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TENSOR PRODUCT TECHNIQUE AND FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Some times it is not easy to find the exact solution of certain partial differential equations. In this paper we use tensor product technique of Banach spaces to find certain solutions of certain fractional differential equations.

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1. Introduction.

In [4], a new definition called $\alpha-$conformable fractional derivative was introduced:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$ let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists then it is called the $\alpha-$ conformable fractional derivative of f at x.

For $x = 0$, $D^{\alpha} f(0) = \lim\limits_{x \to 0} D^{\alpha} f(0)$ if such limit exists.

The new definition satisfies:

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1.$T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in\mathbb{R}$.

2.$T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$.

Further, for $\alpha\in(0,1]$ and and f,g be $\alpha-$differentiable at a point t, with $g(t)\neq 0$. Then

3. $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.

4. $T_\alpha(\frac{f}{g})=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$

We list here the fractional derivatives of certain functions,

5. $T_\alpha(t^p)=p\,t^{p-\alpha}$.

6. $T_\alpha(\sin\frac{1}{\alpha}t^\alpha)=\cos\frac{1}{\alpha}t^\alpha$.

7. $T_\alpha(\cos\frac{1}{\alpha}t^\alpha)=-\sin\frac{1}{\alpha}t^\alpha$.

8. $T_\alpha(e^{\frac{1}{\alpha}t^\alpha})=e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha=1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t)=2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(t)=1$. Hence $T_{\frac{1}{2}}(f)(0)=1$.

But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

Through out this paper, if u is a function of two variables x and t, then we write $D_x^\alpha u$ to denote the $\alpha-$conformable derivative of u with respect to x. Similarly for $D_t^\alpha u$.

In case the function h is a function of 1-variable, then we write $h^{(\alpha)}$ to denote the $\alpha-$derivative of h.

For more on fractional calculus and its applications we refer to [1], [8] and [9].

2. Tensor Product

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x\in X$ and $y\in Y$. The operator $T:X^*\to Y$, defined by $T(x^*)=x^*(x)y$ is a bounded one rank linear operator. We write $x\otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor product. Atoms are used in theory of best approximation in Banach spaces, [6], and [7].

We need the following Lemma in our paper:

Lemma 2.1. [5] If the sum of two atoms is an atom, then either the first components are dependent or the second components are dependent.

That is: if

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3, \tag{1}$$

then we can assume with no loss of generality that either $x_1 = x_2$, or $y_1 = y_2$

3. Main Results

Consider the fractional partial differential equation

$$D_t^{\alpha} u \; + g(x) D_x^{\alpha} u = f(t) \tag{2}$$

If one tries to solve this equation via separation of variables, then it is not possible since the variables cant be separated.

Hence we use the tensor product technique to solve the equation.

Procedure. Let $u(x,t) = P(x)Q(t)$. Substitute in equation (2), to get:

$$P(x)Q^{(\alpha)}(t) + g(x)P^{(\alpha)}(x)Q(t) = f(t) \tag{3}$$

Equation (3) has the form of equation (1). Then using Lemma 2.1 we have two cases to consider:

Case (i) $P(x) = g(x)P^{(\alpha)}(x)$

Case (ii) $Q^{(\alpha)}(t) = Q(t)$

Solution of Case (i).

From $P(x) = g(x)P^{(\alpha)}(x)$, we get

$$P(x) = e^{\int \frac{1}{g(x)} dx^{\alpha}} + c_1 \tag{4}$$

where c is a constant of integration.

Now take equation (4) and substitute in (3) to get:

$$Q^{(\alpha)}(t) + h(x)Q(t) = f(t) \tag{5}$$

Now equation (5) is a linear fractional deferential equation of order α, [10]. The solution is

$$Q(t) = e^{-\int h(x) dt^{\alpha}} [\int f(t) e^{\int h(x) dt^{\alpha}} dt^{\alpha} + c_2 \tag{6}$$

One can find c_1 and c_2 whenever boundary or initial conditions are given or assumed in the equation.

Finally, the atomic solution is the product: $P(x)Q(t)$, where P is given in (4) and Q is given in (6).

Solution of case (ii).

$Q^{(\alpha)}(t) = Q(t)$. Then

$$Q(t) = e^{\frac{t^\alpha}{\alpha}} + c_4 \tag{7}$$

Now substitute (7) in (3) to get

$$P^{(\alpha)}(x) + a(t)b(x)p(x) = h(t) \tag{8}$$

where $a(t) = \frac{Q^{(\alpha)}(t)}{Q(t)}$, $b(x) = \frac{1}{g(x)}$, and $h(t) = \frac{f(t)}{Q(t)}$.

Notice that $a(t)$ and $h(t)$ are known and considered constants with respect to x. Further $b(x) = \frac{1}{g(x)}$ is given in the equation.

Again, equation (8) is a fractional linear equation, [10]. Hence

$$P(x) = e^{-\int a(t)b(x)dx^\alpha}[e^{\int a(t)b(x)dx^\alpha} f(t)dx^\alpha + c_5 \tag{9}$$

The atomic solution of (3) for the case (ii) is $P(x)q(t)$, where $P(x)$ is given in (9) and $Q(t)$ is given in (7).

4. Further Results

Let us consider the equation

$$D_t^\alpha u + W_1(x)R_1(t)D_x^\alpha u = W_2(x)R_2(t) \tag{10}$$

Here $W_1(x)$, $R_1(t)$, $W_2(x)$, $R_2(t)$ are all given.

Again, it is very difficult to use separation of variables to solve this equation.

Let us use Tensor product technique to find certain atomic solutions.

So put $u(x,t) = P(x)Q(t)$. Substitute in equation (10) we get:

$$P(x)Q^{(\alpha)}(t) + W_1(x)P^{(\alpha)}(x)R_1(t)Q(t) = W_2(x)R_2(t) \tag{11}$$

In equation (11), we have the sum of two atoms is an atom. Hence we have two cases:

Case (i) $P(x) = W_1(x)P^{(\alpha)}(x)$

Case (ii) $Q^{(\alpha)}(t) = R_1(t)Q(t)$.

As for case(i) here, it is very similar to case(i) in section III, and so

$$P(x) = e^{\int \frac{1}{W_1(x)} dx^\alpha} + c_6 \tag{12}$$

Now substitute in (11) to get

$$Q^{(\alpha)}(t) + J(x)Q(t) = M(x)R_2(t) \tag{13}$$

which is very similar to equation (5), and so

$$Q(t) = e^{-\int J(x)dt^\alpha}[\int M(x)e^{\int J(x)dt^\alpha} R_2(t)dt^\alpha + 7 \tag{14}$$

So the atomic solution to this case is $P(x).Q(t)$ from (12) and (14) respectively.

The second case is exactly the same as case (ii) in section III.

We end the paper with this problem:

Problem. How can one get an atomic solution if the equation is of the form:

$$D_t^\alpha u + g(x,t)D_x^\alpha u = f(x,t)$$

where g and f are not atomic functions?

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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