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POTENTIALLY RIGHT SIMPLE π-REGULAR SEMIGROUPS ARE RIGHT GROUPS

KOZHUKHOV I.B.* AND KOZHUKHOVA YU.I.

Abstract. We prove that every π-regular semigroup such that all its elements are divided by one another on the left in potential sense, is a right group.

Keywords: π-regular semigroup, potentially right simple semigroup.

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1. Introduction

The simple universal algebras (the simple groups, rings, the simple and congruence-free semigroups) attracted the attention of algebraists always. The "simple" objects may be simple in usual sense. For example, the semigroups without non-trivial right (or left) congruences are exactly either semigroups of no more than two elements or cyclic groups of prime order. However some "simple" objects may have a complicated structure. L.A.Bokut proved that any semigroup may be isomorphically embedded into a congruence-free semigroup [2]. Earlier R.H.Bruck proved that every semigroup has an embedding into a simple semigroup with an identity element ([3], Theorem 8.3).

The notion "simplicity" may be generalized to "potential simplicity". It is well-known that any right simple semigroup (a semigroup having only one right ideal) containing an idempotent, is a right group, i.e. it is isomorphic to the direct product of a group and a right zero semigroup (see [4], 1.27). It was proved in [6] that the finite potentially right simple semigroups are also right groups. The aim of this work is to generalize this result to π-regular semigroups.

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Let $\mathcal{P}$ be a property (of elements, subsets, subsemigroups, congruences…) of a semigroup S. We may consider $\mathcal{P}$ as a condition. We say that the condition $\mathcal{P}$ is fulfilled potentially if there exists an oversemigroup $T \supseteq S$ such that $\mathcal{P}$ is fulfilled in T (see [4], Chap. 10). In the papers [11, 12], the potential divisibility and the potential invertibility of elements of a semigroup were investigated. The potential divisibility of Boolean matrices was considered in [8]. The paper [7] contains some results on potential one-sided congruences of a semigroup and the potential solubility of systems of equations.

Now we provide some definitions and notations. As usual we put for a semigroup S

$$S^1 = \begin{cases} S \ \ if \ \ S \ has \ a \ unity, \\ S \cup \{1\} \ \ if \ \ S \ has \ no \ unity. \end{cases}$$

The symbol $\mathbb{N}$ will denote the set of positive integers. Let X be a set. Denote by $\mathcal{T}(X)$ the semigroup of all transformations $\alpha : X \to X$ with the multiplication $x(\alpha\beta) = (x\alpha)\beta$. The *kernel* of the element $\alpha \in \mathcal{T}(S^1)$ is the set $\ker\alpha = \{(x,y) \in X \times X \mid x\alpha = y\alpha\}$. It is well known that any semigroup S may be isomorphically embedded into the semigroup $\mathcal{T}(S^1)$, the embedding is so: $a \mapsto \varphi_a$ where $x\varphi_a = xa$ for $a \in S,\ \ x \in S^1$. Let a,b be elements of a semigroup S. We say that a is divided by b on the left if $a \in bS^1$ (see [9], 2.1.2), and a is divided potentially by b on the left if $a \in bT^1$ in some oversemigroup $T \supseteq S$. Recall that $\mathcal{R} = \{(a,b) \in S \times S \,\big|\, aS^1 = bS^1\}$ is the Green's relation on the semigroup S. Clearly, $(a,b) \in \mathcal{R}$ if and only if the elements a,b are divided by one another on the left. The generalized Green's relation $\mathcal{R}^*$ (see [5]) is a "potential analogue" of $\mathcal{R}$, i.e. $(a,b) \in \mathcal{R}$ if and only if the elements a,b are divided potentially by one another on the left.

A semigroup S is called *right simple* if $aS = S$ for any $a \in S$. This is eqivalent to the requirement that any elements $a,b \in S$ are divided by one another on the left. A semigroup S is *potentially right simple* if any elements $a,b \in S$ are divided potentially by one another on the left.

Note that if $a \in bT_1^1$ for $T_1 \supseteq S$ and $b \in aT_2^1$ for $T_2 \supseteq S$ then there exists an oversemigroup $T_3 \supseteq S$ such that $a \in bT_3^1$ and $b \in aT_3^1$ (one may take $T_3 = \mathcal{T}(S^1)$). Therefore $(a,b) \in \mathcal{R}^*$ if and only if $(a,b) \in \mathcal{R}$ in some oversemigroup $T \supseteq S$.

Clearly, $\mathcal{R} \subseteq \mathcal{R}^*$. This inclusion may be strong. For example, in the semigroup $(\mathbb{N}, \cdot)$ we have $\mathcal{R}^* = \mathbb{N} \times \mathbb{N}$ but $\mathcal{R} = \{(a,a) \mid a \in \mathbb{N}\}$. The relation $\mathcal{R}^*$ on arbitrary semigroup was characterized by E.G.Shutov in [11] as follows.

Lemma 1 ([11], see also [9], 10.1.6, 10.1.7). The following conditions are equivalent for any elements a,b of a semigroup S:

(i) $(a,b) \in \mathcal{R}^*$ **;**

(ii) $\forall x, y \in S^1$ $(xa = ya \Leftrightarrow xb = yb)$.

2. Preliminaries

For the right simple semigroups we have the following assertion which may be proved easily.

Lemma 2. The following conditions are equivalent for any semigroup S:

(i) S is right simple;

(ii) $\mathcal{R} = S \times S$;

(iii) $\forall a \in S$ $aS = S$.

The next statement is an analogue of Lemma 2 for potentially right semigroups.

Proposition 3. The following conditions are equivalent for any semigroup S:

(i) S is potentially right simple;

(ii) $\mathcal{R}^* = S \times S$;

(iii) $\forall a \in S$ $\exists T \supseteq S$ $aT \supseteq S$.

Proof. (i) $\Rightarrow$ (ii). If $a,b \in S$ then $a \in bT_1^1$, $b \in aT_2^1$ for some oversemigroups $T_1, T_2 \supseteq S$. We seen earlier that $a \in bT_3^1$, $b \in aT_3^1$ for $T_3 \supseteq S$. This means that $(a,b) \in \mathcal{R}^*$.

(ii) $\Rightarrow$ (iii). Let $a,b \in S$. Then $(a,b) \in \mathcal{R}^*$. By Lemma 1, $xa = ya \Leftrightarrow xb = yb$. This implies $\ker a = \ker b$ (we consider a,b as elements of $\mathcal{T}(S^1)$). Then $b \in a \cdot \mathcal{T}(S^1)$. Since b is arbitrary, we have $S \subseteq a \cdot \mathcal{T}(S^1)$.

(iii) $\Rightarrow$ (i). Let $a,b \in S$. In view of (iii) we have $b \in aT$ for some $T \supseteq S$. It means that b is divided potentially by a. Therefore S is potentially right simple.

Lemma 1 shows that every right cancellative semigroup with the unity is potentially right simple. The assumption of the existence of the unity is essential as shows the example of a non-trivial left zero semigroup.

Recall a number of definitions in addition. An element a of a semigroup S is called *regular* if $aba = a$ for some $b \in S$. An element a is a *group element* if it lies in a subgroup of the semigroup S. Let $\mathrm{Reg}\, S$ be the set of all regular elements of the semigroup S, and $\mathrm{Gr}\, S$ be the set of all group elements. A semigroup S is called an *epigroup* (a quasiperiodic semigroup, in another terminology) if

$$\forall a \in S \quad \exists n \in \mathbb{N} \quad a^n \in \mathrm{Gr}\, S$$

(see [10]). A semigroup S is called π*-regular* if

$$\forall a \in S \quad \exists n \in \mathbb{N} \quad a^n \in \mathrm{Reg}\, S$$

(see [1]). Obviously, the following containments of classes of semigroups take place:

$$\textit{finite} \subset \textit{periodic} \subset \textit{epigroups} \subset \pi\textit{-regular}$$

It is seen from the statement (iii) of Lemma 2 that the class of right simple semigroups is closed under taking homomorphic images. At the same time, the example of the Baer – Levi semigroup (see [4], 8.1) shows that this class is not closed under taking subsemigroups, since the Baer – Levi semigroup contains the infinite cyclic semigroup which is not right simple. It is converse for the class of all potentially right simple semigroups: this class is closed under taking of subsemigroups (it follows from the definition) but it is not closed under homomorphic images as shows the following example. Let $S = \{1, a, a^2, ...\}$ is an infinite cyclic monoid. As S is cancellative semigroup with unity then S is potentially right simple. However, for the ideal $I = \{a, a^2, ...\}$ the Rees quotient semigroup $S / I = \{1, 0\}$ is not potentially right simple as shows Proposition 3 (here $(1, 0) \notin \mathcal{R}^*$ by Lemma 1).

The following statement is known. We give the proof for the sake of completeness.

Lemma 4. $\mathcal{R}^* = \mathcal{R}$ in any regular semigroup.

Proof. Let S be a regular semigroup and $a, b \in S$ are such that $(a, b) \in \mathcal{R}^*$. As S is regular then $aca = a$ for some $c \in S$. We have $ac \cdot a = 1 \cdot a$. As $(a, b) \in \mathcal{R}^*$ then, by Lemma 1, $ac \cdot b = 1 \cdot b$. Thus $b \in aS$. It can be prove similarly that $a \in bS$. Therefore $(a, b) \in \mathcal{R}$.

Lemma 5. If a semigroup S is potentially right simple then every its idempotent is a left unity.

Proof. Let $e^2 = e \in S$. Then $e \cdot e = 1 \cdot e$. As $\mathcal{R}^* = S \times S$ then, by Lemma 1, $e \cdot a = 1 \cdot a$ for all $a \in S$.

Lemma 6. Any potentially right simple π-regular semigroup is an epigroup.

Proof. Let S be a semigroup which satisfies Lemma's condition. Take $a \in S$. It follows from the π-regularity that $a^n \in \operatorname{Reg} S$ for some $n \in \mathbb{N}$. This implies $a^n b a^n = a^n$. Clearly $e = ba^n$ is an idempotent. Moreover $a^n = a^n b a^n = a^n e \in Se$. By Lemma 5, e is a left unity, therefore $Se = eSe$. Thus $a^n \in eSe$. Prove that a^n is an invertible element of the semigroup eSe relatively its unity e. Indeed, $ebe \cdot a^n = ba^n = e$, therefore a^n is left invertible. As a^n, $ebe \in eSe$ and $ebe \cdot a^n = e$ (which is the unity of the semigroup eSe), then the element $a^n \cdot ebe$ is an idempotent. By Lemma 5, $a^n \cdot ebe$ is a left unity. Therefore $a^n \cdot ebe \cdot e = e$, hence a^n is right invertible. Thus a^n is invertible in eSe which implies that a^n belongs to the maximal subgroup G_e with the unity e.

3. Main Results

Theorem 7. Any potentially right simple π-regular semigroup is a right group.

Proof. Let S be a potentially right simple π-regular semigroup and $a \in S$. By Lemma 6, S is an epigroup. Therefore $a^n \in G_e$ for some for some $n \in \mathbb{N}$ and a maximal subgroup G_e with the unity e. We may think that $n \geq 2$. Put $g = a^n$ and denote by g^{-1} the inverse element for g in the group G_e. We have $a^n e = a^n$, $g^{-1} a^n = e$. By Lemma 5, e is a left unity. Therefore $a = ea = g^{-1} a^n a = g^{-1} a \cdot a^n$. This implies $eae = e \cdot g^{-1} a \cdot a^n \cdot e = g^{-1} a^n \cdot a = ea = a$, i.e. $a \in eSe$. Show that the element a is invertible in the semigroup eSe. We have $g^{-1} a^{n-1} \cdot a = e$. As $g^{-1} a^{n-1}$, $a \in eSe$ and e is the unity of eSe, then $a \cdot g^{-1} a^{n-1}$ is an idempotent. By Lemma 5, $ag^{-1} a^{n-1}$ is a left unity. Therefore $ag^{-1} a^{n-1} \cdot e = e$. This follows $ag^{-1} a^{n-1} = e$. Thus, a has in eSe a two-sided invertible element $g^{-1} a^{n-1}$. Hence $a \in G_e$. The elements of G_e are regular, therefore $a \in \operatorname{Reg} S$. As a is arbitrary element then S is a regular semigroup. Then, by Lemma 4, we obtain $\mathcal{R} = \mathcal{R}^* = S \times S$. This implies that S is a right simple semigroup. As S is regular then S is a right group.

Conflict of Interests

The author declares that there is no conflict of interests.

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