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REGULAR PROPER *-SEMIGROUP EMBEDDINGS AND INVOLUTIONSTITLE

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Abstract. It is proved that if $(S,*)$ is a proper *-semigroup and if D is 0-characteristic integral domain then $(D[S],*)$ is nil-semisimple provided that S is finite or $i \in D$.Let $(S,*)$ be a finite proper *-semigroup and F be a finite field of characteristic p such that $(F[S],*)$ is a proper *-ring. Then $F[S]$ is a direct product of fields and 2×2 matrix rings over fields. Furthermore, $p \neq 2, p \neq 1$ mod 4.

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1. Introduction

A semigroup with involution $(S,*)$ is called a **-semigroup*. It is called a *p*-semigroup* if the involution * *is proper*. Thus $\forall a,b \in S, aa^* = ab^* = bb^* \Rightarrow a = b$. A ring with involution $(R,*)$ is called a **-ring*. It is called a *p*-ring* if the involution * is proper. Thus $aa^* = 0 \Rightarrow a = 0$ for all $a \in R$. Let $(S,*),(T,*)$ be two *-semigroups. An injective mapping $f : (S,*) -- > (R,*)$ from a *-semigroup $(S,*)$ into a *-ring $(R,*)$ such that for all $a,b \in (S,*), f(ab) = f(a)f(b), f(a^*) = (f(a))^*$ is called a **-embedding*. Let $(S,*)$ be a *-semigroup and consider the semigroup ring $Z[S]$ of S over Z. If $(S,*)$ is a p*-semigroup then $(Z[S],*)$ need not be a p*-ring as in ([6]). Let $(S,*)$ be a *-semigroup. The involution * is called a *maximal proper involution* if for every distinct elements $s_1,...,s_n \in S$, there exists an element s_i such that $s_is_i^* \neq s_is_j^*, j \neq i$, and

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$s_is_i^* = s_ks_l^* \Rightarrow s_i^*s_k = s_i^*s_l; k,l = 1,...,n$. Such a *-semigroup is called an *mp-semigroup*. For example any inverse semigroup is an mp-semigroup under its inverse involution as in ([6]). If $(S,*)$ is an mp-semigroup then $(Z[S],*)$ is a p*-ring and $(S,*)$ is *-embeddable in $(Z[S],*)$, ([6]). Let $(R,*)$ be a *-ring and let n be a fixed positive integer. If for every distinct elements $r_1,...,r_n \in R$ it holds that $\sum r_ir_i^* = 0$ implies that $r_i = 0, i = 1,...,n$ then we say that $(R,*)$ is *n-formally complex*. Let F be a field, let α be an automorphism of order 1 or 2 and let $D \in M_n(F)$ be a diagonal matrix. Then F is $D(\alpha)-$*formally complex* if and only if $\sum d_ia_i\alpha(a_i) = 0$ implies all $a_i = 0$. If D is the identity matrix we say that F is $n-$formally complex and if this true for all n we say that F is formally-complex. On the other hand, if α is the identity then we say that F is $D(\alpha)-$ real and if D is the identity we say that F is $n-$*formally real* and if this is the case for all n we say that F is *formally real*. If $(S,*)$ is an mp-semigroup and $(R,*)$ is formally complex *-ring then $(R[S],*)$ is a p*-ring and $(S,*)$ is *-embeddable in $(R[S],*)$, as in [6]) where it is shown there is a finite p*-semigroup that cannot be *-embedded in any p*-ring. Let $(R,*)$ be a *-ring. An ideal I in R is called a **-ideal* if $I^* = I$. In this case the ring R/I is a *-ring under the involution $(r+I)^* = r^* + I$.

Let F be a field and let α be an automorphism on F of order 1 or 2. Let $R = M_n(R)$ and let $A \in R$. If we apply to every entry in A the automorphism α we get A^{α}. An involution * on R is called $\alpha-$*inner* if there is an invertible matrix P such that for all A in R we have $A^* = P^{-1}A^{\alpha t}P$ and if α is the identity mapping then * is called *inner*.

Let F be a field and let α be an automorphism on F and let two matrices $A,B \in M_n(F)$.We say that the matrices A,B are $\alpha-$*congruent* if there is a matrix C such that $A = CBC^{\alpha t}$. Also we say that a matrix $A \in M_n(F)$ is $\alpha-$*symmetric* if $A = A^{\alpha t}$ and it is called $\alpha-$*antisymmetric* if $A^{\alpha t} = -A$. Here A^{α} is got from the matrix A by applying α to its entries. It is known that if A is a symmetric matrix in $M_n(F), F$ is a field then it is congruent to a diagonal matrix and if A is anti-symmetric invertible matrix then A is congruent to a direct sum of 2 by 2 matrices each of which is of the form $\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \alpha \in F$. See [3] pp. 365-372.

Let $(S,*)$ be a proper *-semigroup of order 5 or less. It was noticed (through a computer program) that once the involution * in the *-semigroup ring $(Z[S],*)$ is not proper then the p*-semigroup $(S,*)$ is not *-embeddable in any ring p*-ring. Up to now there is no proof or disproof for this claim.

In the first part of this note we find a necessary and sufficient condition for a certain class of involutions on $R = M_n(F), F$ is a field, to be proper involutions. In the second part we give a plan to decide if a given proper *-semigroup is *-embeddable in a p*-ring and if so we seek to find a p*-algebra of matrices that *-embeds $(S,*)$ and we look for all involutions *' on S that makes $(S,*')$ *-isomorphic with $(S,*)$. Incase $(S,*)$ is not *-embeddable in a p*-ring we locate the *-subsemigroup $(T,*)$ such that $(S/T,*)$ is *-embeddable in a p*-ring.

2. Preliminaries

We cite the following known facts.

Theorem 1. *(A) Let $(S,*)$ be an mp semigroup and let $(R,*)$ be a formally complex ring. Then $(R[S],*)$ is a proper *-ring and hence it has a zero nil radical, (*[6]*).*

We cite the following version of Wedderburn Theorem from [2] p. 435

Theorem 2. *(B) If R is a non zero left Artinean nil-semisimple ring then it isomorphic with a finite direct sum of finite matrix rings over a division ring.*

We Also cite the following from [5], p.63.

Theorem 3. *(B): If A is a left Noetherian ring, then every nil ideal is nilpotent.*

We also cite the following version of Skolem-Noether theorem; see[2], p.460.

Theorem 4. *(C): Let R be a simple left-Artinian ring and let K be the center of R (so that R is a K-algebra). Let A and B be finite dimensional simple K-algebras of R that contain K. If $\alpha : A \to B$ is a K-algebra isomorphism that leaves K fixed elementwise, then α extends to an inner automorphism of R.*

We cite the following theorem from [1], p136.

Theorem 5. *(D): Let $(R,*)$ be a semi-simple *-ring with involution * such that $\forall x \in R, \exists n(x), (x+x^*)^{n(x)} = x + x^*$. Then R is a subdirect product of fields and 2×2 matrix rings over fields.*

Proposition 6. *Let F be a field and let $P \in M_n(F)$ be a symmetric matrix then there is a diagonal matrix D congruent to P ; i.e.,*

$\exists C \in M_n(F), CPC^t = D$, see [4], for example. If P is antisymmetric then P is congruent to a direct sum of matrices of the form $\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and 0-matrices where $\alpha \in F$.

As a generalization we state a similar proposition whose proof is similar to that of proposition [6] and its proof is omitted.

Proposition 7. *Let F be a field and let α be an automorphism of order 2 on F. Let $P \in M_n(F)$ be an inverteble matrix such that $P^{\alpha t} = P$. Then there is a matrix C and a diagonal matrix D such that $CPC^{\alpha t} = D$.*

3. Main results

Given a semigroup S we can ask how to find all proper involutions on S.For example if S is an inverse semigroup then the inverse operator is one of the proper involutions on S. Similarly given a ring R there is a problem of finding all proper involutions on R. For example if we take a field F and its corresponding matrix ring $R = M_n(F)$ the problem is to find all proper involutions on R. The transpose operator is an involution which need not be proper unless F is n-real. For example the transpose involution is not proper on $R = M_2(Z_2)$.

Let F be a field and let $R = M_n(F)$ be the matrix ring over F and let $Z(R) = \{cI : c \in F\}$ be the center of R. Let * be an involution on R. Let $A \in Z(R)$. Then for all $X \in R, AX = XA$ implies that $A^*X^* = X^*A^*$ and so $A^* \in Z$. Thus for all $c \in F, (cI)^* = c^*I$ and so $*$ induces an automorphism (called the corresponding automorphism) of order at most 2 on F. Conversely we will show that any automorphism α of order at most 2 on F induces an involution * on $R = M_n(F)$ given by $A^* = P^{-1}A^{\alpha t}P$ for all $A \in R$ as shown in the following proposition.

Proposition 8. *(1) Let $*$ be an involution on $R = M_n(F)$ whose corresponding automorphism is the identity on F. Then there is an invertible matrix P such that $A^* = P^{-1}A^tP$ for every matrix A in $M_n(F)$.*

(2) Let $*$ be an involution on $M_n(F)$ whose corresponding automorphism α on F has order 2. Then there is an invertible matrix P such that $A^* = P^{-1}A^{\alpha t}P$ for every matrix $A \in M_n(F)$.

Proof. (1) The operator $h : A -- > A^{*t}$ is an automorphism that fixes the center of $M_n(F)$ elementwise. From Noether-Skolem Theorem it follows that there is an invertible matrix P such that for all $A \in R, h(A) = A^{*t} = PAP^{-1}$. Thus $A^* = Q^{-1}A^tQ, Q = P^t$ for every $A \in M_n(F)$.

(2)The operator $k : A -- > A^{*\alpha t}$ is an automorphism on $M_n(F)$ that fixes the center $Z(R) = \{cI : c \in F\}$ elementwise. From Noether-Skolem Theorem there is an invertible matrix P such that for every matrix A we have $k(A) = A^{*\alpha t} = P^{-1}AP$. Thus for every matrix $A \in R$ we have $A^* = P^{\alpha t}A^{\alpha t}P^{-1\alpha t} = Q^{-1}A^{\alpha t}Q, Q = P^{-1\alpha t}$. ∎

Corollary 9. *Let $*$ be an involution on $R = M_n(F)$ whose corresponding automorphism $\alpha$is of order 1 or 2 on F. Then there is an inverteble matrix P such that $A^* = P^{-1}A^{\alpha t}P$ for every matrix A in $M_n(F)$.*

We can generalize the preceding propositions to division rings. The proof of the following proposition is similar to the proof of proposition 8 and it is omitted.

Proposition 10. *Let $R = M_n(D)$ be a matrix ring on a division ring D. Let * be an involution on R. Let $Z(R)$ be the center of D. Then there is an automorphism α on the ring $Z(R)$ of order 1 or 2 and there is an invertible matrix P such that for all $A \in R$, $A^* = P^{-1}A^{\alpha t}P$.*

We prove the following.

Proposition 11. *Let α be an automorphism of order 1 or 2 on the field F. Let $P \in R$ be an invertible matrix on F. Define $*$ on R as $A^* = P^{-1}A^{\alpha t}P$ for all $A \in R$. Then $*$ is an involution if and only if $P^{\alpha t} = cI$, $c = \pm 1, c^n = 1$.*

Proof. We have for all $A, B \in R, (A+B)^* = A^* + B^*, (AB)^* = B^*A^*$. To make * as an involution we need $A^{**} = A$ to hold on R.Thus $P^{-1}P^{\alpha t}AP^{-1\alpha t}P = A$ for all $A \in R$. Thus $P^{-1}P^{\alpha t} = cI$ or $P^{\alpha t} = cP$ for some nonzero scalar c..Also we notice that $P^{**} = P$ and from $P^* = P^{-1}P^{\alpha t}P = P^{-1}cPP = cP$ we get $P = P^{**} = (cP)^* = c^2P$ and so $c^2 = 1$ and so $c = \pm 1$. From $P^t = cP$ and upon taking determinants we get we get $c^n = 1$. If n is odd we must have $c = 1$ and if n is even we still have $c = \pm 1$. ∎

Remark 1. *If one of the diagonal elements of P in proposition (11) is nonzero then $c = 1$ and $P^t = P$. Otherwise and if all diagonal elements are 0 we have only the condition $c = \pm 1$ and n is even.*

Next we discuss conditions on P that guarantees that the involution * is proper

Proposition 12. *Let F be a field and let let $R = M_n(F)$.*

*(1) Let * be an involution on R defined by $A^* = P^{-1}A^tP$ for all $A \in R$. Let $P^t = P$. If $P^{-1} = QQ^t$ for some matrix Q and if F is formally real then * is a proper involution.*

*(2) Let * be an involution on R defined by $A^* = P^{-1}A^{\alpha t}P$ for all $A \in R$ with $P^t = P$ and let the corresponding automorphism αon F be of order 2 . If $P^{-1} = QQ^{\alpha t}$ for some matrix Q and if F is formally $\alpha-$complex then is a proper involution.*

Proof. (1) For * to be proper we need the condition $AA^* = 0$ to hold if and only if $A = 0$ for all $A \in R$. This is equivalent to require that $AP^{-1}A^tP = 0$ implies that $A = 0$. Or $AP^{-1}A^t = 0$ implies that $A = 0$. Or, $AQQ^tA^t = 0$ implies that $A = 0$. If F is formally real this is equivalent to $AQ = 0$ implies that $A = 0$ which is the case since Q is invertible.

(2) For * to be proper we need the condition $AA^* = 0$ to hold if and only if $A = 0$ for all $A \in R$. This is equivalent to $AP^{-1}A^{\alpha t}P = 0$ if and only if $A = 0$. Or $AP^{-1}A^{\alpha t} = 0$ if and only if $A = 0$. But $P^{-1} = QQ^{\alpha t}$ and so $AP^{-1}A^{\alpha t} = AQQ^{\alpha t}A^{\alpha t} = 0$ implies that AQ and hence $A = 0$ since F is $\alpha-$formally complex. ∎

Proposition 13. *Let $R = M_n(F)$,F being a field. Let * be an involution on R with a corresponding automorphism α and a corresponding matrix $P, P^{\alpha t} = P$. Let D be the corresponding diagonal matrix that is congruent to P as was mentioned in proposition 7. If $\alpha$is the identity mapping then * is proper if and only if F is D-real. If α is of order 2 then * is proper if and only if F is D-complex.*

Proof. We need to show, for * to be proper, that $AP^{-1}A^{\alpha t} = 0$ if and only if $A = 0$. Since $P^{-1} = CDC^{\alpha t}$, we see that we need

$ACDC^{\alpha t}A^{\alpha t} = 0$ if and only if $A = 0$ if and only If $AC = 0$ if and only if $A = 0$.It is clear that we need F to be $D(\alpha)-$complex. ∎

Proposition 14. *Let F be p-characteristic field and let * be a proper involution on $R = M_n(F)$ such that its corresponding automorphism is the identity. Let P be the corresponding matrix for the involution * as in the proof of proposition (11) and let D be a diagonal matrix congruent to P with diagonal entries set $D = \{ d_1, ..., d_n\}$.Then $p \neq 2, P^* = P^t = P$, and F is $D-real$. Conversely if F is $D-real$ then the involution is proper.*

Proof. We have seen in the proof of proposition (11) that $P^t = \pm P$. Assume, to get a contradiction, that $P^* = -P$. Let $Q = P^t$. Define $f : F^n \times F^n \to F^n$ by $f(u,v) = u^t Qv$. Then f is a bilinear form on F^n. In fact, f is alternating because $f(u,v) = (f(u,v))^t \Rightarrow u^t Qv = v^t Q^t u = -v^t Qu = -f(v,u), \forall u,v \in F^n$. Thus $\forall v \neq 0, f(v,v) = 0$. Let us pick one such v and let us form the matrix A whose first row is v^t and whose all other rows are zero rows. Straightforward calculations show that $A^t QA = 0$. Thus $A^t PA = 0$. Thus $A \neq 0, A^*A = P^{-1}A^t PA = 0$, a contradiction with properness of * on R. It follows that $p \neq 2$,for otherwise $P = -P$ and we saw that this contradicts properness of *. To complete the proof let C be an invertible matrix such that $CP^{-1}C^t = D$, a diagonal invertible matrix. Now $\forall A \in R, \exists B \in R, A = BC, AA^* = 0 \Leftrightarrow BC(P^{-1}C^t B^t P) = BDB^t P = 0 \Longleftrightarrow BDB^t = 0$. Thus * is proper if and only if the only solution in $B \in M_n(F)$ for the equation $BDB^t = 0$ is $B = 0$. If we take for B a matrix which is every where 0 except possibly on its first row $\{x_1, ..., x_n\}$ we see that the condition implies the equation $\sum d_i x_i^2 = 0$ has only the trivial solution . Thus F is D-real. ∎

Let * be an involution on $R = M_n(F), n$ is even, with a corresponding matrix P with $P^t = -P$.We give an example that * is not proper.

Example 1. *Let F be any field and let $R = M_2(F)$ and we take the invertible anti-symmetric matrix matrix $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.Let α be an automorphism on F of degree 1 or 2. We define an involution * on R defined by $A^* = P^{-1}A^{\alpha t}P$ for all $A \in R$. This involution is not proper for if we take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then a simple calculation reveals that $AA^* = 0$-matrix although A is not zero.*

Proposition 15. *Let F be a field and let * be a proper involution on $M_n(F)$ with a corresponding matrix P. Then $P^t = P$ and $ch(F) \neq 2$.*

Proof. If $P^t = -P$ then from the fact in the introduction and from the preceding example * is not proper. If the characteristic of the field is 2 then $P^t = -P$ and again the involution is not proper. ∎

Proposition 16. *Let $(S,*)$ be a finite proper *-semigroup and F be a finite field of characteristic $p \neq 0$ such that $(R,*) = (F[S],*)$ is a proper *-ring. Then R is a direct product of fields and 2×2 matrix rings over fields. Furthermore, $p \neq 2, p \neq 1$ mod 4. The converse is also true.*

Proof. $x \in R, y = x + x^*$. Then not all positive powers of y are distinct owing to the finiteness of R. Let $m > 1$ be a positive power of y such that $\exists n > m, y^m = y^n$ such that $m = 2k, n = 2l$. Then, since $y = y^*$, $y^m = (yy^*)^k = y^n = (yy^*)^l$. Using *-cancellation, we get $y^k = y, k > 1$. Thus $\forall x \in R, \exists n(x), (x+x^*)^{n(x)} = x + x^*$ and *Theorem D* applies. The last part follows from the fact that any involution on $M_2(Z_p)$ is transpose-inner and the transpose involution is proper if and only if $p \neq 2, p \neq 1$ mod 4. ∎

Proposition 17. *Let $(R,*) = (M_m(Z_n),*)$ be a proper *-ring. Then $m = 2, \quad n = p_1 ... p_k, \quad p_i \neq p_j (i \neq j), \quad p_i \neq 2, p_i \neq 1$ mod $4, \forall i = 1,...,k$.*

Proof. That $m = 2$ follows from *Theorem D. That $p_i \neq p_j (i \neq j)$* follows from * being proper: $p_1 = p_2 \Rightarrow \frac{n}{p_1}(\frac{n}{p_1})^* = 0 \neq \frac{n}{p_1}$. The proof of the other parts is similar to the proof in proposition 16 . ∎

Proposition 18. *Let $(R,*) = (M_2(Z_p),*)$ be a proper *-ring. Then * is inner.*

Proof. Let $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}., G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then C, D generate the ring R. This is easily seen. Let $C^* = A, D^* = B$. We are looking for a matrix $u = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ such that $C^* = A = u^{-1}Cu = u^{-1}C^t u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad D^* = B = u^{-1}Du = u^{-1}D^t u = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Thus $uA = Cu, uB = Du \Rightarrow uA = CD^{-1}uB = GuB \Rightarrow \begin{pmatrix} z & t \\ -x & -y \end{pmatrix}.A = B$. The last matrix equation gives rise to solutions in x, y, z and t since A and B are invertible. Furthermore the resulting matrix $\begin{pmatrix} z & t \\ -x & -y \end{pmatrix}$,which has the same determinant as that of u, is invertible since A and B are. Thus u is invertible. Thus * is inner at least for the matrices C and D. But C and D generate the whole matrix ring and, for example, $(CD)^* = D^*C^* = u^{-1}D^tuu^{-1}C^tu = u^{-1}(CD)^*u$. Thus * is inner in general. ∎

3.1. ***-Semigroup Embedding in a Proper *-Ring.** We start this subsection with the following remarks:

Although the following remarks are almost routine we present them here for the sake of completeness.

Remark 2. *Let $(R,*)$be an m-characteristic proper *-ring without 1. Then either $m = 0$ or m is square-free. Also $(R,*)$ can be *-embedded in an m-characteristic proper *-ring $(R_1,*)$ with 1.*

Illustration 1. *Let r be a nonzero element of R such that there is a smallest positive integer m with $mr = 0$ and $m = kp^2$,k is not a unit and p is a prime. then kp is not zero. But then $(kpr)(kpr)^* = 0$. From properness of * it follows that $kpr = 0$ which is a contradiction with kpr not zero. To prove the other part we have two cases to consider.*

Illustration 2.

Case 1. *: $m = 0$. In this case we take the Cartesian product $Z\otimes R$ and define addition and multiplication as follows. $(m,r)+(m',r') = (m+m',r+r'),(m,r).(m',r') = (mm',mr'+m'r+rr')$ for every $m,m' \in Z, r,r' \in R$. This makes of $Z\otimes R$ a ring R_1. We define an operator * on R' by $(m,r)^* = (m,r^*)$. Then it is straightforward to see that * is an involution. In fact, it is proper. For, $(m,r)(m,r)^* = (0,0) = (m^2,mr+mr^*+rr^*) \Rightarrow m = 0, rr^* = 0 \Rightarrow r = 0,(m,r) = (0,0)$.*

Illustration 3.

Case 2. *:$m \neq 0$. In this case m is square-free. For if $m = p^2k, p$ is prime, then there exists $0 \neq r \in R, mr = 0, nr \neq 0$,for all positive integers $n < m$. But then $0 \neq pkr, (pkr)(pkr)^* = 0$, a contradiction with the properness of the involution *. Now we form $Z_m\otimes R$. We define addition and multiplication as in Case 1. It is straightforward to see that these operations are well-defined making of $Z_m\otimes R$. a ring denoted by R_2. We define * on R_1 as in Case 1. Then * is an* *involution and it is proper. For,* $(0,0) = (k,r)(k,r)^* = (k^2, kr^* + kr + rr^*) \Rightarrow k^2 = 0 \Rightarrow k = 0$ *for all* $k \in Z_m, r \in R$*. The last implication follows since m is square-free forcing* Z_m *to have a 0-radical. It follows that* $rr^* = 0$ *and so* $r = 0, (m,r) = (0,0)$.

Remark 3. *Let* $(R,*)$ *be an 0-characteristic proper *-ring. Then* $(R,*)$ *can be *-embedded in a 0-characteristic proper *-algebra* $(R_1,*)$ *over Q.*

Illustration 4. *: We may assume that R contains 1. Then R contains a copy of Z. Now we localize R at the multiplicatively closed set* $Z\backslash\{0\}$*.(See* [2] *for definition of localization). The resulting *-ring denoted by* $(R_1,*)$ *contains a copy of Q and it is a proper *-ring. For if* $[(r,m)][(r,m)]^* = [(0,1)]$ *then* $rr^* = 0$ *and so* $r = 0, [(r,m)] = [(0,0)]$.

Now we prove the following.

Proposition 19. *Let* $(R,*)$ *be a *-ring. Let* I_1 *be the ideal generated by all A in* $(R,*)$*such that* AA^* *or* A^*A *is 0 and, for* $k > 1$*, let* I_k *be the ideal generated by all* $A \in (R,*)$ *such that* AA^* *or* A^*A *is in* I_{k-1}*. Then* I_k *is a *-ideal ,* $I_k \subseteq I_{k+1}$*, and if I is the union of all* I_k *,*$k > 0$*, then I is a *-ideal and* $(R/I,*)$ *is a p*-ring.*

Proof. That I_k is a *-ideal and that $I_k \subseteq I_{k+1}$ are trivial to verify. Also I is a *-ideal. If AA^* is in I then it is in some I_k and so A is in I_{k+1} and hence A is in I. Thus (R/I) is a p*-ring. ∎

Corollary 20. *Let* $(S,*)$ *be a *-semigroup, not necessarily a p*-semigroup, and let* $(Z[S],*)$ *be the corresponding *-semigroup ring of* $(S,*)$ *over Z.Let* $I_k, k > 0$*, and I be the ideals as in the preceding proposition. Then* $(Z[S]/I,*)$ *is a p*-ring. If* $(S,*)$ *is a finite p*-semigroup then it is *-embeddable in a p*-ring if and only if there are no distinct elements s,t in S such that* $s-t$ *in any* I_k*.In this case if S is commutative then* $(S,*)$ *is *-embeddable in a subdirect product of fields. Also in this case if* $Z[S]/I$ *is finite then* $(S,*)$ *is *-embeddable in a finite direct product of matrix rings each over a finite field.*

Proof. The proof is a direct consequence of the proposition (19), remarks 3 and 2 and Wedderburn's Theorem since $(S,*)$ in case of S is finite and hence the corresponding algebra is Artinean. For then $(S,*)$ is a finite p*-semigroup such that $(R,*) = (Z[S]/I,*)$ is infinite and there are no distinct elements s,t in S such that $s-t$ is in any I_k. Then $(Q[S]/I,*)$ is isomorphic to a finite direct product of matrices over division ring and hence $(S,*)$ is represented as a p*-semigroup of matrices over a division ring. ∎

Proposition 21. *:Let $(S,*)$be an mp semigroup and let $(D,*)$ be a 0-characteristic integral domain with proper involution *. If S is finite, or if $i \in D$ then $D[S]$ is nil-semisimple while $(D[S],*)$ need not be a proper *ring and the extended involution need not be a proper ring involution.*

Proof. : We can assume that D is contained in the complex number field **C**. Assume first that $i \in D$. Then D is closed under complex conjugation which is a proper involution. Since $(S,*)$ is an mp-semigroup it follows from Theorem A that $(D[S],*)$ is proper * and nil-semisimple. Now assume that $i \notin D$ and assume that S is finite. Let J be a nil ideal in $D[S]$. Since S is finite and the D-module $D[S]$ is isomorphic to the direct sum of $|S|$ copies of the Noetherian left D-modules (each is isomorphic to D), then $D[S]$ is a Noetherian left D-module. Hence it is also a Noetherian left $D[S]$-module and thus it is a left-Noetherian ring. By theorem B, J is nilpotent and there is a positive integer n such that $J^n = 0$. Then $I = J + iJ$ is a nilpotent ideal in $D[i][S]$ which is nil-semisimple. Thus I is 0 and hence J is a 0 ideal. ∎

Proposition 22. *Let $(S,*)$ be a finite mp-semigroup and let F be a 0-chacteristic field. Then $F[S]$ is a finite direct product of matrices over a skew field and $(S,*)$ is *-embeddablle in the *-ring $(F[S],*)$ where * is the natural involution inherited from the involution * in $(S,*)$. If the field F has a non zero characteristic then $F[S]$is a finite direct product of matrices over a field .*

Proof. We can assume without loss of generality that S has an identity element 1 (This easy to prove). Since $F[S]$ is a nil- semisimple ring by proposition 21 and since it is a finite dimensional F-vector space, it follows that it is a finite direct product of matrix rings over a skew field. Let $(S,*)$ be a finite mp-semigroup and let F be a field of 0-characteristic. Then the involution on S gets extended to an involution on $F[S]$ in a natural way: $(\sum a_i s_i)^* = \sum a_i s_i^*$.(But there is no guarantee that this involution is proper on $R[S]$, unless R is formally complex). If $ch(F) \neq 0$ the prime field is Z_p and the subring generated by Z_p and S is finite and has a proper involution and so it is a finite direct sum of matrix rings over a finite skew field (a field then). ∎

Proposition 23. *Let* $(R,*)$ *be a finite proper *-ring. Then* $(R,*)$ *is *-isomorphic with a finite direct product of matrix rings over a field.*

Proof. We show that R has a 0-radical I. For let A be in I. Then AA^* is in I. But then there is a natural number n such that $(AA^*)^n = 0$. By properness of * it follows that $AA^* = 0$ and hence $A = 0$. Thus I is the zero ideal. From Wedderburn Theorem it follows that R is isomorphic with a finite direct product of matrix rings over a skew field. Since R is finite the skew fields are fields. ∎

Proposition 24. *: Let* $(S,*)$ *be a proper *-semigroup *-embeddable in a proper *-ring* $(R,*)$*. Then*

(1) *There is a *-ideal I in* $(Z[S],*)$ *such that* $(Z[S]/I,*)$ *is a p*-ring which *-embeds* $(S,*)$*.*
(2) *If* $ch(R) = 0$ *and S is finite then* $(S,*)$ *is *-embeddable in a finite direct sum of matrix rings over a division ring with proper involution.*
(3) *If* $ch(R) = m \neq 0$ *and S is finite then* $(S,*)$ *is *-embeddable in a finite direct sum of matrix rings over a finite prime- characteristic field with proper involution.*

Proof. (1) There is a natural *-mapping $f : (Z[S],*)-> (R,*)$ given by $f(\sum m_i s_i) = \sum m_i g(s_i)$, where g is the *-embedding of $(S,*)$ into $(R,*)$. If $(Z[S],*)$ is p* then we can take $I = 0$. If there is A not 0 in $Z[S]$ such that AA^* or $A^*A = 0$ then we take the ideal I_1 generated by all such A and we consider the *-ring $Z[S/I_1$.We notice that there can be no two different elements s,t in S such that $s-t$ is in I_1 lest $s-t = 0$ in R which would imply non *-embeddability of $(S,*)$ in $(R,*)$. If this *-ring is p^* then we are done with getting the required p*-ring $Z[S]/I$. Otherwise there is A not in I_1 such that AA^* is in I_1. We take all such A and all B such that B^*B is in I_1 and form the ideal I_2. These are 0 in R of course. Now we form the *-ring R/I_2. There can be no two different elements s,t in S such that $s-t$ is in I_2 lest that would contradict *- embeddability of $(S,*)$ into $(R,*)$. If this *- ring is p^* then we are finished by getting a p^*-ring R/I_2 which *-embeds $(S,*)$. We continue this way. The union of these *-ideals is clearly a *-ideal I and $(R/I,*)$ is a p^*-ring which *-embeds $(S,*)$.

(2) If $ch(R) = 0$ and S is finite we can assume that R contains a copy of Q. Let $R' = \langle Q,S \rangle$ be the set of all rational linear combinations of elements of S in R. Then R' is a proper *-ring which

*-embeds $(S,*)$. Being a homomorphic image of the Artinian ring $Q[S]$, R' is Artinian. Since a proper *-ring has 0 nil-radical, by Wedderburn's Theorem R' is isomorphic to a finite direct sum R_2 of matrix rings over a skew field. We define an involution * on R' as follows. Let f be the isomorphism of R' onto R_2. Take b in R'. Then $b = f(a)$ for a unique element $a \in R'$. Define $b^* = f(a^*)$. We show that * is a proper involution. Let $b, c \in R_2$ and let $b = f(a_1), c = f(a_2)$. Then $(b+c)^* = (f(a_1)+f(a_2))^* = (f(a_1+a_2))^* = f(a_1^*+a_2^*) = f(a_1^*)+f(a_2^*) = (f(a_1))^*+ (f(a_2))^* = b^*+c^*, (bc)^* = (f(a_1a_2))^* = f(a_2^*)f(a_1^*)$
$= (f(a_2))^*(f(a_1))^* = c^*b^*, b^{**} = (f(a_1^*))^* = f(a_1^{**}) = f(a_1) = b$. And if $bb^* = 0$ then $f(a_1)(f(a_1^*) = f(a_1a_1^*) = 0$ and so $a_1a_1^* = 0$ which implies that a_1 and hence $b = 0$.

(3) If $ch(R) = m \neq 0$ and S is finite we can argue similarly that there is a copy of Z_m in R and $R'' = \langle Z_m, S\rangle$ is proper *. Since R'' is finite it is isomorphic to a finite direct sum of matrix rings over a prime characteristic finite field. This is because a finite skew field is a field. The same argument as above applies to show that the involution inherited from S on the finite sum of matrix rings is proper. This completes the proof. ∎

Proposition 25. *Let $(S,*)$ be a simple *-semigroup. Then it is a p*-semigroup and it is *-embeddable in a p*-ring.*

Proof. There is a natural *-homomorphism $f : (S,*) -> (Z[S]/I,*)$ of $(S,*)$ into the proper *-ring $(Z[S]/I,*)$. Now the kernel of f gives rise to a *-ideal in $(S,*)$ which is *-simple. This ideal must be zero and so f is a *-embedding and $(S,*)$ is a p*-semigroup which is *-embedded in a p*-ring. ∎

Strategy 1. *Assume we have a finite proper *-semigroup $(S,*)$ with 1 and assume that we would like to know if $(S,*)$ is *-embeddable in a proper *-ring $(R,*)$ of matrices of characteristic 0. Then we form the algebra $(R,*) = (Q[S],*)$ where * is the natural involution. If $(R,*)$ is p* then we are done. If not then we form the ideal I_1 generated by all $A \in R$ such that AA^* or $A^*A = 0$.Then I_1 is closed under the involution * and so $(R_1,*) = (R/I,*)$ is an algebra with involution and with dimension $n_1 < n = \mid S \mid$. If there are elements $s \neq t$ in S such that $s-t \in I$ then $(S,*)$ is not *-embeddable in a p*-ring of characteristic 0. If there is no such pair we check if $(R_1,*)$ is p*. If it is p* then we are done and If not then we look for all $A \in R$ such* *that* $A \notin I_1$ *such that* AA^* *or* $A^*A \notin I_1$ *and we form the ideal* I_2 *generated by all such A and its involution* A^**.This ideal* I_2 *is closed under involution. Then we form* $(R_2, *) = (R/I_2, *)$ *and with dimension* $n_2 < n_1$*.If there are distinct* $s, t \in S$ *such that* $s - t \in I_2$ *then* $(S, *)$ *is not *-embeddable in a p*-ring of characteristic 0. If there is no such pair we check is* $(R_2, *)$ *is p*. If so then we are done and If not we look for all* $A \neq 0$ *in R such that* AA^* *or* A^*A *is in* I_2 *and form the ideal* I_3 *generated by these A. This is closed under taking * and we form* $(R_3, *) = (R/I_3, *)$*. This has dimension* $n_3 < n_2 < n_1 < n$*. etc. In a finite number of steps either we come up with a p*-algebra of 0-characteristic which *-embeds* $(S, *)$ *or we conclude that there is no such p*-ring. The same procedure we can use to check if there is a p*- ring of any prescribed nonzero characteristic or not.*

Strategy 2. *Assume we have a finite proper *-semigroup* $(S, *)$ *with 1 which is not *-embeddable in a p*-ring with characteristic 0. It is desired to reform (S,*) to a p*-semigroup that is *-embeddable in a p*-ring of characteristic 0. We form as before the p*-ring* $(Q[S]/I, *)$*. Then there is a p*-image* $(T, *)$ *of* $(S, *)$ *in* $(Q[S]/I, *)$*. Then there is a *-congruence ˜ in S such that the p*-semigroup* $(S/\tilde{}, *)$ *is isomorphic with the* $(T, *)$ *inside the p*-ring* $(Q[S]/I, *)$*..*

Conflict of Interests

The author declares that there is no conflict of interests.

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