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AN IMPLICIT ITERATIVE PROCESS FOR SOLUTION SYSTEM OF EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF AN AMENABLE SEMIGROUP AND INFINITE FAMILY OF NON-EXPANSIVE MAPPINGS

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Abstract. In this paper, using δ- strongly monotone and λ- strictly pseudo-contractive (in the terminology of Browder-Petryshyn type) mapping F on a real Hilbert space H, we introduce an implicit iterative scheme to find a common element of the set of solutions of a system of equilibrium problems and the set of fixed points of amenable semigroup of non-expansive mappings and infinite family of non-expansive mappings on H, with respect to a sequence of left regular means defined on an appropriate space of bounded real valued functions of semigroup. Then, we prove the convergence of sequence generated by the suggested algorithm to a unique solution of the variational inequality.

Keywords: fixed point; implicit method; non-expansive mapping; amenable semigroup; equilibrium problem.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let ϕ be a bi-function of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for

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$\phi : C\times C \to \mathbb{R}$ is to determine its equilibrium points, i.e the set

$$EP(\phi) = \{x \in C : \phi(x,y) \geq 0, \forall y \in C\}. \tag{1}$$

Let $\mathscr{J} = \{\phi_i\}_{i\in I}$ be a family of bi-functions from $C\times C$ into $\mathbb{R}$. The system of equilibrium problems for $\mathscr{J} = \{\phi_i\}_{i\in I}$ is to determine common equilibrium points for $\mathscr{J} = \{\phi_i\}_{i\in I}$, i.e the set

$$EP(\mathscr{J}) = \{x \in C : \phi_i(x,y) \geq 0, \forall y \in C,\ \forall i \in I\}. \tag{2}$$

Numerous problems in physics, optimization, and economics reduce into finding some element of $EP(\phi)$. Some methods have been proposed to solve the equilibrium problem; see, for instance, [3, 9, 10, 21]. The formulation (2), extend this formalism to systems of such problems, covering in particular various forms of feasibility problems [2, 8].

A mapping T of C into itself is called non-expansive if $\| Tx - Ty \| \leq \| x - y \|$, for all $x, y \in C$. By $Fix(T)$, we denote the set of fixed point of T i.e., $Fix(T) = \{x \in H : Tx = x\}$. It is well known that $Fix(T)$ is closed convex. Recall that a self-mapping $f : C \to C$ is a contraction on C if there is a constant $\alpha \in (0,1)$ such that

$$\| fx - fy \| \leq \alpha \| x - y \|, \qquad \forall x, y \in C.$$

Assume $A : H \to H$ is strongly positive; that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma} \| x \|^2, \qquad \forall x \in H.$$

Given any $r > 0$ the operator $J_r^{\phi} : H \to C$ defined by

$$J_r^{\phi}(x) = \{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\},$$

is called the resolvent of F, see [9]. It is shown in [9] that, under suitable hypotheses on F (to be stated precisely in Section 2), $J_r^{\phi} : H \to C$ is single- valued and firmly non-expansive and satisfies

$$Fix(J_r^{\phi}) = EP(\phi), \qquad \forall r > 0.$$

Using this result in 2007, Plubtieng and Punpaeng [19] proved the following strong convergence theorem for an implicit iterative sequence $\{x_n\}$ obtained from the viscosity approximation method for finding a common element in $EP(\phi)\cap Fix(T)$ which solves some certain variational inequality.

Theorem 1.1. *Let H be a real Hilbert space and ϕ be a bi-functions from $H\times H$ into $\mathbb{R}$ satisfying*

- (A_1) $\phi(x,x)=0$ *for all* $x\in H$,
- (A_2) ϕ *is monotone, i.e;* $\phi(x,y)+\phi(y,x)\leq 0$ *for all* $x,y\in H$,
- (A_3) *for all* $x,y,z\in H$, $\limsup_{t\to 0}\phi(tz+(1-t)x,y)\leq\phi(x,y)$,
- (A_4) *for all* $x\in H$, $y\to\phi(x,y)$ *is convex and lower semi-continuous.*

For $r>0$, set $J_r^{\phi}: H\to H$ to be the resolvent of ϕ, i.e. $J_r^{\phi}(x)$ is the unique $z\in H$ for which

$$\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\qquad \forall y\in H.$$

Let T be a non-expansive mapping on H such that $EP(\phi)\cap Fix(T)\neq\emptyset$. Let f be a contraction of H into itself with coefficient $\alpha\in(0,1)$ and let A be strongly positive bounded linear mapping on H with coefficient $\overline{\gamma}>0$ and $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)Ty_n, & \forall n\in\mathbb{N},\\ \phi(y_n,y)+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\geq 0, & \forall y\in H,\end{cases}$$

where $y_n=J_{r_n}^{\phi}(x_n)$, $\{r_n\}\subset(0,\infty)$ and $\{\alpha_n\}\subset[0,1]$ satisfying $\lim_{n\to\infty}\alpha_n=0$ and $\liminf_{n\to\infty}r_n>0$. The sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a unique point $x^*\in EP(\phi)\cap Fix(T)$ which solves the variational inequality:

$$\langle(\gamma f-A)x^*,x-x^*\rangle\leq 0,\qquad \forall x\in EP(\phi)\cap Fix(T).$$

Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers in $[0,1]$. For each $n\geq 1$, define a mapping W_n of C into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{3}
$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \cdots, T_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then Colao et al. [7] proved the following strong convergence theorem.

Theorem 1.2. *Let H be a real Hilbert space , $\{T_i\}_{i=1}^{\infty}$ an infinite family of non-expansive mapping of H into H, for $k \in \{1,2,\cdots,M\}$ ϕ_k a bi-function from $H \times H$ into $\mathbb{R}$, A a strongly positive bounded linear mapping on H with coefficient $\overline{\gamma} > 0$ and f an $\alpha-$contraction on H. Moreover, let $\{r_{k,n}\}_{k=1}^{M}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \lambda_n \leq b < 1$, γ be a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

- (B_1) *for every $k \in \{1,2,\cdots,M\}$, the bifunction ϕ_k satisfies $(A_1)-(A_4)$,*
- (B_2) $\mathscr{F} = (\bigcap_{k=1}^{M} EP(\phi_k)) \cap (\bigcap_{i=1}^{\infty} Fix(T_i)) \neq \emptyset$,
- (B_3) *the sequence $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$,*
- (B_4) *the sequences $\{r_{k,n}\}_{k=1}^{M}$ satisfy $\lim_{n\to\infty} r_{k,n} = \hat{r}_k > 0$ for every $k \in \{1,2,\cdots,M\}$.*

Let W_n be the mapping defined by (3) and the sequence $\{x_n\}$ generated by

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n J_{r_{1,n}}^{\phi_1} J_{r_{2,n}}^{\phi_2} \cdots J_{r_{M,n}}^{\phi_M} x_n, \qquad \forall n \in \mathbb{N},$$

Then, the sequence $\{x_n\}$ *converges strongly to* $x^* \in \mathscr{F}$*, where* x^* *is the unique solution of the variational inequality*

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

A mapping F with domain $D(F)$ and range $R(F)$ in H is called δ-strongly monotone if there exists a positive real number $\delta > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \delta \parallel x - y \parallel^2, \qquad \forall x, y \in D(F). \tag{4}$$

F is called λ-strictly pseudo-contractive in the terminology of Browder and Petryshyn [4] if there exists a real number $\lambda \in [0,1)$ such that

$$\parallel Fx - Fy \parallel^2 \leq \parallel x - y \parallel^2 + \lambda \parallel (I-F)x - (I-F)y \parallel^2, \ \forall x, y \in D(F). \tag{5}$$

It is well-known that (5) is equivalent to

$$\langle Fx - Fy, x - y \rangle \leq \parallel x - y \parallel^2 - \frac{1-\lambda}{2} \parallel (I-F)x - (I-F)y \parallel^2. \tag{6}$$

In this paper, motivated and inspired by Lau et al. [11] Colao et al. [7], Piri [14, 15, 16, 17], Piri and Badali [18] and Marino and Xu [13], we introduce an implicit iterative scheme to fined a common element of the set of solutions of a system of equilibrium problems and the set of fixed points of an amenable semigroup of non-expansive mappings and infinite family of non-expansive mappings on a real Hilbert space. Let F be a mapping on real Hilbert space H which is both δ- strongly monotone and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$. Assume S be a semigroup and $\varphi = \{T_t : t \in S\}$ be a non-expansive semigroup on H such that $Fix(\varphi) = \bigcap_{t \in S} Fix(T_t) \neq \emptyset$. Let X be a subspace of $B(S)$ such that $1 \in X$ and the function $t \to \langle T_t(x), y \rangle$ is an element of X for each $x, y \in H$,. Let $\{\mu_n\}$ be a sequence of means on X. We define a sequence $\{x_n\}$ by

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n J_{r_{M,n}}^{\phi_M} \cdots J_{r_{2,n}}^{\phi_2} J_{r_{1,n}}^{\phi_1} x_n, \qquad \forall n \in \mathbb{N},$$

where $\gamma \in \left(0, \frac{1-\sqrt{\frac{2-2\delta}{1-\lambda}}}{\alpha}\right)$. We prove that under assumption on parameters like that in Colao et al. [7], the sequence $\{x_n\}$ strongly converges to $x^* \in \mathscr{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap Fix(\varphi) \cap \bigcap_{k=1}^{M} EP(\phi_k)$, where x^* solves the variational inequality

$$\langle (\gamma f - F)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

Our results improve the corresponding results announced by many others and a consequence for commuting pairs of non-expansive mappings is also presented.

2. Preliminaries

Let S be a semigroup and let $B(S)$ be the space of all bounded real valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st), \qquad (r_s f)(t) = f(ts), \qquad \forall t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its dual. An element μ in X^* is said to be a mean on X if $\| \mu \| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup, see [12]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_{\alpha} \| l_s{}^* \mu_\alpha - \mu_\alpha \| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\varphi = \{T_t : t \in S\}$ of mapping from C into itself is said to be a non-expansive semigroup on C if T_t is non-expansive and $T_{ts} = T_t T_s$ for each $t, s \in S$. By $Fix(\varphi)$ we denote the set of common fixed points of φ, i.e.

$$Fix(\varphi) = \bigcap_{t \in S} \{x \in C : T_t(x) = x\}.$$

Lemma 2.1. [12] *Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup on H such that $\{T_t x : t \in S\}$*

is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, y^* \rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. If we write $T_\mu x$ instead of $\int T_t x d\mu(t)$, then the followings hold.

(i) *T_μ is nonexpansive mapping from C into C.*

(ii) *$T_\mu x = x$ for each $x \in Fix(\varphi)$.*

(iii) *$T_\mu x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C, \qquad Tx_n \to v \qquad imply \qquad Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.2. [1] *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is non-expansive. Then, the mapping $I - T$ is demiclosed at zero.*

Let C be a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\| x - y_0 \| = d(x, C),$$

where $d(x,C) = \inf_{y \in C} \| x - y \|$. The number $d(x,C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximation from x to C is denoted by

$$P_C(x) = \{y \in C : \| x - y \| = d(x, C)\}.$$

This defines a mapping P_C from X into 2^C and is called metric (nearest point) projection onto C. It is well-known that P_C is a non-expansive mapping of H onto C.

Lemma 2.3. [23] *Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let $x \in H$ and $y \in C$. Then, the following are equivalent.*

(i) $y = P_C(x)$,

(ii) $\langle x - y, y - z) \rangle \geq 0, \qquad \forall z \in C.$

Let $\phi : C\times C \to \mathbb{R}$ be a bi-function. Given any $r>0$, the operator $J_r^{\phi} : H\to C$ defined by

$$J_r^{\phi}x=\{z\in C:\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 1,\forall y\in C\}$$

is called the resolvent of ϕ, see [9]. The equilibrium problem for ϕ is to determine its equilibrium points, i.e., the set

$$EP(\phi)=\{x\in C:\phi(x,y)\geq 0,\forall y\in C\}.$$

Let $\mathscr{J}=\{\phi_i\}_{i\in I}$ be a family of bi-functions from $C\times C$ into $\mathbb{R}$. The system of equilibrium problems for $\mathscr{J}$ is to determine common equilibrium points for $\mathscr{J}=\{\phi_i\}_{i\in I}$. i.e, the set

$$EP(\mathscr{J})=\{x\in C:\phi_i(x,y)\geq 0,\forall y\in C,\ \forall i\in I\}.$$

Lemma 2.4. [9] *Let C be a nonempty closed convex subset of H and $\phi : C\times C\to\mathbb{R}$ satisfy*

(A_1) $\phi(x,x)=0$ *for all* $x\in C$,

(A_2) ϕ *is monotone, i.e;* $\phi(x,y)+\phi(y,x)\leq 0$ *for all* $x,y\in C$,

(A_3) *for all* $x,y,z\in C$, $\quad\limsup_{t\to 0}\phi(tz+(1-t)x,y)\leq\phi(x,y)$,

(A_4) *for all* $x\in C$, $y\to\phi(x,y)$ *is convex and lower semi-continuous.*

Given $r>0$, *define the operator* $J_r^{\phi} : H\to C$, *the resolvent of* ϕ, *by*

$$J_r^{\phi}(x)=\{z\in C:\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}.$$

Then,

(1) J_r^{ϕ} *is single valued,*

(2) J_r^{ϕ} *is firmly non-expansive, i.e,* $\| J_r^{\phi}x-J_r^{\phi}y\|^2\leq\langle J_r^{\phi}x-J_r^{\phi}y,x-y\rangle$ *for all* $x,y\in H$,

(3) $Fix(J_r^{\phi})=EP(\phi)$,

(4) $EP(\phi)$ *is closed and convex.*

Lemma 2.5. [7] *Let C be a nonempty closed convex subset of H and $\{r_n\}\subset(0,1)$ be a sequence converging to $r>0$. For a bifunction $\phi : C\times C\to\mathbb{R}$, satisfying conditions $(A_1)-(A_4)$, define $J_{r_n}^{\phi}$ and J_r^{ϕ} for $n\in\mathbb{N}$ as in Lemma . Then for every $x\in H$, we have* $\lim_{n\to\infty}\| J_{r_n}^{\phi}x-J_r^{\phi}x\|=0$.

Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself, where C is a nonempty closed convex subset of a real Hilbert space H. Given a sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $[0,1]$, we define a sequence $\{W_n\}_{n=1}^{\infty}$ of self mappings on C by (3). Then we have the following results.

Lemma 2.6. [20] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1, \forall i \geq 1$. Then*

(1) *W_n is non-expansive and $Fix(W_n) = \bigcap_{i=1}^{n} Fix(T_i)$ for each $n \geq 1$,*

(2) *for each $x \in C$ and for each positive integer j, the limit $\lim_{n\to\infty} U_{n,j}x$ exists.*

(3) *The mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}, \quad \forall x \in C,$$

is a non-expansive mapping satisfying $Fix(W) = \bigcap_{i=1}^{\infty} Fix(T_i)$ and it is called the $W-$ mapping generated by $T_1, T_2, \cdots$ and $\lambda_1, \lambda_2, \cdots$.

Lemma 2.7. [24] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-expansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1, \forall i \geq 1$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \| Wx - W_n x \| = 0.$$

Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. Consider the functional $r_a(., \{x_n\}) : X \to \mathbb{R}$ defined by

$$r_a(., \{x_n\}) = \limsup_{n\to\infty} \| x_n - x \|, \qquad \forall x \in X.$$

The infimum of $r_a(., \{x_n\})$ over K is to be asymptotic radius of $\{x_n\}$ with respect to K and it is denoted by $r_a(K, \{x_n\})$. A point $x \in K$ is said to be asymptotic center of the sequence $\{x_n\}$ with respect to K if

$$r_a(x, \{x_n\}) = \inf\{r_a(y, \{x_n\}) : y \in K\}.$$

The set of all asymptotic center of $\{x_n\}$ with respect to K is denoted by $C_a(K, \{x_n\})$. This set may be empty, a singleton, or infinitely many points.

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$$r_a(x,\{x_n\}) = \inf\{r_a(y,\{x_n\}) : y \in K\}.$$

The set of all asymptotic center of $\{x_n\}$ with respect to K is denoted by $C_a(K,\{x_n\})$. This set may be empty, a singleton, or infinitely many points.

Lemma 2.8. [1] *Let X be uniformly convex Banach space satisfying the Opial's condition and K a nonempty closed convex subset of X. If a sequence $\{x_n\} \subset K$ converges weakly to a point x_0, then x_0 is the asymptotic center of $\{x_n\}$ with respect to K.*

The following Lemma will be frequently used throughout the paper. For the sake of completeness, we include its proof.

Lemma 2.9. [6] *Let H be a real Hilbert space.*

(i) *If $F\colon H \to H$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta, \lambda < 1$ and $\delta > \frac{1+\lambda}{2}$. Then, $I - F$ is contractive with constant $\sqrt{\frac{2-2\delta}{1-\lambda}}$.*

(ii) *If $F\colon H \to H$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta, \lambda < 1$ and $\delta > \frac{1+\lambda}{2}$. Then, for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)$.*

Notation Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. For subset A of H, by $\overline{co}A$, we denote the closed convex hull of A. For $\varepsilon > 0$ and a mapping $T\colon D \to H$, we let $F_\varepsilon(T;D)$ be the set of $\varepsilon-$ approximate fixed points of T, i.e. $F_\varepsilon(T;D) = \{x \in D : \| x - Tx \| \leq \varepsilon\}$. Weak convergence is denoted by $\rightharpoonup$ and strong convergence is denoted by $\to$. F is a mapping on H which is both δ- strongly monotone and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$, and f is a contraction with coefficient $0 < \alpha < 1$. We will also always use γ to mean a number in $\left(0, \frac{1-\sqrt{\frac{2-2\delta}{1-\lambda}}}{\alpha}\right)$.

3. Main results

Consider a mapping Γ_n on H defined by

$$\Gamma_n(x) = \alpha_n \gamma f(x) + (I - \alpha_n F) T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x, \qquad x \in H, n \geq 1,$$

Using Lemma , Lemma and Lemma , we have

$$\begin{aligned}
&\| \Gamma_n(x) - \Gamma_n(y) \| \\
&= \| \alpha_n \gamma (f(x) - f(y)) + (I - \alpha_n F) T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x \\
&\quad - (I - \alpha_n F) T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} y \| \\
&\leq \alpha_n \gamma \alpha \| x - y \| + \left(1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \| x - y \| \\
&= \left(1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \| x - y \| .
\end{aligned}$$

Since $0 < 1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right) < 1$, it follows that Γ_n is a contraction. Therefore, by the Banach contraction principle, Γ_n has a unique fixed point $x_n \in H$ such that

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n.$$

Note that x_n indeed depends on f as well, but we will suppress this dependency of x_n on f for simplicity of notation throughout the rest of this paper. The following is our main result.

Theorem 3.1. *Let S be a semigroup and $\varphi = \{T_t : t \in S\}$ a non-expansive semigroup from H into H such that $Fix(\varphi) = \bigcap_{t \in S} Fix(T_t) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y \rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Let $\mathscr{J} = \{\phi_k : k = 1, 2, \cdots, M\}$ be a finite family of bi-functions from $H \times H$ into $\mathbb{R}$ which satisfy $(A_1) - (A_4)$ and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of non-expansive mappings of H into H such that $T_i(Fix(\varphi)) \subset Fix(\varphi)$ for each $i \in \mathbb{N}$ and*

$\mathscr{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap Fix(\varphi) \cap EP(\mathscr{J}) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$, $\{\lambda_n\}$ a sequence in $(0,b]$ for some $b \in (0,1)$, $\{r_{k,n}\}$ sequences in $(0,\infty)$. Suppose the following conditions are satisfied:

(B_1) $\lim_{n\to\infty} \alpha_n = 0$,

(B_4) $\lim_{n\to\infty} r_{k,n} = \hat{r}_k$, for every $k \in \{1,2,\cdots,M\}$.

Let W_n be the mapping defined by (3) and for every $k \in \{1,2,\cdots,M\}$ and $n \in \mathbb{N}$, let $J_{r_{k,n}}^{\phi_k}$ be the resolvent generated by ϕ_k and $r_{k,n}$ in Lemma . If $\{x_n\}$ is the sequence generated by

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n J_{r_{M,n}}^{\phi_M} \cdots J_{r_{2,n}}^{\phi_2} J_{r_{1,n}}^{\phi_1} x_n, \qquad \forall n \in \mathbb{N}. \tag{7}$$

Then $\{x_n\}$ converges strongly to $x^* \in \mathscr{F}$, where x^* is the unique solution of the variational inequality:

$$\langle (\gamma f - F)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

Proof. By taking $\mathscr{J}_n^k = J_{r_{k,n}}^{\phi_k} \cdots J_{r_{2,n}}^{\phi_2} J_{r_{1,n}}^{\phi_1}$ for $k \in \{1,2,\cdots,M\}$ and $\mathscr{J}_n^0 = I$ for all $n \in \mathbb{N}$, we shall equivalently write scheme (7) as follows:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n \mathscr{J}_n^M x_n, \qquad \forall n \in \mathbb{N}.$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded.

Proof of Step 1. Let $p \in \mathscr{F}$. Using Lemma , Lemma and Lemma , we have

$$\begin{aligned}
&\| x_n - p \|^2 \\
&\quad = \langle \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n \mathscr{J}_n^M x_n - p, x_n - p \rangle \\
&\quad = \alpha_n \gamma \langle f(x_n) - f(p), x_n - p \rangle + \alpha_n \langle \gamma f(p) - F(p), x_n - p \rangle \\
&\qquad + \langle (I - \alpha_n F) T_{\mu_n} W_n \mathscr{J}_n^M x_n - (I - \alpha_n F) p, x_n - p \rangle \\
&\quad \leq \alpha_n \gamma \alpha \| x_n - p \|^2 + \alpha_n \langle \gamma f(p) - F(p), x_n - p \rangle \\
&\qquad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{2 - 2\delta}{1 - \lambda}} \right) \right) \| x_n - p \|^2 .
\end{aligned}$$

Thus,

(8) $$\| x_n - p \|^2 \leq \frac{1}{1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha}\langle \gamma f(p) - F(p), x_n - p\rangle.$$

Hence

(9) $$\| x_n - p \| \leq \frac{1}{1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha} \| \gamma f(p) - F(p) \| .$$

Therefore $\{x_n\}$ is bounded.

Step 2. For every $k \in \{1,2,\cdots,M\}$, we have $\lim_{n\to\infty} \| x_n - J^{\phi_k}_{r_k,n} x_n \| = 0$.

Proof of Step 2. Let $p \in \mathscr{F}$ since, for each $k \in \{1,2,\cdots,M\}$, $J^{\phi_k}_{r_k,n}$ is firmly non-expansive, we have

$$\begin{aligned}
\| J^{\phi_k}_{r_k,n} x_n - p \|^2 &= \| J^{\phi_k}_{r_k,n} x_n - J^{\phi_k}_{r_k,n} p \|^2 \\
&\leq \langle J^{\phi_k}_{r_k,n} x_n - J^{\phi_k}_{r_k,n} p, x_n - p\rangle \\
&= \frac{1}{2}\left[\| J^{\phi_k}_{r_k,n} x_n - p \|^2 + \| x_n - p \|^2 - \| x_n - J^{\phi_k}_{r_k,n} x_n \|^2 \right].
\end{aligned}$$

It follows that

(10) $$\| x_n - J^{\phi_k}_{r_k,n} x_n \|^2 \leq \| x_n - p \|^2 - \| J^{\phi_k}_{r_k,n} x_n - p \|^2 .$$

Moreover set $l_n = 2\langle \gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}^M_n x_n, x_n - p\rangle$ and note that, by using the inequality

$$\| x + y \| 62 \leq \| x \|^2 + 2\langle y, x + y\rangle,$$

we obtain

$$\begin{aligned}
\| x_n - p \|^2 &= \| \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n \mathscr{J}^M_n x_n - p \|^2 \\
&\leq \| T_{\mu_n} W_n \mathscr{J}^M_n x_n - p \|^2 + \alpha_n l_n \\
&= \| T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - p \|^2 + \alpha_n l_n \\
&\leq \| J^{\phi_1}_{r_1,n} x_n - p \|^2 + \alpha_n l_n.
\end{aligned}$$

Applying the last to (10), we obtain

$$\| x_n - J^{\phi_1}_{r_1,n} x_n \| \leq \alpha_n l_n$$

and since $\{l_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we get

$$\lim_{n\to\infty} \| x_n - J^{\phi_1}_{r_1,n} x_n \| = 0.$$

Now we assume that $\overline{k} \in \{1,2,\cdots,M\}$ and for every $k \in \{1,2,\cdots,\overline{k}\}$, $\lim_{n\to\infty} \| x_n - J^{\phi_k}_{r_k,n} x_n \| = 0$. We shall prove that $\lim_{n\to\infty} \| x_n - J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n \| = 0$. Indeed

$$\begin{aligned} \| x_n - p \|^2 &\leq \| T_{\mu_n} W_n J^{\phi_M}_{r_M,n} \cdots J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - p \|^2 + \alpha_n l_n \\ &\leq \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - p \|^2 + \alpha_n l_n. \end{aligned} \tag{11}$$

Observe that

$$\begin{aligned} &\| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - p \| \\ &\quad \leq \| J^{\phi_{\overline{k}-1}}_{r_{\overline{k}-1},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - x_n \| + \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n - p \| \\ &\quad \leq \| J^{\phi_{\overline{k}-2}}_{r_{\overline{k}-2},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - x_n \| + \| J^{\phi_{\overline{k}-1}}_{r_{\overline{k}-1},n} x_n - x_n \| + \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n - p \| \\ &\quad\quad \vdots \\ &\quad \leq \sum_{k=1}^{\overline{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| + \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n - p \| . \end{aligned}$$

Inequality (11) becomes then

$$\begin{aligned} &\| x_n - p \|^2 \\ &\quad \leq \left[\sum_{k=1}^{\overline{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| + 2 \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n - p \| \right] \sum_{k=1}^{\overline{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| \\ &\quad\quad + \| J^{\phi_{\overline{k}}}_{r_{\overline{k}},n} x_n - p \|^2 + \alpha_n l_n. \end{aligned} \tag{12}$$

It follows from (10) and (12) that

$$\begin{aligned}
&\| x_n - J^{\phi_{\bar{k}}}_{r_{\bar{k}},n} x_n \| \\
&\quad \leq \left[\sum_{k=1}^{\bar{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| + 2 \| J^{\phi_{\bar{k}}}_{r_{\bar{k}},n} x_n - p \| \right] \sum_{k=1}^{\bar{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| \\
&\qquad + \alpha_n l_n.
\end{aligned}$$

By our assumption, we have that

$$\lim_{n\to\infty} \sum_{k=1}^{\bar{k}-1} \| J^{\phi_k}_{r_k,n} x_n - x_n \| = 0.$$

Then, from the last and condition (B_1), we derive

$$\lim_{n\to\infty} \| x_n - J^{\phi_{\bar{k}}}_{r_{\bar{k}},n} x_n \| = 0.$$

Step 3. $\lim_{n\to\infty} \| x_n - T_{\mu_n} W_n x_n \| = 0$.

Proof of Step 3. put $M_n = 2\langle \gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}^M_n x_n, x_n - T_{\mu_n} W_n x_n \rangle$ and note that

$$\begin{aligned}
\| x_n - T_{\mu_n} W_n x_n \|^2 &= \| \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n} W_n \mathscr{J}^M_n x_n - T_{\mu_n} W_n x_n \|^2 \\
&\leq \| T_{\mu_n} W_n \mathscr{J}^M_n x_n - T_{\mu_n} W_n x_n \|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}^M_n x_n, x_n - T_{\mu_n} W_n x_n \rangle \\
&\leq \| \mathscr{J}^M_n x_n - x_n \|^2 + \alpha_n M_n.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\| \mathscr{J}^M_n x_n - x_n \| \\
&\quad = \| J^{\phi_M}_{r_M,n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - x_n \| \\
&\quad \leq \| J^{\phi_{M-1}}_{r_{M-1},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - x_n \| + \| J^{\phi_M}_{r_M,n} x_n - x_n \| \\
&\quad \leq \| J^{\phi_{M-2}}_{r_{M-2},n} \cdots J^{\phi_2}_{r_2,n} J^{\phi_1}_{r_1,n} x_n - x_n \| + \| J^{\phi_{M-1}}_{r_{M-1},n} x_n - x_n \| + \| J^{\phi_M}_{r_M,n} x_n - x_n \| \\
&\qquad \vdots \\
&\quad \leq \sum_{k=1}^{M} \| J^{\phi_k}_{r_k,n} x_n - x_n \| .
\end{aligned}$$

Thus, by Step 2, condition (B_1) and boundedness of M_n, we have

$$\lim_{n\to\infty} \| x_n - T_{\mu_n} W_n x_n \| = 0.$$

Step 4. $\lim_{n\to\infty} \| x_n - T_t x_n \| = 0$, for all $t \in S$.

Proof of Step 4. Let $p \in \mathscr{F}$ and set $M_0 = \frac{1}{1-\sqrt{\frac{2-2\delta}{1-\lambda}-\gamma\alpha}} \| \gamma f(p) - F(p) \|$ and $D = \{y \in H : \| y - p \| \leq M_0\}$, we remark that D is bounded closed convex set, $\{x_n\} \subset D$ and it is invariant under $\{J^{\phi_k}_{r_k,n} : k = 1,2,\cdots,M,\ n \in \mathbb{N}\}$, φ and W_n for all $n \in \mathbb{N}$. We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \| T_{\mu_n} y - T_t T_{\mu_n} y \| = 0, \qquad \forall t \in S \tag{13}$$

Let $\varepsilon > 0$. By [5, Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{co} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D), \qquad \forall t \in S. \tag{14}$$

Also by [5, Corollary 1.1], there exists a natural number N such that

$$\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y - T_t \left(\frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \right) \| \leq \delta, \tag{15}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\| \mu_n - l^*_{t^i}\mu_n \| \leq \frac{\delta}{(M_0 + \|p\|\,)}$ for $n \geq N_0$ and $i = 1,2,\cdots,N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D} \| T_{\mu_n} y - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y d\mu_n(s) \| \\
&\quad = \sup_{y\in D} \sup_{\|z\|=1} | \langle T_{\mu_n} y, z\rangle - \langle \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y d\mu_n(s), z\rangle | \\
&\quad = \sup_{y\in D} \sup_{\|z\|=1} | \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_s y, z\rangle - \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_{t^i s} y, z\rangle | \\
&\quad \leq \frac{1}{N+1} \sum_{i=0}^{N} \sup_{y\in D} \sup_{\|z\|=1} | (\mu_n)_s \langle T_s y, z\rangle - (l^*_{t^i}\mu_n)_s \langle T_s y, z\rangle | \\
&\quad \leq \max_{i=0,1,2,\cdots,N} \| \mu_n - l^*_{t^i}\mu_n \| (M_0 + \| p \|) \leq \delta, \qquad \forall n \geq N_0.
\end{aligned} \tag{16}$$

By Lemma , we have

$$\int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y d\mu_n(s) \in \overline{co} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S \right\}. \tag{17}$$

It follows from (14), (15), (16) and (17) that

$$T_{\mu_n} y \in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s} y : s \in S\right\} + B_\delta$$

$$\subset \overline{co} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D),$$

for all $y \in D$ and $n \geq N_0$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in D} \| T_t(T_{\mu_n} y) - T_{\mu_n} y \| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get (13).

Let $t \in S$ and $\varepsilon > 0$. Then, there exists $\delta > 0$, which satisfies (14). Take

$$L_0 = \left[\left(1 + \sqrt{\frac{2-2\delta}{1-\lambda}} + \gamma\alpha\right) M_0 + \| \gamma f(p) - F(p) \|\right].$$

From (13) and condition (B_1), there exists $N_1 \in \mathbb{N}$ such that $T_{\mu_n} y \in F_\delta(T_t; D)$, $\forall y \in D$ and $\alpha_n \leq \frac{\delta}{L_0}$ for all $n \geq N_1$. By Lemma , Lemma and Lemma , we have

$$\begin{aligned}
&\| \gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}_n^M x_n \| \\
&\leq \gamma \| f(x_n) - f(p) \| + \| \gamma f(p) - F(p) \| + \| F(p) - F T_{\mu_n} W_n \mathscr{J}_n^M x_n \| \\
&\leq \gamma\alpha \| x_n - p \| + \| \gamma f(p) - F(p) \| + \left(1 + \sqrt{\frac{2-2\delta}{1-\lambda}}\right) \| x_n - p \| \\
&= \left(1 + \sqrt{\frac{2-2\delta}{1-\lambda}} + \gamma\alpha\right) \| x_n - p \| + \| \gamma f(p) - F(p) \| = L_0.
\end{aligned}$$

It follows that

$$\alpha_n \| \gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}_n^M x_n \| \leq \alpha_n L_0 \leq \delta \qquad \forall n \geq N_1.$$

Therefore, we have

$$x_n = T_{\mu_n} W_n \mathscr{J}_n^M x_n + \alpha_n [\gamma f(x_n) - F T_{\mu_n} W_n \mathscr{J}_n^M x_n]$$

$$\in F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D),$$

for all $n \geq N_1$. This shows that

$$\| x_n - T_t(x_n) \| \leq \varepsilon, \qquad \forall n \geq N_1,$$

Since $\varepsilon > 0$ is arbitrary the proof of Step 4 is complete.

Step 5. There exists a unique $x^* \in \mathscr{F}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - F)x^*, x_n - x^*\rangle \leq 0. \tag{18}$$

Proof of Step 5. Let $Q = P_{\mathscr{F}}$. Then $Q(I - F + \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned}
&\| Q(I-F+\gamma f)x - Q(I-F+\gamma f)y \| \\
&\leq \| (I-F+\gamma f)x - (I-F+\gamma f)y \| \\
&\leq \| (I-F)x - (I-F)y \| + \gamma \| f(x) - f(y) \| \\
&\leq \left(\sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right) \| x - y \|,
\end{aligned}$$

and hence $Q(I - F + \gamma f)$ is a contraction due to $\left(\sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right) \in (0,1)$.
Therefore, by Banachs contraction principal, $P_{\mathscr{F}}(I - F + \gamma f)$ has a unique fixed point x^*. Then using Lemma , x^* is the unique solution of the variational inequality:

$$\langle(\gamma f - F)x^*, x - x^*\rangle \leq 0, \qquad \forall x \in \mathscr{F}. \tag{19}$$

We can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - F)x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle(\gamma f - F)x^*, x_{n_j} - x^*\rangle.$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup z^*$. In terms of Lemma and Step 4, we conclude that $z^* \in Fix(\varphi)$.

Consider the set of the asymptotic center of $\{x_{n_j}\}$ with respect to H,

$$C_a(H, \{x_{n_j}\}) = \{x \in H : \limsup_{j\to\infty} \| x_{n_j} - x \| = \inf_{y\in H} \limsup_{j\to\infty} \| x_{n_j} - y \|\}.$$

For $z \in C_a(H, \{x_{n_j}\})$, we have

$$\limsup_{j\to\infty} \| x_{n_j} - z \| \leq \limsup_{j\to\infty} \| x_{n_j} - T_t x_{n_j} \|, \quad \forall t \in S.$$

By Step 4, we get $x_{n_j} \to z$. It follows from Step 4 and Lemma that $z \in Fix(\varphi)$. By our assumption, we have $T_i z \in Fix(\varphi)$ for all $i \in \mathbb{N}$ and then $W_n z \in Fix(\varphi)$, hence $T_{\mu_n} W_n z = W_n z$. Using Lemma and Step 3, we get

$$\begin{aligned}
&\limsup_{j\to\infty} \| x_{n_j} - Wz \| \\
&\quad \le \limsup_{j\to\infty} \| x_{n_j} - T_{\mu_{n_j}} W_{n_j} x_{n_j} \| + \limsup_{j\to\infty} \| T_{\mu_{n_j}} W_{n_j} x_{n_j} - T_{\mu_{n_j}} W_{n_j} z \| \\
&\qquad + \limsup_{j\to\infty} \| T_{\mu_{n_j}} W_{n_j} z - Wz \| \\
&\quad \le \limsup_{j\to\infty} \| x_{n_j} - T_{\mu_{n_j}} W_{n_j} x_{n_j} \| + \limsup_{j\to\infty} \| x_{n_j} - z \| \\
&\qquad + \limsup_{j\to\infty} \| W_{n_j} z - Wz \| \\
&\quad \le \limsup_{j\to\infty} \| x_{n_j} - z \| .
\end{aligned}$$

This is enough to prove that $W(C_a(H,\{x_{n_j}\})) \subset C_a(H,\{x_{n_j}\})$. Using Lemma , Lemma and Step 2, we have

$$\begin{aligned}
&\limsup_{j\to\infty} \| x_{n_j} - J^{\phi_k}_{r_k} z \| \\
&\quad \le \limsup_{j\to\infty} \| x_{n_j} - J^{\phi_k}_{r_{k,n_j}} x_{n_j} \| + \limsup_{j\to\infty} \| J^{\phi_k}_{r_{k,n_j}} x_{n_j} - J^{\phi_k}_{r_{k,n_j}} z \| \\
&\qquad + \limsup_{j\to\infty} \| J^{\phi_k}_{r_{k,n_j}} z - J^{\phi_k}_{\hat{r}_k} z \| \\
&\quad \le \limsup_{j\to\infty} \| x_{n_j} - z \| .
\end{aligned}$$

This is enough to prove that $J^{\phi_k}_{r_k}(C_a(H,\{x_{n_j}\})) \subset C_a(H,\{x_{n_j}\})$.
Since $x_{n_j} \rightharpoonup z^*$ Lemma implies that $C_a(H,\{x_{n_j}\}) = \{z^*\}$; therefore, $z^* \in Fix(W) \cap (\bigcap_{k=1}^{M} Fix(J^{\phi_k}_{r_{k,n}}))$. In terms of Lemma and Lemma, we conclude that $z^* \in (\bigcap_{i=1}^{\infty} Fix(T_i)) \cap EP(\mathscr{J})$. Since $z^* \in Fix(\varphi)$; therefore, $z^* \in \mathscr{F}$. Applying (19), we have

$$\limsup_{n\to\infty} \langle (\gamma f - F)x^*, x_n - x^* \rangle = \langle (\gamma f - F)x^*, z^* - x^* \rangle \le 0.$$

Step 6. The sequence $\{x_n\}$ converges strongly to x^*.

Proof of Step 6. Since $x^* \in \mathscr{F}$ from (8), we have

$$\limsup_{n\to\infty} \| x_n - x^* \|^2 \leq \frac{1}{1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha} \limsup_{n\to\infty} \langle (\gamma f - F)x^*, x_n - x^* \rangle.$$

Then, using Step 5, we have $x_n \to x^*$ as $n \to \infty$.

Corollary 3.2. *Let H be a real Hilbert space H, $\{T_i\}_{i=1}^{\infty}$ an infinite family of non-expansive mapping of H into H, for $k \in \{1,2,\cdots,M\}$ ϕ_k a bi-functions from $H\times H$ into $\mathbb{R}$, λ a real number in $[0.1)$, A a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$ such that $\overline{\gamma} > \frac{1+\lambda}{2}$, ζ a real number in $(0, \frac{1-\sqrt{\frac{2-2\overline{\gamma}}{1-\lambda}}}{\alpha})$. Moreover, let $\{r_{k,n}\}_{k=1}^{M}$, $\{\varepsilon_n\}$ and λ_n be real sequences such that $r_{k,n} > 0$ and $0 < \lambda_n \leq b < 1$. Assume that,*

- (B_1) *for every $k \in \{1,2,\cdots,M\}$, the bifunction ϕ_k satisfies $(A_1)-(A_4)$.*
- (B_2) $\mathscr{F} = (\bigcap_{k=1}^{M} EP(\phi_k)) \cap (\bigcap_{i=1}^{\infty} Fix(T_i)) \neq \emptyset$.
- (B_3) *the sequence $\{\alpha_n\}$satisfies $\lim_{n\to\infty} \alpha_n = 0$,*
- (B_4) *the sequences $\{r_{k,n}\}_{k=1}^{M}$ satisfy $\lim_{n\to\infty} r_{k,n} = \hat{r}_k > 0$ for every $k \in \{1,2,\cdots,M\}$.*

Let W_n be the mapping defined by (3) and the sequence $\{x_n\}$ generated by

$$x_n = \alpha_n \zeta f(x_n) + (I - \alpha_n A) W_n J_{r_{M,n}}^{\phi_M} \cdots J_{r_{2,n}}^{\phi_2} J_{r_{1,n}}^{\phi_1} x_n, \qquad \forall n \in \mathbb{N},$$

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathscr{F}$, where x^* is the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

Proof. Take $\varphi = \{I\}$ then, using Lemma we have $T_{\mu_n} = I$. Because A is strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$, we have

$$\langle Ax - Ay, x - y \rangle \geq \overline{\gamma} \| x - y \|^2 .$$

Therefore, A is $\overline{\gamma}-$strongly monotone. On the other hand

$$\begin{aligned}
&\| (I-A)x-(I-A)y \|^2 \\
&= \langle (x-y)-(Ax-Ay),(x-y)-(Ax-Ay)\rangle \\
&= \langle x-y,x-y\rangle - 2\langle Ax-Ay,x-y\rangle + \langle Ax-Ay,Ax-Ay\rangle \\
&\leq \| x-y \|^2 -2\langle Ax-Ay,x-y\rangle + \| A \| \| x-y \|^2
\end{aligned}$$

Since A is strongly positive if and only if $\frac{1}{\|A\|}A$ is strongly positive. We may assume, with no loss of generality, that $\| A \|= 1$. Therefore,

$$\begin{aligned}
\langle Ax-Ay,x-y\rangle &\leq \| x-y \|^2 - \frac{1}{2} \| (I-A)x-(I-A)y \|^2 \\
&\leq \| x-y \|^2 - \frac{1-\lambda}{2} \| (I-A)x-(I-A)y \|^2
\end{aligned}$$

This show that A is $\lambda-$ strictly pseudo-contractive of Browder-Petryshyn type. Now apply Theorem to conclude the result.

Corollary 3.3. *Let S and T be non-expansive mappings on H with $ST = TS$, $\mathscr{J} = \{\phi_k : k = 1,2,\cdots,M\}$ a finite family of bi-functions from $H\times H$ into $\mathbb{R}$ which satisfy $(A_1)-(A_4)$, $\{T_i\}_{i=1}^{\infty}$ an infinite family of non-expansive mappings of H into H such that $T_i(Fix(T)\cap Fix(S)) \subset Fix(T)\cap Fix(S)$ for each $i\in\mathbb{N}$, $\mathscr{F} = \bigcap_{i=1}^{\infty} Fix(T_i)\cap Fix(T)\cap Fix(S)\cap EP(\mathscr{J}) \neq \emptyset$, $\{\alpha_n\}$ a sequence in $(0,1)$, $\{\lambda_n\}$ a sequence in $(0,b]$ for some $b\in(0,1)$, $\{r_{k,n}\}$ sequences in $(0,\infty)$. Suppose the following conditions are satisfied:*

- (B_1) $\lim_{n\to\infty}\alpha_n = 0$,
- (B_4) $\lim_{n\to\infty} r_{k,n} = \hat{r}_k$, *for every* $k\in\{1,2,\cdots,M\}$.

Let W_n be the mapping defined by (3) and for every $k\in\{1,2,\cdots,M\}$ and $n\in\mathbb{N}$, let $J_{r_{k,n}}^{\phi_k}$ be the resolvent generated by ϕ_k and $r_{k,n}$ in Lemma . If $\{x_n\}$ is the sequence generated by

$$x_n = \alpha_n\gamma f(x_n) + (I-\alpha_n F)\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j W_n J_{r_{M,n}}^{\phi_M}\cdots J_{r_{2,n}}^{\phi_2} J_{r_{1,n}}^{\phi_1} x_n, \qquad \forall n\in\mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $x^* \in \mathscr{F}$, where x^* is the unique solution of the variational inequality:

$$\langle(\gamma f - F)x^*, x - x^*\rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

Proof. Let $S = \mathbb{N}\bigcup\{o\} \times \mathbb{N}\bigcup\{o\}$ and let $T(i,j) = S^i T^j$ for all $(i,j) \in S$. Since S^i and T^j are non-expansive for each $(i,j) \in S$ and $ST = TS$. Therefore, $\varphi = \{T(i,j) : (i,j) \in S\}$ is a non-expansive semigroup on H. Now for each $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(i,j)$ for each $f \in B(S)$. Then, $\{\mu_n\}$ is strongly left regular sequence of means on $B(S)$; for more details, see [22]. Next, for each $y \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n}(y) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j(y).$$

Therefore, it follows from Theorem that the sequence $\{x_n\}$ converges strongly, as $n \to \infty$ to a point $z \in \mathscr{F}$, which solves the variational inequality:

$$\langle(\gamma f - F)x^*, x - x^*\rangle \leq 0, \qquad \forall x \in \mathscr{F}.$$

Conflict of Interests

The author declares that there is no conflict of interests.

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