Available online at http://scik.org
J. Semigroup Theory Appl. 2014, 2014:1
ISSN: 2051-2937

ON (m,n)-IDEALS OF LEFT ALMOST SEMIGROUPS

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Abstract. In this paper, we study (m,n)-ideals of an $\mathscr{LA}$-semigroup in detail. We characterize $(0,2)$-ideals of an $\mathscr{LA}$-semigroup S and prove that A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S. We also show that an $\mathscr{LA}$-semigroup S is 0-$(0,2)$-bisimple if and only if S is right 0-simple. Furthermore we study 0-minimal (m,n)-ideals in an $\mathscr{LA}$-semigroup S and prove that if R (L) is a 0-minimal right (left) ideal of S, then either $R^mL^n = \{0\}$ or R^mL^n is a 0-minimal (m,n)-ideal of S for $m,n \geq 3$. Finally we discuss (m,n)-ideals in an (m,n)-regular $\mathscr{LA}$-semigroup S and show that S is $(0,1)$-regular if and only if $L = SL$ where L is a $(0,1)$-ideal of S.

Keywords: $\mathscr{LA}$-semigroups, left invertive law, left identity, (m,n)-ideals.

2010 AMS Subject Classification: 20M10.

1. Introduction

A left almost semigroup ($\mathscr{LA}$-semigroup) is a groupoid S satisfying the left invertive law $(ab)c = (cb)a$ for all $a,b,c \in S$. This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$. The concept of an $\mathscr{LA}$-semigroup was

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Received April 16, 2014

first given by Kazim and Naseeruddin in 1972 [2]. An $\mathcal{LA}$-semigroup satisfies the medial law $(ab)(cd) = (ac)(bd)$ for all $a,b,c,d \in S$. Since $\mathcal{LA}$-semigroups satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [11]. If an $\mathcal{LA}$-semigroup S contains a left identity (unitary $\mathcal{LA}$-semigroup), then it satisfies the paramedial law $(ab)(cd) = (dc)(ba)$ and the identity $a(bc) = b(ac)$ for all $a,b,c,d \in S$ [6].

An $\mathcal{LA}$-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An $\mathcal{LA}$-semigroup is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures. It has been investigated in [6] that if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes a commutative semigroup. The connection of a commutative inverse semigroup with an $\mathcal{LA}$-semigroup has been given by Yousafzai et al. in [12] as, a commutative inverse semigroup $(S, .)$ becomes an $\mathcal{LA}$-semigroup $(S, *)$ under $a*b = ba^{-1}r^{-1}$, $\forall\ a,b,r \in S$. An $\mathcal{LA}$-semigroup S with left identity becomes a semigroup under the binary operation "$\circ_e$" defined as, $x \circ_e y = (xe)y$ for all $x,y \in S$ [13]. An $\mathcal{LA}$-semigroup is the generalization of a semigroup theory [6] and has vast applications in collaboration with semigroups like other branches of mathematics. Khan et al. studied an intra-regular class of an $\mathcal{LA}$-semigroup in [3] and proved some interesting problems by using different ideals. They proved that the set of all two-sided ideals of intra-regular $\mathcal{LA}$-semigroup forms a semilattice structure. They characterized an intra-regular $\mathcal{LA}$-semigroup by using left, right, two-sided and bi-ideals. An $\mathcal{LA}$-semigroup is the generalization of a semigroup theory [6]. Many interesting results on $\mathcal{LA}$-semigroups have been investigated in [4, 8, 9, 10].

In this paper, we investigate two classes of ideals called the (m,n)-ideals and 0-minimal ideals of an $\mathcal{LA}$-semigroup and their characterizations. First we study $(0,2)$-ideals of an $\mathcal{LA}$-semigroup S and prove that A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S. Further, we characterize $(0,2)$-bi-ideals in unitary $\mathcal{LA}$-semigroups and proceed to prove that A is a 0-minimal $(0,2)$-bi-ideal of a unitary $\mathcal{LA}$-semigroup S with zero. Then either $A^2 = \{0\}$ or A is right 0-simple. We also study some interesting results in (m,n)-ideals and investigate that if A is an (m,n)-ideal of S and B is an (m,n)-ideal of A such that B is idempotent. Then B is an (m,n)-ideal of S. The concept of (m,n)-regular $\mathcal{LA}$-semigroups is indeed an important and interesting part of the paper. In this respect, we prove that if S is a unitary (m,n)-regular $\mathscr{LA}$-semigroup such that $m=n$. Then for every $R \in \mathfrak{R}_{(m,0)}$ and $L \in \mathfrak{L}_{(0,n)}$, $R \cap L = R^m L \cap RL^n$.

2. Preliminaries and examples

If S is an $\mathscr{LA}$-semigroup with product $\cdot : S \times S \longrightarrow S$, then $ab \cdot c$ and $(ab)c$ both denote the product $(a \cdot b) \cdot c$.

If there is an element 0 of an $\mathscr{LA}$-semigroup $(S,\cdot)$ such that $x \cdot 0 = 0 \cdot x = x \; \forall \; x \in S$, we call 0 a *zero element* of S.

Example 1. *Let $S = \{a,b,c,d,e\}$ with a left identity d. Then the following multiplication table shows that $(S,\cdot)$ is a unitary $\mathscr{LA}$-semigroup with a zero element a.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Example 2. *Let $S = \{a,b,c,d\}$. Then the following multiplication table shows that $(S,\cdot)$ is an $\mathscr{LA}$-semigroup with a zero element a.*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	d	d	c
c	a	c	c	c
d	a	c	c	c

The above $\mathscr{LA}$-semigroup S has commutative powers, that is $aa \cdot a = a \cdot aa$ for all $a \in S$ which is called a locally associative $\mathscr{LA}$-semigroup [7]. Note that S has no associative powers for all $a \in S$ because $(bb \cdot b)b \neq b(bb \cdot b)$ for $b \in S$.

Assume that S is an $\mathscr{LA}$-semigroup. Let us define $a^1 = a$ and $a^m = ((((aa)a)a)...a)a = a^{m-1}a$ for all $a \in S$ where $m \geq 1$. It is easy to see that $a^m = a^{m-1}a = aa^{m-1}$ for all $a \in S$ and $m \geq 3$ if S has a left identity. Also, we can show by induction, $(ab)^m = a^m b^m$ and $a^m a^n = a^{m+n}$ hold for all $a, b \in S$ and $m, n \geq 3$.

A subset A of an $\mathscr{LA}$-semigroup S is called a *right* (*left*) ideal of S if $AS \subseteq A$ $(SA \subseteq A)$, and is called an *ideal* of S if it is both left and right ideal of S.

A subset A of an $\mathscr{LA}$-semigroup S is called an *$\mathscr{LA}$-subsemigroup* of S if $A^2 \subseteq A$.

The concept of (m,n)-ideals of a semigroup and an $\mathscr{LA}$-semigroup was given in [5] and [1] respectively.

An $\mathscr{LA}$-subsemigroup A of an $\mathscr{LA}$-semigroup S is said to be an *(m,n)-ideal* of S if $A^m S \cdot A^n \subseteq A$ where m,n are non-negative integers such that $m = n \neq 0$. Here A^m or A^n are suppressed if $m = 0$ or $n = 0$, that is $A^0 S = S$ or $SA^0 = S$. Note that if $m = n = 1$, then an (m,n)-ideal A of an $\mathscr{LA}$-semigroup S is called a *bi-ideal* of S. If we take $m = 0$ or $n = 0$, then an (m,n)-ideal A of an $\mathscr{LA}$-semigroup S becomes a left or a right ideal of S.

An (m,n)-ideal A of an $\mathscr{LA}$-semigroup S with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only (m,n)-ideal of S properly contained in A.

An $\mathscr{LA}$-semigroup S with zero is said to be 0-$(0,2)$-*bisimple* if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S.

An $\mathscr{LA}$-semigroup S with zero is said to be *nilpotent* if $S^l = \{0\}$ for some positive integer l.

Let m,n be non-negative integers and S be an $\mathscr{LA}$-semigroup. We say that S is *(m,n)-regular* if for every element $a \in S$ there exists some $x \in S$ such that $a = (a^m x)a^n$. Note that a^0 is defined as an operator element such that $a^0 y = y$ and $za^0 = z$ for any $y,z \in S$.

3. 0-minimal $(0,2)$-bi-ideals in unitary $\mathscr{LA}$-semigroups

If S is a unitary $\mathscr{LA}$-semigroup, then it is easy to see that $S^2 = S$, $SA^2 = A^2S$ and $A \subseteq SA$ $\forall\, A \subseteq S$. Note that every right ideal of a unitary $\mathscr{LA}$-semigroup S is a left ideal of S but the converse is not true in general. Example 1 shows that there exists a subset $\{a,b,e\}$ of S which is a left ideal of S but not a right ideal of S. It is easy to see that SA and SA^2 are the left and right ideals of a unitary $\mathscr{LA}$-semigroup S. Thus SA^2 is an ideal of a unitary $\mathscr{LA}$-semigroup S.

Lemma 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Let A be a $(0,2)$-ideal of S, then $SA \cdot A = AA \cdot S = SA^2 \subseteq A$ and $A \cdot SA = S \cdot AA = SS \cdot AA = SA^2 \subseteq A$. Hence A is an ideal of a left ideal SA of S.

Conversely, assume that A is a left ideal of a left ideal L of S, then

$$SA^2 = AA \cdot S = SA \cdot A \subseteq SL \cdot A \subseteq LA \subseteq A,$$

and clearly A is an $\mathscr{LA}$-subsemigroup of S, therefore A is a $(0,2)$-ideal of S. ∎

Corollary 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S.*

Lemma 2. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-bi-ideal of S if and only if A is an ideal of some right ideal of S.*

Proof. Let A be a $(0,2)$-bi-ideal of S, then $SA^2 \cdot A = A^2S \cdot A = AS \cdot A^2 \subseteq SA^2 \subseteq A$ and $A \cdot SA^2 = SS \cdot AA^2 = A^2A \cdot SS = SA \cdot A^2 \subseteq SA^2 \subseteq A$. Hence A is an ideal of some right ideal SA^2 of S.

Conversely, assume that A is an ideal of a right ideal R of S, then

$$SA^2 = A \cdot SA = A \cdot (SS)A = A \cdot (AS)S \subseteq A \cdot (RS)R \subseteq AR \subseteq A,$$

and $(AS)A \subseteq (RS)A \subseteq RA \subseteq A$, which shows that A is a $(0,2)$-ideal of S. ∎

Theorem 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then the following statements are equivalent.*

(i) *A is a* $(1,2)$*-ideal of* S;

(ii) *A is a left ideal of some bi-ideal of* S;

(iii) *A is a bi-ideal of some ideal of* S;

(iv) *A is a* $(0,2)$*-ideal of some right ideal of* S;

(v) *A is a left ideal of some* $(0,2)$*-ideal of* S.

Proof. $(i) \Longrightarrow (ii)$. It is easy to see that $SA^2 \cdot S$ is a bi-ideal of S. Let A be a $(1,2)$-ideal of S, then

$$(SA^2 \cdot S)A = (SA^2 \cdot SS)A = (SS \cdot A^2S)A = (S \cdot A^2S)A = A^2S \cdot A$$
$$= AS \cdot A^2 \subseteq A,$$

which shows that A is a left ideal of a bi-ideal $SA^2 \cdot S$ of S.

$(ii) \Longrightarrow (iii)$. Let A be a left ideal of a bi-ideal B of S, then

$$(A \cdot SA^2)A = (S \cdot AA^2)A \subseteq [S(SA \cdot AA)]A = [S(AA \cdot AS)]A$$
$$= [AA \cdot S(AS)]A = [\{S(AS) \cdot A\}A]A = [(AS \cdot A)A]A$$
$$\subseteq [(BS \cdot B)A]A \subseteq BA \cdot A \subseteq A,$$

which shows that A is a bi-ideal of an ideal SA^2 of S.

$(iii) \Longrightarrow (iv)$. Let A be a bi-ideal of an ideal I of S, then

$$SA^2 \cdot A^2 = (A^2 \cdot AA)S = (A \cdot A^2A)S \subseteq [A \cdot (AI)A]S = AA \cdot S$$
$$= SA \cdot A \subseteq SI \cdot S \subseteq I,$$

which shows that A is a $(0,2)$-ideal of a right ideal SA^2 of S.

$(iv) \Longrightarrow (v)$. It is easy to see that SA^3 is a $(0,2)$-ideal of S. Let A be a $(0,2)$-ideal of a right ideal R of S, then

$$A \cdot SA^3 = A(SS \cdot A^2A) = A(AA^2 \cdot S) \subseteq A[(SA \cdot AA)S] = A[(AA \cdot AS)S]$$
$$= (AA)[(A \cdot AS)S] = [S \cdot A(AS)]A^2 = [A \cdot S(AS)]A^2$$
$$\subseteq RS \cdot A^2 \subseteq RA^2 \subseteq A,$$

which shows that A is a left ideal of a $(0,2)$-ideal SA^3 of S.

$(v) \Longrightarrow (i)$. Let A be a left ideal of a $(0,2)$-ideal O of S, then

$$AS \cdot A^2 = (AA \cdot SS)A = SA^2 \cdot A \subseteq SO^2 \cdot A \subseteq OA \subseteq A,$$

which shows that A is a $(1,2)$-ideal of S. ∎

Lemma 3. *Let S be a unitary $\mathscr{LA}$-semigroup and A be an idempotent subset of S. Then A is a $(1,2)$-ideal of S if and only if there exist a left ideal L and a right ideal R of S such that $RL \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-ideal of S such that A is idempotent. Setting $L = SA$ and $R = SA^2$, then

$$\begin{aligned} RL &= SA^2 \cdot SA = A^2S \cdot SA = (SA \cdot SS)A^2 = (SS \cdot AS)A^2 \\ &= [S(AA \cdot SS)]A^2 = [S(SS \cdot AA)]A^2 = [S\{A(SS \cdot A)\}]A^2 \\ &= [A(S \cdot SA)]A^2 \subseteq AS \cdot A^2 \subseteq A. \end{aligned}$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right ideal and L be a left ideal of S such that $RL \subseteq A \subseteq R \cap L$, then $AS \cdot A^2 = AS \cdot AA \subseteq RS \cdot SL \subseteq RL \subseteq A$. ∎

Assume that S is a unitary $\mathscr{LA}$-semigroup with zero. Then it is easy to see that every left (right) ideal of S is a $(0,2)$-ideal of S. Hence if O is a 0-minimal $(0,2)$-ideal of S and A is a left (right) ideal of S contained in O, then either $A = \{0\}$ or $A = O$.

Lemma 4. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is a 0-minimal ideal of S and O is an $\mathscr{LA}$-subsemigroup of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-ideal of S contained in a 0-minimal ideal A of S. Then $SO^2 \subseteq O \subseteq A$. Since SO^2 is an ideal of S, therefore by minimality of A, $SO^2 = \{0\}$ or $SO^2 = A$. If $SO^2 = A$, then $A = SO^2 \subseteq O$ and therefore $O = A$. Let $SO^2 = \{0\}$, then $O^2S = SO^2 = \{0\} \subseteq O^2$, which shows that O^2 is a right ideal of S, and hence an ideal of S contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then $SO^2 = O^2S = \{0\}S = \{0\} = O^2$. Now if $O = A$, then $SO^2 = SS \cdot OO = SA \cdot SA \subseteq A = O$, which shows that O is a $(0,2)$-ideal of S contained in A. ∎

Corollary 2. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is a 0-minimal left ideal of S and O is an $\mathscr{LA}$-subsemigroup of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Lemma 5. *Let S be a unitary $\mathscr{LA}$-semigroup with zero and O be a 0-minimal $(0,2)$-ideal of S. Then $O^2 = \{0\}$ or O is a 0-minimal right (left) ideal of S.*

Proof. Let O be a 0-minimal $(0,2)$-ideal of S, then

$$S(O^2)^2 = SS \cdot O^2O^2 = O^2O^2 \cdot S = SO^2 \cdot O^2 \subseteq OO^2 \subseteq O^2,$$

which shows that O^2 is a $(0,2)$-ideal of S contained in O, therefore by minimality of O, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $OS = OO \cdot SS = SO^2 \subseteq O$, which shows that O is a right ideal of S. Let R be a right ideal of S contained in O, then $R^2S = RR \cdot S \subseteq RS \cdot S \subseteq R$. Thus R is a $(0,2)$-ideal of S contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$. ∎

The following Corollary follows from Lemma 4 and Corollary 2.

Corollary 3. *Let S be a unitary $\mathscr{LA}$-semigroup. Then O is a minimal $(0,2)$-ideal of S if and only if O is a minimal left ideal of S.*

Theorem 2. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a minimal $(2,1)$-ideal of S if and only if A is a minimal bi-ideal of S.*

Proof. Let A be a minimal $(2,1)$-ideal of S. Then

$$\begin{aligned}
[(A^2S\cdot A)^2S](A^2S\cdot A) &= [\{(A^2S\cdot A)(A^2S\cdot A)\}S](A^2S\cdot A)\\
&\subseteq [\{(AS\cdot A)(AS\cdot A)\}S](AS\cdot A)\\
&= [\{(AS\cdot AS)(AA)\}S](AS\cdot A)\\
&= [(A^2S\cdot AA)S](AS\cdot A)\\
&\subseteq [(AS\cdot AS)S](AS\cdot A)\\
&= (A^2S\cdot S)(AS\cdot A)\\
&\subseteq (AS\cdot S)(AS\cdot A) = (AS\cdot AS)(SA)\\
&= A^2S\cdot SA = AS\cdot SA^2 = (SA^2\cdot S)A\\
&= (A^2S\cdot S)A = (SS\cdot AA)A = A^2S\cdot A,
\end{aligned}$$

and similarly we can show that $(A^2S\cdot A)^2\subseteq A^2S\cdot A$. Thus $A^2S\cdot A$ is a $(2,1)$-ideal of S contained in A, therefore by minimality of A, $A^2S\cdot A = A$. Now

$$\begin{aligned}
AS\cdot A &= (AS)(A^2S\cdot A) = [(A^2S\cdot A)S]A = (SA\cdot A^2S)A\\
&= [A^2(SA\cdot S)]A \subseteq A^2S\cdot A = A,
\end{aligned}$$

It follows that A is a bi-ideal of S. Suppose that there exists a bi-ideal B of S contained in A, then $B^2S\cdot B\subseteq BS\cdot B\subseteq B$, so B is a $(2,1)$-ideal of S contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-ideal of S, then it is easy to see that A is a $(2,1)$-ideal of S. Let C be a $(2,1)$-ideal of S contained in A, then

$$\begin{aligned}
[(C^2S\cdot C)S](C^2S\cdot C) &= (SC\cdot C^2S)(C^2S\cdot C) = (SC^2\cdot CS)(C^2S\cdot C)\\
&= [C(SC^2\cdot S)](C^2S\cdot C) = [(C^2S\cdot C)(SC^2\cdot SS)]C\\
&= [(C^2S\cdot C)(S\cdot C^2S)]C = [(C^2S\cdot C)(C^2S)]C\\
&= [C^2\{(C^2S\cdot C)S\}]C \subseteq C^2S\cdot C.
\end{aligned}$$

This shows that $C^2S\cdot C$ is a bi-ideal of S, and by minimality of A, $C^2S\cdot C = A$. Thus $A = C^2S\cdot C\subseteq C$, and therefore A is a minimal $(2,1)$-ideal of S. ∎

Theorem 3. *Let A be a* 0*-minimal* $(0,2)$*-bi-ideal of a unitary* $\mathscr{LA}$*-semigroup S with zero. Then exactly one of the following cases occurs*:

(i) $A = \{0,a\}$, $a^2 = 0$;

(ii) $\forall\, a \in A\backslash\{0\}$, $Sa^2 = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a \in A\backslash\{0\}$, then $Sa^2 \subseteq A$. Also Sa^2 is a $(0,2)$-bi-ideal of S, therefore $Sa^2 = \{0\}$ or $Sa^2 = A$.

Let $Sa^2 = \{0\}$. Since $a^2 \in A$, we have either $a^2 = a$ or $a^2 = 0$ or $a^2 \in A\backslash\{0,a\}$. If $a^2 = a$, then $a^3 = a^2a = a$, which is impossible because $a^3 \in a^2S = Sa^2 = \{0\}$. Let $a^2 \in A\backslash\{0,a\}$, we have

$$S\cdot\{0,a^2\}\{0,a^2\} = SS\cdot a^2a^2 = Sa^2\cdot Sa^2 = \{0\} \subseteq \{0,a^2\},$$

and

$$[\{0,a^2\}S]\{0,a^2\} = \{0,a^2S\}\{0,a^2\} = a^2S\cdot a^2 \subseteq Sa^2 = \{0\} \subseteq \{0,a^2\}.$$

Therefore $\{0,a^2\}$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $\{0,a^2\} \neq \{0\}$ and $\{0,a^2\} \neq A$. This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-ideal of S. Therefore $a^2 = 0$ and $A = \{0,a\}$.

If $Sa^2 \neq \{0\}$, then $Sa^2 = A$. ∎

Corollary 4. *Let A be a* 0*-minimal* $(0,2)$*-bi-ideal of a unitary* $\mathscr{LA}$*-semigroup S with zero such that $A^2 \neq 0$. Then $A = Sa^2$ for every $a \in A\backslash\{0\}$.*

Lemma 6. *Let S be a unitary* $\mathscr{LA}$*-semigroup. Then every right ideal of S is a* $(0,2)$*-bi-ideal of S.*

Proof. Assume that A is a right ideal of S, then

$$SA^2 = AA\cdot SS = AS\cdot AS \subseteq AA \subseteq AS \subseteq A,\ AS\cdot A \subseteq A,$$

and clearly $A^2 \subseteq A$, therefore A is a $(0,2)$-bi-ideal of S. ∎

The converse of Lemma 6 is not true in general. Example 1 shows that there exists a $(0,2)$-bi-ideal $A = \{a,c,e\}$ of S which is not a right ideal of S.

Theorem 4. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Then $Sa^2 = S$ $\forall$ $a \in S\backslash\{0\}$ if and only if S is* 0-$(0,2)$-*bisimple if and only if S is right* 0-*simple.*

Proof. Assume that $Sa^2 = S$ for every $a \in S\backslash\{0\}$. Let A be a $(0,2)$-bi-ideal of S such that $A \neq \{0\}$. Let $a \in A\backslash\{0\}$, then $S = Sa^2 \subseteq SA^2 \subseteq A$. Therefore $S = A$. Since $S = Sa^2 \subseteq SS = S^2$, we have $S^2 = S \neq \{0\}$. Thus S is 0-$(0,2)$-bisimple. The converse statement follows from Corollary 4.

Let R be a right ideal of 0-$(0,2)$-bisimple S. Then by Lemma 6, R is a $(0,2)$-bi-ideal of S and so $R = \{0\}$ or $R = S$.

Conversely, assume that S is right 0-simple. Let $a \in S\backslash\{0\}$, then $Sa^2 = S$. Hence S is 0-$(0,2)$-bisimple. ∎

Theorem 5. *Let A be a* 0-*minimal* $(0,2)$-*bi-ideal of a unitary $\mathscr{LA}$-semigroup S with zero. Then either $A^2 = \{0\}$ or A is right* 0-*simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-ideal of S such that $A^2 \neq \{0\}$. Then by using Corollary 4, $Sa^2 = A$ for every $a \in A\backslash\{0\}$. Since $a^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$, we have $a^4 = (a^2)^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$. Let $a \in A\backslash\{0\}$, then

$$\begin{aligned}(Aa^2)S \cdot Aa^2 &= a^2A \cdot S(Aa^2) = [(S \cdot Aa^2)A]a^2 \subseteq [(S \cdot A)A]a^2 \\ &= (AA \cdot SS)a^2 = SA^2 \cdot a^2 \subseteq Aa^2,\end{aligned}$$

and

$$\begin{aligned}S(Aa^2)^2 &= S(Aa^2 \cdot Aa^2) = S(a^2A \cdot a^2A) = S[a^2(a^2A \cdot A)] \\ &= (aa)[S(a^2A \cdot A)] = [(a^2A \cdot A)S]a^2 \\ &\subseteq (AA \cdot SS)a^2 = SA^2 \cdot a^2 \subseteq Aa^2,\end{aligned}$$

which shows that Aa^2 is a $(0,2)$-bi-ideal of S contained in A. Hence $Aa^2 = \{0\}$ or $Aa^2 = A$. Since $a^4 \in Aa^2$ and $a^4 \in A\backslash\{0\}$, we get $Aa^2 = A$. Thus by using Theorem 4, A is right 0-simple. ∎

4. (m,n)-ideals in unitary $\mathscr{LA}$-semigroups

In this section, we characterize a unitary $\mathscr{LA}$-semigroup in terms of (m,n)-ideals with the assumption that $m,n \geq 5$. If we take $m,n \geq 2$, then all the results of this section can be trivially followed for a locally associative unitary $\mathscr{LA}$-semigroup. If S is a unitary $\mathscr{LA}$-semigroup, then it is easy to see that $SA^m = A^mS$ and $A^mA^n = A^nA^m$ for $m,n \geq 3$ such that $A^0 = e$ if occurs, where e is a left identity of S.

Lemma 7. *Let S be a unitary $\mathscr{LA}$-semigroup. If R and L are the right and left ideals of S respectively, then RL is an (m,n)-ideal of S.*

Proof. Let R and L be the right and left ideals of S respectively, then

$$\begin{aligned}
(RL)^mS\cdot(RL)^n &= (R^mL^m\cdot S)(R^nL^n) = (R^mL^m\cdot R^n)(SL^n) \\
&= (L^mR^m\cdot R^n)(SL^n) = (R^nR^m\cdot L^m)(SL^n) \\
&= (R^mR^n\cdot L^m)(SL^n) = (R^{m+n}L^m)(SL^n) \\
&= S(R^{m+n}L^m\cdot L^n) = S(L^nL^m\cdot R^{m+n}) \\
&= SS\cdot L^{m+n}R^{m+n} = SL^{m+n}\cdot SR^{m+n} \\
&= R^{m+n}S\cdot L^{m+n}S = SR^{m+n}\cdot SL^{m+n},
\end{aligned}$$

and

$$\begin{aligned}
SR^{m+n}\cdot SL^{m+n} &= (S\cdot R^{m+n-1}R)(S\cdot L^{m+n-1}L) \\
&= [S(R^{m+n-2}R\cdot R)][S(L^{m+n-2}L\cdot L)] \\
&= [S(RR\cdot R^{m+n-2})][S(LL\cdot L^{m+n-2})] \\
&\subseteq (SS\cdot RR^{m+n-2})(SS\cdot LL^{m+n-2}) \\
&\subseteq (SR\cdot SR^{m+n-2})(SL\cdot SL^{m+n-2}) \\
&\subseteq (R^{m+n-2}S\cdot RS)(L\cdot SL^{m+n-2})
\end{aligned}$$

$$\subseteq (R^{m+n-2}S\cdot R)(S\cdot LL^{m+n-2})$$
$$= (RS\cdot R^{m+n-2})(SL^{m+n-1})$$
$$\subseteq RR^{m+n-2}\cdot SL^{m+n-1}$$
$$\subseteq SR^{m+n-1}\cdot SL^{m+n-1},$$

therefore

$$(RL)^m S\cdot(RL)^n \subseteq SR^{m+n}\cdot SL^{m+n} \subseteq SR^{m+n-1}\cdot SL^{m+n-1} \subseteq ... \subseteq SR\cdot SL$$
$$\subseteq (SS\cdot R)L = (RS\cdot S)L \subseteq RL,$$

and also

$$RL\cdot RL = LR\cdot LR = (LR\cdot R)L = (RR\cdot L)L \subseteq (RS\cdot S)L \subseteq RL.$$

This shows that RL is an (m,n)-ideal of S. ∎

Theorem 6. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. If S has the property that it contains no non-zero nilpotent (m,n)-ideals and R (L) is a 0-minimal right $(left)$ ideal of S, then either $RL=\{0\}$ or RL is a 0-minimal (m,n)-ideal of S.*

Proof. Assume that R (L) is a 0-minimal right (left) ideal of S such that $RL \neq \{0\}$, then by lemma 7, RL is an (m,n)-ideal of S. Now we show that RL is a 0-minimal (m,n)-ideal of S. Let $\{0\} \neq M \subseteq RL$ be an (m,n)-ideal of S. Note that since $RL \subseteq R\cap L$, we have $M \subseteq R\cap L$. Hence $M \subseteq R$ and $M \subseteq L$. By hypothesis, $M^m \neq \{0\}$ and $M^n \neq \{0\}$. Since $\{0\} \neq SM^m = M^m S$, therefore

$$\{0\} \neq M^m S \subseteq R^m S = R^{m-1}R\cdot S = SR\cdot R^{m-1} = SR\cdot R^{m-2}R$$
$$= RR^{m-2}\cdot RS \subseteq RR^{m-2}\cdot R = R^m,$$

and

$$R^m \subseteq SR^m = SS \cdot RR^{m-1} = R^{m-1}R \cdot S = (R^{m-2}R \cdot R)S$$
$$= (RR \cdot R^{m-2})S = SR^{m-2} \cdot RR \subseteq SR^{m-2} \cdot R$$
$$= (SS \cdot R^{m-3}R)R = (RR^{m-3} \cdot SS)R = (RS \cdot R^{m-3}S)R$$
$$\subseteq (R \cdot R^{m-3}S)R = (R^{m-3} \cdot RS)R \subseteq R^{m-3}R \cdot R = R^{m-1},$$

therefore $\{0\} \neq M^mS \subseteq R^m \subseteq R^{m-1} \subseteq ... \subseteq R$. It is easy to see that M^mS is a right ideal of S. Thus $M^mS = R$ since R is 0-minimal. Also

$$\{0\} \neq SM^n \subseteq \{0\} \neq SL^n = S \cdot L^{n-1}L = L^{n-1} \cdot SL \subseteq L^{n-1}L = L^n,$$

and

$$L^n \subseteq SL^n = SS \cdot LL^{n-1} = L^{n-1}L \cdot S = (L^{n-2}L \cdot L)S = SL \cdot L^{n-2}L$$
$$\subseteq L \cdot L^{n-2}L = L^{n-2} \cdot LL \subseteq L^{n-2}L = L^{n-1} \subseteq ... \subseteq L,$$

therefore $\{0\} \neq SM^n \subseteq L^n \subseteq L^{n-1} \subseteq ... \subseteq L$. It is easy to see that SM^n is a left ideal of S. Thus $SM^n = L$ since L is 0-minimal. Therefore

$$M \subseteq RL = M^mS \cdot SM^n = M^nS \cdot SM^m = (SM^m \cdot S)M^n$$
$$= (SM^m \cdot SS)M^n = (S \cdot M^mS)M^n = (M^m \cdot SS)M^n$$
$$= M^mS \cdot M^n \subseteq M.$$

Thus $M = RL$, which means that RL is a 0-minimal (m,n)-ideal of S. ∎

Theorem 7. *Let S be a unitary $\mathscr{LA}$-semigroup. If R (L) is a 0-minimal right $(left)$ ideal of S, then either $R^mL^n = \{0\}$ or R^mL^n is a 0-minimal (m,n)-ideal of S.*

Proof. Assume that R (L) is a 0-minimal right (left) ideal of S such that $R^mL^n \neq \{0\}$, then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Hence $\{0\} \neq R^m \subseteq R$ and $\{0\} \neq L^n \subseteq L$, which shows that $R^m = R$ and $L^n = L$ since R (L) is a 0-minimal right (left) ideal of S. Thus by lemma 7, $R^mL^n = RL$ is an (m,n)-ideal of S. Now we show that R^mL^n is a 0-minimal (m,n)-ideal of S. Let $\{0\} \neq M \subseteq R^mL^n = RL \subseteq R \cap L$ be an (m,n)-ideal of S. Hence $\{0\} \neq SM^2 = MM \cdot SS = MS \cdot MS \subseteq RS \cdot RS \subseteq$

R and $\{0\} \neq SM \subseteq SL \subseteq L$. Thus $R = SM^2 = MM \cdot SS = SM \cdot M \subseteq SM$ and $SM = L$ since R (L) is a 0-minimal right (left) ideal of S. Therefore

$$M \subseteq R^m L^n \subseteq (SM)^m (SM)^n = S^m M^m \cdot S^n M^n = SS \cdot M^m M^n$$
$$= M^n M^m \cdot S = SM^m \cdot M^n = M^m S \cdot M^n \subseteq M,$$

Thus $M = R^m L^n$, which shows that $R^m L^n$ is a 0-minimal (m,n)-ideal of S. ∎

Theorem 8. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is an (m,n)-ideal of S and B is an (m,n)-ideal of A such that B is idempotent. Then B is an (m,n)-ideal of S.*

Proof. It is trivial that B is an $\mathscr{LA}$-subsemigroup S. Secondly, since $A^m S \cdot A^n \subseteq A$ and $B^m A \cdot B^n \subseteq B$, then

$$B^m S \cdot B^n = (B^m B^m \cdot S)(B^n B^n) = (B^n B^n)(S \cdot B^m B^m)$$
$$= [(S \cdot B^m B^m) B^n] B^n = [(B^n \cdot B^m B^m)(SS)] B^n$$
$$= [(B^m \cdot B^n B^m)(SS)] B^n = [S(B^n B^m \cdot B^m)] B^n$$
$$= [S(B^n B^m \cdot B^{m-1} B)] B^n = [S(B B^{m-1} \cdot B^m B^n)] B^n$$
$$= [S(B^m \cdot B^m B^n)] B^n = [B^m (SS \cdot B^m B^n)] B^n$$
$$= [B^m (B^n B^m \cdot SS)] B^n = [B^m (S B^m \cdot B^n)] B^n$$
$$= [B^m \{(SS \cdot B^{m-1} B) B^n\}] B^n = [B^m (B^m S \cdot B^n)] B^n$$
$$\subseteq [B^m (A^m S \cdot A^n)] B^n \subseteq B^m A \cdot B^n \subseteq B,$$

which shows that B is an (m,n)-ideal of S. ∎

Lemma 8. *Let $\langle a \rangle_{(m,n)} = a^m S \cdot a^n$, then $\langle a \rangle_{(m,n)}$ is an (m,n)-ideal of a unitary $\mathscr{LA}$-semigroup S.*

Proof. Assume that S is a unitary $\mathscr{LA}$-semigroup and m,n are non-negative integers, then

$$\begin{aligned}\langle a\rangle_{(m,n)} S\cdot\langle a\rangle_{(m,n)} &= [(a^m S\cdot a^n)S](a^m S\cdot a^n) = (a^n\cdot a^m S)[S(a^m S\cdot a^n)]\\ &= [\{S(a^m S\cdot a^n)\}(a^m S)]a^n = [a^m[\{S(a^m S\cdot a^n)\}S]]\, a^n\\ &\subseteq a^m S\cdot a^n = \langle a\rangle_{(m,n)},\end{aligned}$$

and similarly we can show that $\left(\langle a\rangle_{(m,n)}\right)^2\subseteq\langle a\rangle_{(m,n)}$. ∎

Theorem 9. *Let S be a unitary $\mathscr{LA}$-semigroup and $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold*:

(*i*) $\left(\langle a\rangle_{(1,0)}\right)^m S = a^m S$;
(*ii*) $S\left(\langle a\rangle_{(0,1)}\right)^n = Sa^n$;
(*iii*) $\left(\langle a\rangle_{(1,0)}\right)^m S\cdot\left(\langle a\rangle_{(0,1)}\right)^n = (a^m S)a^n$.

Proof. (i). As $\langle a\rangle_{(1,0)} = aS$, we have

$$\begin{aligned}\left(\langle a\rangle_{(1,0)}\right)^m S &= (aS)^m S = (aS)^{m-1}(aS)\cdot S = S(aS)\cdot(aS)^{m-1}\\ &= (aS)(aS)^{m-1} = (aS)[(aS)^{m-2}(aS)]\\ &= (aS)^{m-2}(aS\cdot aS) = (aS)^{m-2}(a^2S)\\ &= ... = (aS)^{m-(m-1)}(a^{m-1}S)\ [\text{if } m \text{ is odd}]\\ &= ... = (a^{m-1}S)(aS)^{m-(m-1)}\ [\text{if } m \text{ is even}]\\ &= a^m S.\end{aligned}$$

Analogously, we can prove (ii) and (iii) is simple. ∎

Corollary 5. *Let S be a unitary $\mathscr{LA}$-semigroup and let $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold*:

(*i*) $\left(\langle a\rangle_{(1,0)}\right)^m S = Sa^m$;
(*ii*) $S\left(\langle a\rangle_{(0,1)}\right)^n = a^n S$;
(*iii*) $\left(\langle a\rangle_{(1,0)}\right)^m S\cdot\left(\langle a\rangle_{(0,1)}\right)^n = (Sa^m)(a^n S)$.

Let $\mathfrak{L}_{(0,n)}$, $\mathfrak{R}_{(m,0)}$ and $\mathfrak{A}_{(m,n)}$ denote the sets of $(0,n)$-ideals, $(m,0)$-ideals and (m,n)-ideals of an $\mathscr{LA}$-semigroup S respectively.

Theorem 10. *If S is a unitary $\mathscr{LA}$-semigroup, then the following statements hold*:

(i) *S is $(0,1)$-regular if and only if $\forall\, L \in \mathfrak{L}_{(0,1)}$, $L = SL$*;

(ii) *S is $(2,0)$-regular if and only if $\forall\, R \in \mathfrak{R}_{(2,0)}$, $R = R^2S$ such that every R is semiprime*;

(iii) *S is $(0,2)$-regular if and only if $\forall\, U \in \mathfrak{A}_{(0,2)}$, $U = U^2S$ such that every U is semiprime.*

Proof. (i). Let S be $(0,1)$-regular, then for $a \in S$ there exists $x \in S$ such that $a = xa$. Since L is $(0,1)$-ideal, therefore $SL \subseteq L$. Let $a \in L$, then $a = xa \in SL \subseteq L$. Hence $L = SL$. Converse is simple.

(ii). Let S be $(2,0)$-regular and R be $(2,0)$-ideal of S, then it is easy to see that $R = R^2S$. Now for $a \in S$ there exists $x \in S$ such that $a = a^2x$. Let $a^2 \in R$, then

$$a = a^2x \in RS = R^2S \cdot S = SS \cdot R^2 = R^2S = R,$$

which shows that every $(2,0)$-ideal is semiprime.

Conversely, let $R = R^2S$ for every $R \in \mathfrak{R}_{(2,0)}$. Since Sa^2 is a $(2,0)$-ideal of S such that $a^2 \in Sa^2$, therefore $a \in Sa^2$. Thus

$$\begin{aligned} a \in Sa^2 &= (Sa^2)^2S = (Sa^2 \cdot Sa^2)S = (a^2S \cdot a^2S)S = [a^2(a^2S \cdot S)]S \\ &= (a^2 \cdot Sa^2)S = (S \cdot Sa^2)a^2 \subseteq Sa^2 = a^2S, \end{aligned}$$

which implies that S is $(2,0)$-regular.

Analogously, we can prove (iii). ∎

Lemma 9. *If S is a unitary $\mathscr{LA}$-semigroup, then the following statements hold*:

(i) *If S is $(0,n)$-regular, then $\forall\, L \in \mathfrak{L}_{(0,n)}$, $L = SL^n$*;

(ii) *If S is $(m,0)$-regular, then $\forall\, R \in \mathfrak{R}_{(m,0)}$, $R = R^mS$*;

(iii) *If S is (m,n)-regular, then $\forall\, U \in \mathfrak{A}_{(m,n)}$, $U = (U^mS)U^n$.*

Proof. It is simple. ∎

Corollary 6. *If S is a unitary $\mathscr{LA}$-semigroup, then the following statements hold*:

(i) *If* S *is* $(0,n)$*-regular, then* $\forall\, L \in \mathfrak{L}_{(0,n)}$, $L = L^n S$;

(ii) *If* S *is* $(m,0)$*-regular, then* $\forall\, R \in \mathfrak{R}_{(m,0)}$, $R = SR^m$;

(iii) *If* S *is* (m,n)*-regular, then* $\forall\, U \in \mathfrak{A}_{(m,n)}$, $U = U^{m+n}S = SU^{m+n}$.

Theorem 11. *Let* S *be a unitary* (m,n)*-regular* $\mathscr{LA}$*-semigroup such that* $m = n$*. Then for every* $R \in \mathfrak{R}_{(m,0)}$ *and* $L \in \mathfrak{L}_{(0,n)}$, $R \cap L = R^m L \cap RL^n$.

Proof. It is simple. ∎

Theorem 12. *Let* S *be a unitary* (m,n)*-regular* $\mathscr{LA}$*-semigroup. If* M (N) *is a* 0*-minimal* $(m,0)$*-ideal* $((0,n)$*-ideal*$)$ *of* S *such that* $MN \subseteq M \cap N$*, then either* $MN = \{0\}$ *or* MN *is a* 0*-minimal* (m,n)*-ideal of* S*.*

Proof. Let M (N) be a 0-minimal $(m,0)$-ideal $((0,n)$-ideal$)$ of S. Let $O = MN$, then clearly $O^2 \subseteq O$. Moreover

$$O^m S \cdot O^n = (MN)^m S \cdot (MN)^n = (M^m N^m)S \cdot M^n N^n \subseteq (M^m S)S \cdot SN^n$$
$$= SM^m \cdot SN^n = M^m S \cdot SN^n \subseteq MN = O,$$

which shows that O is an (m,n)-ideal of S. Let $\{0\} \neq P \subseteq O$ be a non-zero (m,n)-ideal of S. Since S is (m,n)-regular, therefore by using Lemma 9, we have

$$\{0\} \neq P = P^m S \cdot P^n = (P^m \cdot SS)P^n = (S \cdot P^m S)P^n = (P^n \cdot P^m S)(SS)$$
$$= (P^n S)(P^m S \cdot S) = P^n S \cdot SP^m = P^m S \cdot SP^n.$$

Hence $P^m S \neq \{0\}$ and $SP^n \neq \{0\}$. Further $P \subseteq O = MN \subseteq M \cap N$ implies that $P \subseteq M$ and $P \subseteq N$. Therefore $\{0\} \neq P^m S \subseteq M^m S \subseteq M$ which shows that $P^m S = M$ since M is 0-minimal. Likewise, we can show that $SP^n = N$. Thus we have

$$P \subseteq O = MN = P^m S \cdot SP^n = P^n S \cdot SP^m = (SP^m \cdot SS)P^n$$
$$= (S \cdot P^m S)P^n = P^m S \cdot P^n \subseteq P.$$

This means that $P = MN$ and hence MN is 0-minimal. ∎

Theorem 13. *Let* S *be a unitary* (m,n)*-regular* $\mathscr{LA}$*-semigroup. If* M (N) *is a* 0*-minimal* $(m,0)$*-ideal* $((0,n)$*-ideal*$)$ *of* S*, then either* $M \cap N = \{0\}$ *or* $M \cap N$ *is a* 0*-minimal* (m,n)*-ideal of* S*.*

Proof. Once we prove that $M\cap N$ is an (m,n)-ideal of S, the rest of the proof is same as in Theorem 11. Let $O=M\cap N$, then it is easy to see that $O^2\subseteq O$. Moreover $O^mS\cdot O^n\subseteq M^mS\cdot N^n\subseteq MN^n\subseteq SN^n\subseteq N$. But, we also have

$$\begin{aligned} O^mS\cdot O^n &\subseteq M^mS\cdot N^n=(M^m\cdot SS)N^n=(S\cdot M^mS)N^n=(N^n\cdot M^mS)S \\ &=(M^m\cdot N^nS)(SS)=(M^mS)(N^nS\cdot S)=M^mS\cdot SN^n \\ &=M^mS\cdot N^nS=N^n(M^mS\cdot S)=N^n\cdot SM^m=N^n\cdot M^mS \\ &=M^m\cdot N^nS=M^m\cdot SN^n\subseteq M^mN\subseteq M^mS\subseteq M. \end{aligned}$$

Thus $O^mS\cdot O^n\subseteq M\cap N=O$ and therefore O is an (m,n)-ideal of S.. ∎

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The work of first author is supported by the NNSF (Grant No. 11371335) grant of China. The second author is highly thankful to CAS-TWAS President's Fellowship.

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