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FIXED POINT ITERATIONS OF SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, the problem of finding fixed points of semigroups of nonexpansive mappings is investigated based on an iterative algorithms. Strong convergence theorems of fixed points are obtained.

Keywords: semigoup; nonexpansive mapping; fixed point; Hilbert space.

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1. Introduction-Preliminaries

Recently, iterative algorithms have been investigated for many problems, such as economy, mechanics, transportation and optimization; see [1-11] and the references therein. In this paper, we always assume that H is a real Hilbert space. Let T be a nonlinear mapping with the domain $D(T)$. A point $x \in D(T)$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in D(T) : Tx = x\}$. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(A).$$

Recall that a family $S = \{T(s)|s \geq 0\}$ of mappings from H into itself is called a one-parameter nonexpansive semigroup if it satisfies the following conditions:

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(i) $T(0)x = x$, $\forall x \in H$;

(ii) $T(s+t)x = T(s)T(t)x$, $\forall s,t \geq 0$ and $\forall x \in H$;

(iii) $\|T(s)x - T(s)y\| \leq \|x-y\|$, $\forall s \geq 0$ and $\forall x,y \in H$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(S)$ the set of common fixed points of S, that is, $F(S) = \bigcap_{0\leq s<\infty} F(T(s))$. Let C be a nonempty closed and convex subset of H. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. If T enjoys a nonempty fixed point set, Browder [12] proved the following well-known strong convergence theorem.

let T be a nonexpansive mapping on C. Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1-t)Tz_t$ for $t \in (0,1)$. Then as $t \to 0$, $\{z_t\}$ converges strongly to a element of $F(T)$ nearest to u.

Halpern [13] considered the following explicit iteration:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

and proved the following theorem.

Let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.2). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u.

In 1977, Lions [14] improved the result of Halpern, still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}^2} = 0$.

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. This was overcome in 1992 by Wittmann [15], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Recall that a mapping $f : H \to H$ is an α-contraction if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.$$

Recall that An operator A is strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \forall x \in H$.

Lemma 1.1 [16] *Let D be a nonempty bounded closed convex subset of a Hilbert space H and let $S = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on D. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in D} \|\frac{1}{t}\int_0^t T(s)xds - T(h)\frac{1}{t}\int_0^t T(s)xds\| = 0.$$

Lemma 1.2 [17] *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed, i.e. if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ strongly converges to y, then $(I - T)x = y$.*

Lemma 1.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C(i.e., for $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations: $\langle x - z, y - z\rangle \leq 0, \forall y \in C$.*

Lemma 1.4. *Let H be a Hilbert space, f a α-contraction, and A a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\alpha$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A-\gamma f$ *is strongly monotone with coefficient* $\bar{\gamma}-\alpha\gamma$. $\langle x-y,(I-f)x-(I-f)y\rangle \geq 0, \quad x,y\in H.$

Lemma 1.6 *Assume A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient* $\bar{\gamma}>0$ *and* $0<\rho\leq \|A\|^{-1}$. *Then* $\|I-\rho A\|\leq 1-\rho\bar{\gamma}$.

Lemma 1.7 *Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the following condition:*

$$\alpha_{n+1}\leq (1-\gamma_n)\alpha_n+\gamma_n\sigma_n, \quad \forall n\geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence of real numbers such that*

(i) $\lim_{n\to\infty}\gamma_n=0$ *and* $\sum_{n=0}^{\infty}\gamma_n=\infty$,

(ii) *either* $\limsup_{n\to\infty}\sigma_n\leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n|<\infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ *converges to zero.*

2. Main results

Theorem 2.1. *Let H be a real Hilbert space H, C a closed and convex subset of H. Let* $S=\{T(s):0\leq s<\infty\}$ *be a nonexpansive semigroup such that* $F(S)\neq\emptyset$. *Let* $\{s_n\}$ *be a positive real divergent sequence and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0,1)$ *satisfying the following conditions* $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$. *Let f be an* α*-contraction and let A be a strongly positive linear bounded self-adjoint operator with the coefficient* $\bar{\gamma}>0$. *Assume that* $0<\gamma<\bar{\gamma}/\alpha$. *Then sequence* $\{x_n\}$ *defined by*

$$x_0\in C, \quad x_{n+1}=Proj_C\Big(\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)\frac{1}{s_n}\int_0^{s_n}T(s)x_n ds\Big), \quad n\geq 0.$$

strongly converges to $x^*\in F(S)$.

Proof. We first prove that the sequence $\{x_n\}$ is bounded. $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$, we may assume, with no loss of generality, that $\frac{\alpha_n}{1-\beta_n}<\|A\|^{-1}$ for all $n\geq 0$. From Lemma 1.6, we know that $\|(1-\beta_n)I-\alpha_n A\| \le (1-\beta_n-\alpha_n\bar{\gamma})$. Picking $p \in F(S)$, we have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\le \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds-p)\| \\
&\le \alpha_n\|\gamma f(x_n)-Ap\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\gamma})\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds-p\| \\
&\le \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\| \\
&\le [1-\alpha_n(\bar{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|.
\end{aligned}$$

By simple inductions, we see that

$$\|x_n-p\| \le \max\{\|x_0-p\|, \frac{\|Ap-\gamma f(p)\|}{\bar{\gamma}-\gamma\alpha}\},$$

which yields that the sequence $\{x_n\}$ is bounded. Now, we are in a position to prove that

$$\limsup_{n\to\infty}\langle(\gamma f-A)x^*, y_n-x^*\rangle \le 0,$$

where $y_n = \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds$. Putting $z_0 = P_{F(S)}x_0$, we see that the closed ball M of center z_0 and radius $\max\{\|z_0-p\|, \frac{\|Az_0-\gamma f(z_0)\|}{\bar{\gamma}-\gamma\alpha}\}$ is $T(s)$-invariant for each $s \in [0,\infty)$ and contain $\{x_n\}$. Therefore, we assume, without loss of generality, $S = \{T(s) : 0 \le s < \infty\}$ is a nonexpansive semigroup on M. It follows from Lemma 1.1 that $\lim_{n\to\infty}\|y_n - T(h)y_n\| = 0$ for all $0 \le h < \infty$. Taking a suitable subsequence $\{y_{n_i}\}$ of $\{y_n\}$, we see that

$$\limsup_{n\to\infty}\langle(\gamma f-A)x^*, y_n-x^*\rangle = \lim_{i\to\infty}\langle(\gamma f-A)x^*, y_{n_i}-x^*\rangle.$$

Since the sequence $\{y_n\}$ is also bounded, we may assume that $y_{n_i} \rightharpoonup \bar{x}$. From the demiclosedness principle, we have $\bar{x} \in F(S)$. Therefore, we have

$$\limsup_{n\to\infty}\langle(\gamma f-A)x^*, y_n-x^*\rangle = \langle(\gamma f-A)x^*, \bar{x}-x^*\rangle \le 0.$$

On the other hand, we have $\|x_{n+1}-y_n\| \le \alpha_n\|\gamma f(x_n)-Ax_n\|+\beta_n\|x_n-y_n\|$. From the assumption $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ that $\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0$, which gives that $\limsup_{n\to\infty}\langle(\gamma f-$

$A)x^*, x_{n+1} - x^*\rangle \leq 0.$

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n \Big(\gamma \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \Big) \\
&\quad + \beta_n \|x_n - x^*\| \|x_{n+1} - x^*\| + \|(1-\beta_n)I - \alpha_n A\| \|y_n - x^*\| \|x_{n+1} - x^*\| \\
&\leq \alpha_n \alpha \gamma \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad + \beta_n \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \| \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle. \\
&\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|x_{n+1} - x^*\|^2 + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1} - x^*\|^2 \leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] \|x_n - x^*\|^2 + 2\alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle.$$

In view of Lemma 1.7, we obtain the desired conclusion easily. This completes the proof.

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