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ENDOMORPHISMS AND AUTOMORPHISMS OF CERTAIN SEMIGROUPS

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Abstract. Let V be an arbitrary vector space over $\mathbb{C}$ and let $0 \neq \mu \in V^*$ be a linear functional on V. We equip V with a multiplication converting it into a semigroup, denoted by V_μ. In this note the semigroup structure of V_μ are investigated and in particular, the endomorphisms and automorphisms of V_μ are characterized.

Keywords: semigroup, semigroup homomorphism, semigroup automorphism, algebra homomorphism, algebra automorphism.

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1. Introduction and preliminaries

A semigroup is an algebraic structure consisting of a set together with an associative binary operation. A semigroup with an identity element is called a monoid. A monoid in which every element has an inverse is called a group. A semigroup homomorphism between two semigroups T and T' is a function $\varphi : T \longrightarrow T'$ such that the equation $\varphi(ab) = \varphi(a)\varphi(b)$ is hold for all elements $a, b \in T$. A semigroup homomorphism from T into itself is called an endomorphism. For semigroup T, a bijective endomorphism of T is called a semigroup automorphism. The set

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of all semigroup automorphisms of T is denoted by $AutT$. In the case where T is an algebra, $Hom(T)$ is the set of all algebra endomorphisms of T.

In this paper let V be a non-zero vector space over $\mathbb{C}$ and let μ be a non-zero linear functional on V. For each $a,b \in V$ define $a \cdot b = \mu(a)b$. One can simply verify that "$\cdot$" converts V into an associative algebra. We denote $(V,\cdot)$ by V_μ that is a semigroup. Note that V_μ is not a monoid in general. Indeed V_μ is a monoid if and only if $dimV = 1$. Also if $dimV > 1$ then $Z(V_\mu) = \{0\}$, where $Z(V_\mu)$ is the center of V_μ. It follows that V_μ is not an abelian semigroup. Indeed V_μ is abelian if and only if $dimV = 1$. Since μ is linear it is clear that $\mu(ab) = \mu(\mu(a)b) = \mu(a)\mu(b), (a,b \in V_\mu)$. In particular one can simply verify that $Hom(V_\mu,\mathbb{C}) = \{\mu, 0\}$, where $Hom(V_\mu,\mathbb{C})$ is the set of all algebra homomorphisms from V_μ into $\mathbb{C}$.

Some basic properties of V_μ such as Arens regularity, $n-$weak amenability, minimal idempotents and ideal structure are investigated in [2], in the case where V is a Banach Space. The number of roots of a polynomial equation with coefficients in V_μ is investigated in [1], in the case where V is a vector space.

In this note our purpose is to characterize the semigroup endomorphisms and automorphisms of V_μ. In particular we characterize the algebra endomorphisms and automorphisms of V_μ. It is worthwhile mentioning that the study of the semigroup endomorphisms and automorphisms of V_μ is very interesting. Also the study of these products has significance in two respects. First, the products exhibit many properties that are not shared in general. Second, the semigroup V_μ can serve as a source of examples (or counterexamples) for various purposes in semigroup theory.

The following examples are some different endomorphisms of V_μ that are worthy of consideration.

(1) $\varphi : V_\mu \longrightarrow V_\mu,\ \ \varphi(a) = e$, where e is a constant element of V_μ satisfying, $\mu(e) = 1$.

(2) Let $n \in \mathbb{N}$ and let $\varphi : V_\mu \longrightarrow V_\mu,\ \ \varphi(a) = a^n$(note that V_μ is not abelian and also φ is not linear).

(3) $\varphi : V_\mu \longrightarrow V_\mu,\ \ \varphi(a) = \mu(a)e$, where e is a constant element of V_μ, satisfying, $\mu(e) = 1$.

(4) $\varphi : V_\mu \longrightarrow V_\mu$, $\varphi(a) = a + \mu(a)c$, where c is a constant element of $ker\mu = \{v \in V |\ \mu(v) = 0\}$.

2. Main Results

In this section we characterize the semigroup endomorphisms and automorphisms of V_μ. Also we characterize the algebra endomorphisms and automorphisms of V_μ.

Theorem 2.1. *Let V be a non-zero vector space and let $\mu \in V^*$ be a non-zero linear functional. Then the map $\varphi : V_\mu \longrightarrow V_\mu$ is a semigroup endomorphism if and only if one of the following statements is hold.*

(1) $\varphi = 0$.

(2) $\varphi(a) = c$ *for all* $a \in V_\mu$*, where* c *is a constant element of* V_μ *satisfying,* $\mu(c) = 1$.

(3) $\varphi(0) = 0,\ \mu \circ \varphi = 0$ *on* $ker\mu$ *and* $\varphi(\frac{b}{\mu(a)}) = \frac{\varphi(b)}{\mu\circ\varphi(a)}$ *for all* $a \in (ker\mu)^C$ *and* $b \in V_\mu$.

Proof. Let $\varphi : V_\mu \longrightarrow V_\mu$ be a semigroup endomorphism. So

$$\begin{aligned}\varphi(\mu(a)b) &= \varphi(ab) = \varphi(a)\varphi(b)\\ &= \mu(\varphi(a))\varphi(b),\ (a,b \in V_\mu).\end{aligned}$$

It follows that

$$\varphi(\mu(a)b) = \mu(\varphi(a))\varphi(b),\ (a,b \in V_\mu). \tag{2.1}$$

Upon substituting $b = 0$ in (2.1) we conclude that $\varphi(0) = \mu(\varphi(a))\varphi(0)$. So $(1-\mu\circ\varphi(a))\varphi(0) = 0$ for all $a \in V_\mu$, which is equivalent to $\varphi(0) = 0$ or $\mu\circ\varphi(a) = 1$ for all $a \in V_\mu$. Let $\mu\circ\varphi(a) = 1$ for all $a \in V_\mu$. It follows that $\varphi(\mu(a)b) = \mu(\varphi(a))\varphi(b) = \varphi(b),\ (a,b \in V_\mu)$. Choosing $a = 0$ we conclude that $\varphi(0) = \varphi(b)$ for all $b \in V_\mu$. Let $\varphi(0) = c$. So $\varphi(b) = c$ for all $b \in V_\mu$ and for some $c \in \mu^{-1}(\{1\})$, Providing 2.

In the case where $\varphi(0) = 0$ let $e \in \mu^{-1}(\{1\})$. Upon substituting $a = e$ in (2.1) we conclude that $\varphi(b) = \varphi(\mu(e)b) = \mu(\varphi(e))\varphi(b)$. It follows that $(1-\mu\circ\varphi(e))\varphi(b) = 0$ for all $b \in V_\mu$. Which is equivalent to $\varphi = 0$(providing 1) or $\mu\circ\varphi(e) = 1$. Our proof in the case where $\varphi(0) = 0$

and $\varphi \neq 0$ reveals that, the condition $\mu(e) = 1$ implies $\mu \circ \varphi(e) = 1$. So if $\mu(a) \neq 0$ then

$$\begin{aligned}1 = \mu \circ \varphi(\frac{a}{\mu(a)}) &= \mu \circ \varphi(a\frac{a}{\mu(a)^2}) \\ &= \mu(\varphi(a)\varphi(\frac{a}{\mu(a)^2})) \\ &= \mu \circ \varphi(a)\mu \circ \varphi(\frac{a}{\mu(a)^2}).\end{aligned}$$

It follows that $\mu \circ \varphi(a) \neq 0$ for all $a \in (ker\mu)^C$. Hence

$$\begin{aligned}\varphi(b) = \varphi(\mu(a)\frac{b}{\mu(a)}) &= \varphi(a\frac{b}{\mu(a)}) \\ &= \varphi(a)\varphi(\frac{b}{\mu(a)}) \\ &= \mu \circ \varphi(a)\varphi(\frac{b}{\mu(a)}),\ (a \in (ker\mu)^C,\ b \in V_\mu).\end{aligned}$$

So $\varphi(\frac{b}{\mu(a)}) = \frac{\varphi(b)}{\mu\circ\varphi(a)}$, $(a \in (ker\mu)^C,\ b \in V_\mu)$. Also if $a \in ker\mu$ then the equation (2.1) implies that $0 = \varphi(0) = \mu \circ \varphi(a)\varphi(b)$ for all $b \in V_\mu$. Since $\varphi \neq 0$ so $\mu \circ \varphi(a) = 0,\ (a \in ker\mu)$. It follows that $\mu \circ \varphi = 0$ on $ker\mu$, providing 3.

An straightforward calculation can be applied to show that the converse is hold.

Deepening in the proof of the previous theorem one can conclude the following results.

Corollary 2.1. *Let V be a non-zero vector space and let $\mu \in V^*$ be a non-zero linear functional. If $\varphi : V_\mu \longrightarrow V_\mu$ is a non-constant semigroup endomorphism then the following statements are hold*

(1) $\varphi(0) = 0$.

(2) $\varphi(ker\mu) \subseteq ker\mu$.

(3) $\mu(a) = 1$ *implies* $\mu \circ \varphi(a) = 1$.

Corollary 2.2. *Let V be a non-zero vector space and let $\mu \in V^*$ be a non-zero linear functional. If $\varphi : V_\mu \longrightarrow V_\mu$ is a semigroup automorphism then the following statements are hold*

(1) $\varphi(0) = 0$.

(2) $\varphi(ker\mu) = ker\mu$.

Proof. Since φ is an automorphism so φ is a non-constant endomorphism. So the fact that $\varphi(0) = 0$ follows from Corollary 2.1. Also since φ and φ^{-1} are semigroup endomorphisms, Corollary 2.1 implies that $\varphi(ker\mu) \subseteq ker\mu$ and also $\varphi^{-1}(ker\mu) \subseteq ker\mu$. So $\varphi(ker\mu) \subseteq ker\mu$ and $ker\mu \subseteq \varphi(ker\mu)$. It follows that $\varphi(ker\mu) = ker\mu$.

Corollary 2.3. *Let V be a non-zero vector space and let $\mu \in V^*$ be a non-zero linear functional. Then the bijective map $\varphi : V_\mu \longrightarrow V_\mu$ is a semigroup automorphism if and only if the following statements are hold*

(1) $\varphi(0) = 0,\ \mu \circ \varphi = 0$ *on* $ker\mu$.
(2) $\varphi(\frac{\varphi^{-1}(b)}{\mu(a)}) = \frac{b}{\mu\circ\varphi(a)}$ *for all* $a \in (ker\mu)^C$ *and* $b \in V_\mu$.

Theorem 2.2. *Let V be a non-zero vector space, let $\mu \in V^*$ be a non-zero linear functional, and let $\varphi : V_\mu \longrightarrow V_\mu$ be a non-zero linear map. Then the following statements are equivalent.*

(1) $\varphi \in Hom(V_\mu)$.
(2) $\mu = \mu \circ \varphi$.
(3) there exists a linear map $\phi : V_\mu \longrightarrow ker\mu$ *satisfying,* $\varphi(a) = a - \phi(a)$, $a \in V_\mu$.

Proof. $1 \longrightarrow 2$. Let $\varphi \in Hom(V_\mu)$. So $\mu(a)\varphi(b) = \varphi(ab) = \varphi(a)\varphi(b) = \mu(\varphi(a))\varphi(b) = \mu \circ \varphi(a)\varphi(b),\ (a,b \in V_\mu)$. It follows that $(\mu(a) - \mu\circ\varphi(a))\varphi(b) = 0$ for all $a,b \in V_\mu$. Since $\varphi \neq 0$, so $\mu(a) = \mu\circ\varphi(a)$ for all $a \in V_\mu$. Hence $\mu = \mu\circ\varphi$.
$2 \longrightarrow 3$. Let $\mu(a) = \mu\circ\varphi(a)$ for all $a \in V_\mu$. So $a - \varphi(a) \in ker\mu,\ a \in V_\mu$. Hence there exists a map $\phi : V_\mu \longrightarrow ker\mu$ satisfying $\phi(a) = a - \varphi(a)$ for all $a \in V_\mu$. Since φ is linear, clearly ϕ is linear and also $\varphi(a) = a - \phi(a),\ a \in V_\mu$.
$3 \longrightarrow 1$. Let $\phi : V_\mu \longrightarrow ker\mu$ be a linear map and let $\varphi = I - \phi$. So

$$\begin{aligned}\varphi(ab) &= ab - \phi(ab) = \mu(a)b - \phi(\mu(a)b) = \mu(a)b - \mu(a)\phi(b) = \mu(a)(b - \phi(b))\\ &= \mu(a - \phi(a))(b - \phi(b)) = \mu(\varphi(a))\varphi(b)\\ &= \varphi(a)\varphi(b),\ (a,b \in V_\mu).\end{aligned}$$

It follows that $\varphi \in Hom(V_\mu)$.

Let $Ism(V_\mu)$ be the set of all algebra automorphisms of V_μ. The following examples are two kinds of algebra automorphisms of V_μ.

(1) $\varphi : V_\mu \longrightarrow V_\mu,\ \varphi(a) = a - \mu(a)c$, where $c \in ker\mu$.

(2) $\varphi : V_\mu \longrightarrow V_\mu,\ \varphi(a) = -a + 2\mu(a)e$, where $e \in \mu^{-1}(\{1\})$.

We characterize and unify the members of $Ism(V_\mu)$.

Theorem 2.3. *Let V be a non-zero vector space and let $\mu \in V^*$ be a non-zero linear functional. Then $\varphi \in Ism(V_\mu)$ if and only if there exists a linear map*
$\phi : V_\mu \longrightarrow ker\mu$, satisfying the following properties,
(1) $\varphi(a) = a - \phi(a),\ a \in V_\mu$.
(2) $\phi = \varphi \circ \phi \circ \varphi^{-1}$.

Proof. Let $\varphi \in Ism(V_\mu)$. Then by Theorem 2.2 there exists a linear map $\phi : V_\mu \longrightarrow ker\mu$ satisfying $\varphi(a) = a - \phi(a)$ for all $a \in V_\mu$. So $\varphi(\varphi^{-1}(a)) = \varphi^{-1}(a) - \phi(\varphi^{-1}(a))$. It follows that

$$a = \varphi^{-1}(a) - \phi \circ \varphi^{-1}(a),\ a \in V_\mu. \tag{2.2}$$

On the other hand, we have $a = \varphi^{-1}(\varphi(a)) = \varphi^{-1}(a - \phi(a)) = \varphi^{-1}(a) - \varphi^{-1} \circ \phi(a),\ a \in V_\mu$. It follows that

$$a = \varphi^{-1}(a) - \varphi^{-1} \circ \phi(a). \tag{2.3}$$

By (2.2) and (2.3), we can conclude that $\phi \circ \varphi^{-1} = \varphi^{-1} \circ \phi$. So $\phi = \varphi \circ \phi \circ \varphi^{-1}$. For the converse, let $\varphi = I - \phi$ and $\phi = \varphi \circ \phi \circ \varphi^{-1}$, for some linear map $\phi : V_\mu \longrightarrow ker\mu$. Since ϕ is linear so φ is linear. Also by Theorem 2.2, $\varphi \in Hom(V_\mu)$. Since φ is bijective so $\varphi \in Ism(V_\mu)$.

The following example shows that the inclusion $Ism(V_\mu) \subset AutV_\mu$ is proper.

Example 2.1. *Let $V = \mathbb{C}$ and let $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ be the identity map. It is clear that $\mu \in V^*$ and $V_\mu = \mathbb{C}$ with the usual product. Define $\varphi : V_\mu \longrightarrow V_\mu$ by $\varphi(z) = z|z|$. One can simply verify that $\varphi \in AutV_\mu$ and also $\varphi^{-1}(z) = \frac{z}{\sqrt{|z|}}$, $z \neq 0$ and $\varphi^{-1}(0) = 0$. Clearly φ is not linear. So $Ism(V_\mu) \subsetneqq AutV_\mu$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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