $\mathrm{ID}\rho\cap S_{\alpha_1}\neq\emptyset$ and $\mathrm{ID}\rho\cap S_{\alpha_2}\neq\emptyset$. And let $x_{\alpha_1}=\mathrm{ID}\rho\cap S_{\alpha_1}$, $x_{\alpha_2}=\mathrm{ID}\rho\cap S_{\alpha_2}$. By Definition 1.2, there exist $(x_{\alpha_i},x)\in\rho$ or $(x,x_{\alpha_i})\in\rho$, $i=1,2$ such that

$$x_{\alpha_i}ax=0, xax_{\alpha_i}=0.$$

So a converges to the point 0. Thus, according to Definition 3.1, $Cay(S_{\alpha_1},\rho_\alpha)$ also converges to the point 0. Hence $E(Cay(S,\rho))=E(\bigcup\limits_{\alpha\in I}C(Cay(S_\alpha,\rho_\alpha);0))$. This completes the proof of Theorem 3.2.

Proceeding as Theorem 3.2, we can prove the following two theorems.

Theorem 3.3. *Assume that* $\{S_\alpha\}_{\alpha\in I}$ *is a class of semigroups such that for any* $\alpha\in I$*,* S_α *has a zero element* 0*, and for any* $\alpha\neq\beta\in I$*,* $S_\alpha\cap S_\beta=\{0\}$*. Let* T_α *be an ideal extension of* S_α *and* S*,* T *the* 0*-direct union of* $\{S_\alpha\}_{\alpha\in I}$*,* $\{T_\alpha\}_{\alpha\in I}$ *respectively. Set* $\rho\subseteq T^1\times T^1$*. If there exists* $\alpha\in I$ *such that* $ID\rho\subseteq S_\alpha$ *and* $ID\rho\cap S_\alpha\neq\emptyset$*. If* $ID\rho\cap S_\alpha$ *has a nonzero element, then*

$$Cay(S,\rho)=Cay(S_\alpha,\rho_\alpha)\bigcup\cup_{\beta\neq\alpha}C(Cay(S_\beta,\rho_\beta);0).$$

Theorem 3.4. *Assume that* $\{\Gamma_\alpha\}_{\alpha\in I}$ *is a class of semigroups such that* $E_\alpha\cap E_\beta=\emptyset$ *and* $V_\alpha\cap V_\beta=\{0\}$ *for any* $\alpha\neq\beta\in I$*. For any* $\alpha\in I$*, we can define a semigroup on the set* V_α*, and let* T_α *be an ideal extension of* S_α*. It is clear that for any* $\alpha\neq\beta\in I$*,* $T_\alpha\cap T_\beta=\{0\}$*. Let* $\rho_\alpha\in T^1_\alpha\times T^1_\alpha$

and $\rho = \bigcup_{\alpha \in I} \rho_\alpha$. *If there exist* $\alpha \neq \beta \in I$ *such that both* $ID\rho_\alpha$ *and* $ID\rho_\beta$ *have at least one nonzero element, then* $Cay(S,\rho) = \bigcup_{\alpha \in I} C(Cay(S_\alpha, \rho_\alpha); 0)$. .

Conflict of Interests

The author declares that there is no conflict of interests.

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