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TENSOR PRODUCT C-SEMIGROUPS OF OPERATORS

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Abstract. In this paper, we introduce tensor product C-semigroups of operators on Banach spaces. The basic properties are presented. The generator and the resolvent of the generator of such semigroups are studied. The compactness of tensor product C-semigroups is also discussed.

Keywords: C-semigroups; compact operators; tensor product.

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1. Introduction

Let X be a Banach space and let $L(X)$ be the space of bounded linear operators on X. By a one parameter semigroup of operators on X we mean a map: $T : [0,\infty) \to L(X)$ such that

(1) $T(0) = I$, the identity operator on X,

(2) $T(s+t) = T(s)T(t)$, for all $s,t \geq 0$.

The linear operator A defined by

$$\mathfrak{D}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \text{ exists} \right\}$$

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and

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d}{dt} T(t)x \mid_{t=0}, \text{ for all } x \in \mathfrak{D}(A)$$

is called the infinitesimal generator of the semigroup $T(t)$, where $\mathfrak{D}(A)$ is the domain of A; see [16] and the references therein. Semigroups of operators are a main tool to solve the abstract Cauchy problem.

Definition 1.1. Let C be an invertible linear operator on X. A map $T(t) : [0,\infty) \to L(X)$ is called C-semigroup if

(1) $T(0) = C$,
(2) $CT(s+t) = T(s)T(t)$, for all $s,t \in [0,\infty)$.

Let $T(t)$ be a C-semigroup on X. The operator A defined by $Ax = C^{-1}\left(\lim_{t \to 0^+} \frac{T(t)x - Cx}{t}\right)$ with

$$\mathfrak{D}(A) = \{x \in X : \lim_{t \to 0^+} \frac{T(t)x - Cx}{t} \text{ exists}\}$$

is called the generator of $T(t)$. The notion of C-semigroups were introduced in 1987 by Davis and Pang. We refer authors to [3] and [5] for the basic structure of one parameter C-semigroups.

2. Tensor product of C-semigroups

Let X be a Banach space and $L(X)$ be the space of all bounded linear operators on X.

Definition 2.1. A map $T(s,t) : [0,\infty) \times [0,\infty) \to L(X)$ is called a two-parameter semigroup of bounded linear operators on X if

(1) $T(0,0) = I$, where I is the identity operator on X,
(2) $T((s_1,t_1) + (s_2,t_2)) = T(s_1,t_1)\, T(s_2,t_2)$, for all s_1, s_2, t_1 and $t_2 \geq 0$.

Basic properties and structure of two parameter semigroups were studied in [2] and [19].

In [20], Jafanda studied very specific two-parameter semigroups associated with differentiabilty. The problem of tensor product semi-semigroups of different parameters, were studied in [1].

Now, for two Banach spaces X and Y we use $X \overset{\wedge}{\otimes} Y$ to denote the completed projective tensor product of X and Y. We refer authors to [1] and [13] for a good account on tensor products of Banach spaces and tensor products of operators.

Definition 2.2. Let X and Y be two Banach spaces. Let $T(s)$ and $S(t)$ be two semigroups in $L(X)$ and $L(Y)$ respectively. Define a two-parameter semigroup as a vector valued function of two variables $F : [0,\infty) \times [0,\infty) \to L(X \overset{\wedge}{\otimes} Y)$, by $F(s,t) = T(s) \otimes S(t)$, where $T(s) \otimes S(t)(x \otimes y) = T(s)x \otimes S(t)y$. Then $F(s,t)$ is called a tensor product semigroup.

Tensor products of one-parameter semigroups of operators were studied in [1]. Let us recall the following result from [1].

Theorem 2.3. *Let $T(s) \overset{\wedge}{\otimes} S(t) : X \overset{\wedge}{\otimes} Y \to X \overset{\wedge}{\otimes} Y$ be a semigroup of class c_0. If A_1 and A_2 are the infinitesimal generators of $T(s)$ and $S(t)$ respectively, then the infinitesimal generator of $T(s) \otimes S(t)$ is the linear transformation $L : \mathbb{R}^{+^2} \to L(X \overset{\wedge}{\otimes} Y)$, defined by*

$$\begin{aligned} L(s,t)(x \otimes y) &= \left(\overline{A_1 \otimes I} \quad \overline{I \otimes A_2}\right) \begin{pmatrix} s \\ t \end{pmatrix} (x \otimes y) \\ &= s\left(\overline{A_1 \otimes I}\right)(x \otimes y) + t\left(\overline{I \otimes A_2}\right)(x \otimes y). \end{aligned}$$

Here, $\overline{A}$ denotes the closed extension of A. We refer authors to [13] for more details on tensor product operators and closed extension of operators.

Now we introduce C-tensor product semigroups of operators.

Definition 2.4. Let $T(s)$ and $S(t)$ be two maps from $[0,\infty)$ into $L(X)$ and $L(Y)$ respectively, and C_1, C_2 be two invertible operators on $L(X)$ and $L(Y)$ respectively. Then we say $T(s) \otimes S(t)$ is a $C_1 \otimes C_2-$tensor product semigroup in $L\left(X \overset{\wedge}{\otimes} Y\right)$ if

(1) $T(0) \otimes S(0) = C_1 \otimes C_2$,
(2) $(C_1 \otimes C_2) \circ (T \otimes S)((s_1,t_1) + (s_2,t_2)) = (T(s_1) \otimes S(t_1)) \circ (T(s_2) \otimes S(t_2))$.

For simplicity, a tensor product $C_1 \otimes C_2$-semigroup $T(s) \otimes S(t)$ will be called a $C_1 \otimes C_2-$semigroup from now on.

Proposition 2.5. *If $T(s) \otimes S(t)$ is a $C_1 \otimes C_2$-semigroup, then $T(s)$ and $S(t)$ are C_1-semigroup and C_2-semigroup respectively.*

Proof. Since $T(s)\otimes S(t)$ is a tensor product $C_1\otimes C_2$-semigroup, we have

$$T(0)\otimes S(0)=C_1\otimes C_2,$$

which implies from [1] that there exists a nonzero $\lambda\in\mathbb{R}$ such that $T(0)=\lambda C_1$, and $S(0)=\frac{1}{\lambda}C_2$. With no loss of generality, we can assume that $T(0)=C_1$ and $S(0)=C_2$. Moreover,

$$\begin{aligned}(C_1\otimes C_2)\circ(T\otimes S)((s_1,t_1)+(s_2,t_2)) &= (C_1\otimes C_2)\circ(T\otimes S)((s_1+s_2,t_1+t_2))\\ &= (T(s_1)\otimes S(t_1))\circ(T(s_2)\otimes S(t_2))\\ &= T(s_1)T(s_2)\otimes S(t_1)S(t_2).\end{aligned}$$

Hence, we have

$$\begin{aligned}T(s_1)T(s_2)\otimes S(t_1)S(t_2) &= (C_1\otimes C_2)\circ(T\otimes S)((s_1+s_2,t_1+t_2))\\ &= C_1T(s_1+s_2)\otimes C_2S(t_1+t_2),\end{aligned}$$

which implies that $C_1T(s_1+s_2)=\lambda T(s_1)T(s_2)$, and $C_2S(t_1+t_2)=\frac{1}{\lambda}S(t_1)S(t_2)$. Assume that $C_1T(s_1+s_2)=T(s_1)T(s_2)$ and $C_2S(t_1+t_2)=S(t_1)S(t_2)$. It follows that $T(s)$ is a C_1-semigroup and $S(t)$ is a C_2-semigroup.

Theorem 2.6. *Let $T(s)$ and $S(t)$ be C_1-semigroup and C_2-semigroup on $L(X)$ and $L(Y)$, respectively. Let A_1 be the infinitesimal generator of $T(s)$ and A_2 be the infinitesimal generator of $S(t)$. Then the infinitesimal generator of the $C_1\otimes I$-semigroup $T(s)\overset{\wedge}{\otimes}I : X\overset{\wedge}{\otimes}Y\to X\overset{\wedge}{\otimes}Y$ is $\overline{A_1\otimes I}$, and the infinitesimal generator of the $I\otimes C_2-$semigroup $I\overset{\wedge}{\otimes}S(t) : X\overset{\wedge}{\otimes}Y\to X\overset{\wedge}{\otimes}Y$ is $\overline{I\otimes A_2}$.*

Proof. Let $z=x\otimes y$ for some $x\otimes y\in\mathfrak{D}(A_1)\otimes Y$. Let A be the infinitesimal generator of $T(s)\overset{\wedge}{\otimes}I$. Then $Az=(A_1\otimes I)z$. This means that

$$A\underset{\mathfrak{D}(A_1)\otimes Y}{|}=A_1\otimes I.$$

In other words, A is an extension of $A_1\otimes I$ from the subspace $\mathfrak{D}(A_1)\otimes Y$ to the domain $\mathfrak{D}(A)$. Being the infinitesimal generator of a one parameter $C-$semigroup, then [16], A is closed. Thus, A is a closed extension of $A_1\otimes I$. But $A_1\otimes I$ is closable [13]. Since the closure of an operator is the smallest closed extension, then $A_1\otimes I\subset\overline{A_1\otimes I}\subset A$. On the other hand since the closure of a closable operator is its maximal extension we have $A \subset \overline{A_1 \otimes I}$. Hence $A = \overline{A_1 \otimes I}$.

Similarly, one can show that $\overline{I \otimes A_2}$ generates $I \overset{\wedge}{\otimes} S(t)$.

Definition 2.7. The infinitesimal generator of a $C_1 \otimes C_2$-semigroup $T(s) \otimes S(t)$ is $\left(C_1^{-1} \otimes C_2^{-1}\right)$. $\mathscr{L}(0,0)$, where $\mathscr{L}(0,0)$ is the derivative of $T(s) \otimes S(t)$ at $(0,0)$.

Theorem 2.8. *Let $T(s) \otimes S(t)$ be a $C_1 \otimes C_2$-semigroup. Then the infinitesimal generator of $T(s) \otimes S(t)$ is the linear transformation $A : \mathbb{R}^{+^2} \to L(X \otimes Y)$ defined by*

$$\begin{aligned} A(a,b)(x \otimes y) &= \begin{pmatrix} A_1 \otimes I & I \otimes A_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} (x \otimes y) \\ &= a(A_1 \otimes I)(x \otimes y) + b(I \otimes A_2)(x \otimes y), \end{aligned}$$

where A_1 and A_2 are the infinitesimal generators of $T(s)$ and $S(t)$ respectively.

Proof. Let $F = T(s) \otimes S(t)$. The infinitesimal generator of F is $\left(C_1^{-1} \otimes C_2^{-1}\right)$. $\mathscr{L}(0,0)$, where $\mathscr{L}(0,0)$ is the derivative of F at $(0,0)$. But the derivative of F at $(0,0)$ is $\left(\frac{\partial F}{\partial s} \big|_{s=0} \quad \frac{\partial F}{\partial t} \big|_{t=0} \right)$. Now we have

$$\begin{aligned} \frac{\partial F}{\partial s} \Big|_{s=0} &= \lim_{s \to 0^+} \frac{F(s,0) - F(0,0)}{s} (x \otimes y) \\ &= \lim_{s \to 0^+} \frac{T(s)x - C_1 x}{s} \otimes C_2 y \\ &= C_1 A_1 x \otimes C_2 y. \end{aligned}$$

Similarly, we have $\frac{\partial F}{\partial t} \big|_{t=0} = C_1 x \otimes C_2 A_2 y$. It follows that $\mathscr{L}(0,0) = (C_1 \otimes C_2)\begin{pmatrix} A_1 \otimes I & I \otimes A_2 \end{pmatrix}$. Hence, the infinitesimal generator of $T(s) \otimes S(t)$ is $\begin{pmatrix} A_1 \otimes I & I \otimes A_2 \end{pmatrix}$.

From Theorem 2.8, we find the following result immediately.

Lemma 2.9. *If $T(t)$ and $S(t)$ are C_1-semigroup and C_2-semigroup respectively, with generators A_1 and A_2, then the generator of the one parameter semigroup $T(at) \otimes S(bt)$ is $aA_1 \otimes I + bI \otimes A_2$.*

Lemma 2.10. *If $T(t)$ and $S(t)$ are C_1-semigroup and C_2-semigroup respectively, with infinitesimal generators A_1 and A_2 then the infinitesimal generator of the one parameter semigroup $e^{-\lambda t} T(at) \otimes S(bt)$ is $[aA_1 \otimes I] + [bI \otimes A_2] - \lambda I \otimes I$.*

Proof. Let $z = x \otimes y$. Define

$$J = C_1^{-1} \otimes C_2^{-1} \lim_{t \to 0^+} \frac{e^{-\lambda t} T(at) \otimes S(bt) - C_1 \otimes C_2}{t} z.$$

It follows that

$$\begin{aligned} J &= C_1^{-1} \otimes C_2^{-1} \lim_{t \to 0^+} \frac{e^{-\lambda t} T(at) \otimes S(bt) - e^{-\lambda t} C_1 \otimes C_2 + e^{-\lambda t} C_1 \otimes C_2 - C_1 \otimes C_2}{t} z \\ &= C_1^{-1} \otimes C_2^{-1} \lim_{t \to 0^+} e^{-\lambda t} \frac{T(at) \otimes S(bt) - C_1 \otimes C_2}{t} z \\ &\quad + C_1^{-1} \otimes C_2^{-1} \lim_{t \to 0^+} C_1 \otimes C_2 \frac{e^{-\lambda t} I \otimes I - I \otimes I}{t} z \\ &= (aA_1 \otimes I + bI \otimes A_2) z - \lambda I \otimes I z. \end{aligned}$$

Thus, the infinitesimal generator of the one parameter semigroup $e^{-\lambda t} T(at) \otimes S(bt)$ is $[aA_1 \otimes I] + [bI \otimes A_2] - \lambda I \otimes I$.

Lemma 2.11. *Let $T(at) \otimes S(bt)$ be a $C_1 \otimes C_2$-semigroup with $\|T(s)\| \leq M_1 e^{w_1 s}$ and $\|S(t)\| \leq M_2 e^{w_2 t}$. If $Re(\lambda) > (a+b)\max(w_1, w_2)$, then $\lim_{t \to \infty} e^{-\lambda t} T(at) \otimes S(bt) = 0$.*

Proof. Note that

$$\begin{aligned} \left\| e^{-\lambda t} T(at) \otimes S(bt) \right\| &= \left\| e^{-\lambda t} T(at) \right\| \|S(bt)\| \\ &\leq \left\| e^{-\lambda t} \right\| M_1 M_2 e^{t(aw_1 + bw_2)} \\ &= M_1 M_2 e^{-t(Re(\lambda) - aw_1 - bw_2)}, \end{aligned}$$

which tends to zero as $t \to \infty$, since $Re(\lambda) > (a+b)\max(w_1, w_2)$.

The proof of the following two lemmas is standard, and is therefore omitted.

Lemma 2.12. *Let $T(t)$ be a one parameter C-semigroup. Then for any $x \in X$, we have $\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} T(s) x ds = T(t)x$.*

Lemma 2.13. *Let $T(t)$ be a one parameter $C-$ semigroup whose infinitesimal generator is A. Then for any $x \in X$, $s \geq 0$ we have $\int_0^s T(t) x dt \in D(A)$ with $A \int_0^s T(t) x dt = T(s)x - Cx$.*

Theorem 2.14. *Let $T(s) \otimes S(t)$ be a $C_1 \otimes C_2$-semigroup whose infinitesimal generator is $(A_1 \otimes I \quad I \otimes A_2)$, with $\|T(s)\| \leq M_1 e^{w_1 s}$ and $\|S(t)\| \leq M_2 e^{w_2 t}$. If $\lambda \in \rho\left((A_1 \otimes I \quad I \otimes A_2)\binom{a}{b}\right)$, where*

$(a,b) \in \mathbb{R}^{+^2}$ *and* $Re(\lambda) > (a+b)\max(w_1,w_2)$, *then*

$$R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)(x \otimes y) = C_1^{-1} \otimes C_2^{-1} \int_0^\infty e^{-\lambda t}\left(T(at) \otimes S(bt)\right)(x \otimes y)\, dt,$$

and

$$\left\| R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)\right\| \leq \frac{M\left\|C_1^{-1}\right\|\left\|C_2^{-1}\right\|}{Re(\lambda) - aw_1 - bw_2}.$$

Proof. From Lemma 2.10, the infinitesimal generator of the one parameter C-semigroup $e^{-\lambda t}T(at) \otimes S(bt)$ is $([aA_1 \otimes I] + [bI \otimes A_2] - \lambda I \otimes I)$. This equals to

$$(A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix} - \lambda I \otimes I.$$

Let $A = (A_1 \otimes I \quad I \otimes A_2)\binom{a}{b} - \lambda I \otimes I$. It follows from Lemma 2.13 that

$$A\int_0^t e^{-\lambda s}\left(T(as) \otimes S(bs)\right)(x \otimes y)\, ds = e^{-\lambda t}T(at) \otimes S(bt)(x \otimes y) - C_1 \otimes C_2 (x \otimes y).$$

From Lemma 2.11, we see that s $\lim\limits_{t \to \infty} e^{-\lambda t}T(at) \otimes S(bt) = 0$. Thus, taking the limit as $t \to \infty$ for both sides the right hand side becomes$-C_1 \otimes C_2(x \otimes y)$. Hence, we conclude

$$\left((A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix} - \lambda I \otimes I\right)\int_0^\infty e^{-\lambda s}\left(T(as) \otimes S(bs)\right)(x \otimes y)\, ds = -C_1 \otimes C_2(x \otimes y).$$

This implies that

$$\left(\lambda I \otimes I - (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)\left(C_1^{-1} \otimes C_2^{-2}\right)\int_0^\infty e^{-\lambda s}\left(T(as) \otimes S(bs)\right)(x \otimes y)\, ds = x \otimes y.$$

It follows that

$$R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)(x \otimes y) = C_1^{-1} \otimes C_2^{-1} \int_0^\infty e^{-\lambda t}\left(T(at) \otimes S(bt)\right)(x \otimes y)\, dt.$$

Moreover, we have

$$
\begin{aligned}
\left\| R\left(\lambda, (A_1 \otimes I, I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)\right\| &= \left\| C_1^{-1} \otimes C_2^{-1} \int_0^{\infty} e^{-\lambda t} \left(T(at) \otimes S(bt)\right)(x \otimes y)\, dt \right\| \\
&\leq \|C_1^{-1}\| \|C_2^{-1}\| \int_0^{\infty} M_1 M_2 e^{-Re(\lambda)t + aw_1 + bw_2} dt \\
&= M_1 M_2 \|C_1^{-1}\| \|C_2^{-1}\| \int_0^{\infty} e^{-Re(\lambda)t + aw_1 + bw_2} dt \\
&= \frac{M_1 M_2 \left\|C_1^{-1}\right\| \left\|C_2^{-1}\right\|}{Re(\lambda) - aw_1 - bw_2}.
\end{aligned}
$$

As required.

3. Compact tensor product C-semigroups

In this section, necessary conditions and sufficient conditions for C-tensor product semigroups to be compact are obtained.

Definition 3.1. An operator T on a Banach space X is said to be compact if for every bounded sequence x_n in X the sequence Tx_n has a convergent subsequence.

Remark 3.2. An operator $T \in L(X)$ is compact iff T takes any bounded set to a relatively compact set. Hence, every finite rank operator is compact.

Definition 3.3. A C-semigroup $T(t)$ is called compact, if $T(t)$ is a compact operator on X for all $t \in (0, \infty)$.

The following is a known result in [8].

Theorem 3.4. *For any bounded linear operators A and B on a Banach spaces X and Y respectively, one has $A \otimes B$ is compact iff both A and B are compact.*

As a consequence we get the following.

Theorem 3.5. *Let $T(s) \otimes S(t)$ be a $C_1 \otimes C_2-$semigroup. Then $T(s) \otimes S(t)$ is compact iff $T(s)$ and $S(t)$ are compact.*

In the following Theorem, we need C_1, and C_2 to be bounded.

Theorem 3.6. *Let $T(s)$ be a compact C_1-semigroup with infinitesimal generator A_1 such that $\|T(s)\| \leq M_1 e^{w_1 s}$ and $S(t)$ be a compact C_2-semigroup with infinitesimal generator A_2 such that $\|S(t)\| \leq M_2 e^{w_2 t}$. Then $(C_1 \otimes C_2)^2 R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\binom{a}{b}\right)$ is compact for all $\lambda \in \rho\left((A_1 \otimes I \quad I \otimes A_2)\binom{a}{b}\right)$.*

Proof. Let $\lambda \in \rho\left((A_1 \otimes I \quad I \otimes A_2)\binom{a}{b}\right)$, such that $Re(\lambda) > (a+b)\max(w_1, w_2)$. Then by Theorem 14, we have

$$(C_1 \otimes C_2) R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right)(x \otimes y) = \int_0^\infty e^{-\lambda s} T(as) \otimes S(bs)(x \otimes y)\, ds.$$

Define

$$\begin{aligned} R_t\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right) &= (C_1 \otimes C_2)\int_t^\infty e^{-\lambda s} T(as) \otimes S(bs)\, ds \\ &= \int_t^\infty e^{-\lambda s} C_1 T(as) \otimes C_2 S(bs)\, ds \\ &= T(at) \otimes S(bt) \int_t^\infty e^{-\lambda s} T(a(s-t)) \otimes S(b(s-t))\, ds. \end{aligned}$$

Since $Re(\lambda) > (a+b)\max(w_1, w_2)$, we have $\int_t^\infty e^{-\lambda s} T(a(s-t)) \otimes S(b(s-t))\, ds$ is bounded. Since $T(s)$ and $S(t)$ are compact, we have by Theorem 2.20, we get $T(at) \otimes S(bt)$ is compact, and since the composition of a compact and a bounded operators is compact we get, $R_t\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\binom{a}{b}\right)$ is compact for all $t > 0$. Further, let

$$J = R_t\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right) - (C_1 \otimes C_2)^2 R\left(\lambda, (A_1 \otimes I \quad I \otimes A_2)\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

It follows that

$$\begin{aligned} \|J\| &= \left\|(C_1 \otimes C_2)\int_t^\infty e^{-\lambda s} T(as) \otimes S(bs)\, ds - (C_1 \otimes C_2)\int_0^\infty e^{-\lambda s} T(as) \otimes S(bs)\, ds\right\| \\ &\leq \|C_1 \otimes C_2\| \int_0^t \left\|e^{-\lambda s} T(as) \otimes S(bs)\right\| ds. \end{aligned}$$

On the other hand, we have $\left\|e^{-\lambda s} T(as) \otimes S(bs)\right\| \leq e^{-Re(\lambda)s}\|T(as)\|\,\|S(bs)\|$. Further, we have $\|T(s)\| \leq M_1 e^{w_1 s}$ and $\|S(t)\| \leq M_2 e^{w_2 t}$. Thus, we get

$$\|J\| \leq M_1 M_2 \|(C_1 \otimes C_2)\| \int_0^t e^{-s(Re(\lambda) - aw_1 - bw_2)} ds.$$

And since $\lim_{t\to 0^+}\int_0^t e^{-s(Re(\lambda)-aw_1-bw_2)}ds=0$, and $R_t\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ is compact for all $t>0$, and since the uniform limit of compact operators is compact, then

$$(C_1\otimes C_2)^2 R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)$$

is compact for all $\lambda\in\mathbb{C},\ \ Re(\lambda)>(a+b)\max(w_1,w_2)$.

Now let μ be any element in $\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$. Then from the resolvent identity we have

$$(C_1\otimes C_2)^2 R(\mu,A)=(C_1\otimes C_2)^2 R(\lambda,A)+(\lambda-\mu)(C_1\otimes C_2)^2 R(\mu,A)R(\lambda,A),$$

for any $\lambda\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$, where $A=(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}$. Thus, if

$$\lambda\in\rho\left((A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)$$

and $Re(\lambda)>(a+b)\max(w_1,w_2)$ we get

$$(C_1\otimes C_2)^2 R\left(\mu,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)$$

is compact. Hence, it is compact for all $\mu\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$.

Theorem 3.7. *Let $T(s)\otimes S(t)$ be a $C_1\otimes C_2$-semigroup on $X\otimes Y$ with $\|T(s)\|\le M_1e^{w_1s}$ and $\|S(t)\|\le M_2e^{w_2t}$. If $R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ is compact for all $\lambda\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ and $T(t)\otimes S(t)$ is uniformly continuous on $(0,\infty)$, then $T(s)\otimes S(t)$ is compact for all $s,t>0$.*

Proof. Since $R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ is compact for all λ and $T(at)\otimes S(bt)\in L(X\otimes Y)$ for all $t>0$, this implies that $\lambda R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)T(at)\otimes S(bt)$ is compact. Now for $\lambda\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ with $Re(\lambda)>(a+b)\max(w_1+w_2)$ we have by Theorem 2.14

$$R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)=C_1^{-1}\otimes C_2^{-1}\int_0^\infty e^{-\lambda s}T(as)\otimes S(bs)\,ds.$$

Let

$$J=\lambda R\left(\lambda,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)T(at)\otimes S(bt)-T(at)\otimes S(bt).$$

It follows that

$$\begin{aligned} \|J\| &= \left\| \lambda C_1^{-1} \otimes C_2^{-1} \int_0^\infty e^{-\lambda s} \left(T\left(as\right) \otimes S\left(bs\right)\right) \left(T\left(at\right) \otimes S\left(bt\right)\right) ds - T\left(at\right) \otimes S\left(bt\right) \right\| \\ &\leq \|\lambda\| \int_0^\infty \left\| e^{-\lambda s} \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| ds \\ &\leq \|\lambda\| \int_0^\infty e^{-Re(\lambda)s} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| ds. \end{aligned}$$

By dividing the integral to to integrals, we get

$$\begin{aligned} \|J\| \leq \; & \|\lambda\| \int_0^c e^{-Re(\lambda)s} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| ds \\ & + \|\lambda\| \int_c^\infty e^{-Re(\lambda)s} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| ds. \end{aligned}$$

It follows that

$$\begin{aligned} \|J\| &\leq \|\lambda\| \sup_{0 \leq s \leq c} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| \int_0^c e^{-Re(\lambda)s} ds \\ & \quad + \|\lambda\| \int_c^\infty e^{-Re(\lambda)s} M_1 M_2 \left(e^{w_1 a(s+t) + b w_2 (s+t)} + e^{a w_1 t + b w_2 t} \right) ds \\ &= \sup_{0 \leq s \leq c} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\| \|\lambda\| \left(\frac{1}{Re\left(\lambda\right)} - \frac{e^{-Re(\lambda)s}}{Re\left(\lambda\right)} \right) \\ & \quad + \|\lambda\| M_1 M_2 e^{(a w_1 + b w_2)t} \left(\frac{e^{-c(Re(\lambda) - a w_1 - b w_2)}}{Re\left(\lambda\right) - a w_1 - b w_2} + \frac{e^{-Re(\lambda)c}}{Re\left(\lambda\right)} \right). \end{aligned}$$

Since $T\left(t\right) \otimes S\left(t\right)$ is uniformly continuous, we have

$$\sup_{0 \leq s \leq c} \left\| \left(T\left(a\left(s+t\right)\right) \otimes S\left(b\left(s+t\right)\right) ds - T\left(at\right) \otimes S\left(bt\right)\right) \right\|$$

can be made less than any $\varepsilon > 0$. This implies

$$\limsup_{Re(\lambda) \to \infty} \left\| \lambda R \left(\lambda, \left(A_1 \otimes I \quad I \otimes A_2\right) \begin{pmatrix} a \\ b \end{pmatrix} \right) T\left(at\right) \otimes S\left(bt\right) - T\left(at\right) \otimes S\left(bt\right) \right\| \leq \varepsilon$$

for every $c > 0$. Since c is arbitrary we have

$$\lim_{Re(\lambda) \to \infty} \left\| \lambda R \left(\lambda, \left(A_1 \otimes I \quad I \otimes A_2\right) \begin{pmatrix} a \\ b \end{pmatrix} \right) T\left(at\right) \otimes S\left(bt\right) - T\left(at\right) \otimes S\left(bt\right) \right\| = 0.$$

Thus, $T\left(at\right) \otimes S\left(bt\right)$ is compact being the limit of a compact operator. Now $T\left(at\right)$ and $S\left(bt\right)$ are compact. Thus, $T\left(s\right) \otimes S\left(t\right)$ is compact.

Theorem 3.8. *Let $T(s)\otimes S(t)$ be a $C_1\otimes C_2-$semigroup on $X\otimes Y$ whose infinitesimal generator is $(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}$. If $T(t)\otimes S(t)$ is differentiable and*

1. *There exists $\lambda_0\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ such that $R\left(\lambda_0,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ is compact,*
2. *$T(t)\otimes S(t)$ is uniformly continuous on $(0,\infty)$,*

then $T(s)\otimes S(t)$ is compact for all $s,t>0$.

Proof. Let $\lambda_0\in\rho\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$. Then using the rescaled semigroup $S(t)=e^{-\lambda_0 t}T(at)\otimes S(bt)$ we may assume without loss of generality that $\lambda_0=0$. Define

$$B(t)(x\otimes y)=\int_0^t T(as)\otimes S(bs)(x\otimes y)\,ds.$$

Then $B\in L(X\otimes Y)$ and we have

$$\begin{aligned}(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}B(t)(x\otimes y) &= (A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\int_0^t T(as)\otimes S(bs)(x\otimes y)\,ds\\ &= (T(at)\otimes S(bt)-C_1\otimes C_2)(x\otimes y).\end{aligned}$$

For all $x\otimes y\in\mathfrak{D}(C_1\otimes C_2)$. Hence

$$\begin{aligned}-(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}B(t)(x\otimes y) &= \left(0-(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}B(t)(x\otimes y)\right)\\ &= (C_1\otimes C_2-T(at)\otimes S(bt))(x\otimes y).\end{aligned}$$

It follows that

$$B(t)(x\otimes y)=R\left(0,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)(C_1\otimes C_2-T(at)\otimes S(bt))(x\otimes y).$$

Since $\mathfrak{D}\left((A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ and Range$(C_1\otimes C_2)$ are dense in $X\otimes Y$, then

$$B(t)=R\left(0,(A_1\otimes I \quad I\otimes A_2)\begin{pmatrix}a\\b\end{pmatrix}\right)(C_1\otimes C_2-T(at)\otimes S(bt)).$$

But $R\left(0,(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}\right)$ is compact. Thus, $B(t)$ is compact for all $t>0$. Let $A=(A_1\otimes I \quad I\otimes A_2)\binom{a}{b}$. Then we have,

$$\begin{aligned}B'(t)(x\otimes y) &= \lim_{h\to 0}\frac{B(h+t)-B(t)}{h}\\ &= \lim_{n\to\infty} n\left(B\left(\frac{1}{n}+t\right)-B(t)\right).\end{aligned}$$

Thus, we have

$$B'(t)(x\otimes y)=\lim_{n\to\infty} nR(0,A)\left(T\left(a\left(t+\frac{1}{n}\right)\right)\otimes S\left(b\left(t+\frac{1}{n}\right)\right)-T(at)\otimes S(bt)\right)(x\otimes y).$$

Define

$$D_n(t)(x\otimes y)=nR(0,A)\left(T\left(a\left(t+\frac{1}{n}\right)\right)\otimes S\left(b\left(t+\frac{1}{n}\right)\right)-T(at)\otimes S(bt)\right)(x\otimes y).$$

Since $R(0,A)$ is compact then $D_n(t)(x\otimes y)$ is compact for all $t>0$ and $n\in\mathbb{N}$. But

$$B'(t)(x\otimes y)=\frac{d}{dt}\int_0^t T(as)\otimes S(bs)(x\otimes y)\,ds=T(at)\otimes S(bt)(x\otimes y).$$

Since $T(t)\otimes S(t)$ is uniformly continuous, it follows that $T(at)\otimes S(bt)$ is compact for all $t>0$. That is $T(at)$ and $S(bt)$ are compact, which implies $T(s)\otimes S(t)$ is compact.

The following result is standard, and the proof is therefore omitted.

Theorem 3.9. *Let $T(s)\otimes S(t)$ be a $C_1\otimes C_2$-semigroup such that $\|T(s)\otimes S(t)\|\le Me^{w(s+t)}$. If $T(t)\otimes S(t)$ is compact for all $t>t_0>0$, then $(C_1\otimes C_2)(T(t)\otimes S(t))$ is continuous in the uniform topology for all $t>t_0$.*

Theorem 3.10. *Let $T(s)\otimes S(t)$ be a $C_1\otimes C_2$-semigroup satisfying*

(1) $(C_1\otimes C_2)(T(t)\otimes S(t))(x\otimes y)=(T(t)\otimes S(t))(C_1\otimes C_2)(x\otimes y)$, for all $x\otimes y\in X\otimes Y$,

(2) $T(t)\otimes S(t)$ is compact for all $t>t_0>0$.

Then $T(t)\otimes S(t)$ is uniformly continuous for all $t>0$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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