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HYPERBOLICITY OF TENSOR PRODUCT SEMIGROUPS

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Abstract: In this paper, tensor product semigroup $(T(s)\otimes S(t))_{s,t\geq 0}$ is defined on $X \overset{\alpha}{\otimes} Y$, the completion of the tensor product of Banach spaces X and Y with respect to a uniform cross norm α. Its stability properties are discussed and used to examine its hyperbolic structure.

Keywords: spectral inclusion theorem; uniform exponential stability; growth bounds; spectral bound.

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1. Introduction

One of the most interesting parts of the theory of linear operator semigroups is their asymptotic behaviour, because it interacts with partial differential equations, geometry of Banach spaces, harmonic analysis, complex function theory and spectral theory. For a systematic study of the stability theory of linear operator semigroups, we refer to the survey article by R. Chill and Y. Tomilov [1] and the book by J. Van Neerven [9]. Hyperbolicity is one of the basic concepts in

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the qualitative study of differential equations. For classical theory of hyperbolicity we refer to the lecture notes by N. A. Coppel [2].

Tensor products have important applications, for example they are useful in the approximation of multivariate functions of combinations of univariate ones. For definitions and basic results on tensor products we refer to the text by Raymon A Rayon [8]. Tensor product semigroup (TPS) is introduced and studied by R. Khahil and others [6].In [7] the author has studied tensor product dynamical systems on Banach spaces.

In the second section the spectrum of the TPS is related with the spectra of its component semigroups. In the third section stability concepts of TPS are discussed. In the fourth section formulae relating the resolvent of the generator of TPS are obtained.In the last section hyperbolicity of TPS is charecterised and an example is provided to illustrate the result obtained.

2. SPECTRAL INCLUSION THEOREM FOR TPS

If X and Y are Banach spaces, we write $X \overset{\alpha}{\otimes} Y$ to denote either one of the tensor products the projective or the injective. We recall some definitions and results from [6].

Definition 1.1. Let X and Y be Banach spaces and $(T(s))_{s\geq 0}$, $(S(t))_{t\geq 0}$ be one parameter families of operators in $L(X)$, $L(Y)$ respectively. The family $(T(s)\otimes S(t))_{s,t\geq 0}$ is called a tensor product semigroup (TPS) on the Banach space $X \overset{\alpha}{\otimes} Y$ if

i) $T(0)\otimes S(0) = I_{X \overset{\alpha}{\otimes} Y}$

ii) $T(s_1+s_2)\otimes S(t_1+t_2) = (T(s_1)\otimes S(t_1))(T(s_2)\otimes S(t_2))$.

Definition 1.2. A TPS is called a C_0 - TPS if

$\lim_{(s,t)\to(0^+,0^+)} ||(T(s)\otimes S(t))z - z|| = 0$ for all $z \in X \overset{\alpha}{\otimes} Y$.

Theorem 1.3. *Suppose that $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are one parameter C_0-semigroups on the Banach spaces X, Y with infinitesimal generators A_1, A_2 respectively. Then the infinitesimal generators of the C_0 - semigroups $(T(s)\otimes I)_{s\geq 0}$, $(I\otimes S(t))_{t\geq 0}$ are respectively $\overline{A_1\otimes I}$ and $\overline{I\otimes A_2}$. Moreover the infinitesimal generator of the C_0 - semigroup $(T(s)\otimes S(t))_{s,t\geq 0}$ is the linear transformation*

$L : R^2_+ \to L(X \overset{\alpha}{\otimes} Y), (a,b) \to ((\overline{A_1\otimes I}, \overline{I\otimes A_2})(a,b)) = (a\overline{A_1\otimes I} + b\overline{I\otimes A_2})$ *where A_1 and A_2 are*

the infinitesimal generators of the one parameter C_0-semigroups $(T(\hat{s}))_{s\geq 0}, (S(\hat{t}))_{t\geq 0}$ respectively. Here$(T(\hat{s}))_{s\geq 0} = \beta T(s)$ and
$(S(\hat{t}))_{t\geq 0} = \frac{1}{\beta}S(t)$ for some unique $0 \neq \beta \in R$ and for all $s,t \geq 0$. This infinitesimal generator will be denoted by $A = (\overline{A_1 \otimes I}, \overline{I \otimes A_2})$.

Lemma 1.4. *Let $0 \neq (a,b) \in {R^{+}}^{2}$. Then the infinitesimal generator of the one parameter C_0 - semigroup $(T(as) \otimes S(bs))_{s\geq 0}$ is the linear operator*
$a\overline{A_1 \otimes I} + b\overline{I \otimes A_2}$.

Proposition 1.5. *Let $(T(s) \otimes S(t))_{s,t\geq 0}$ be a TPS on the Banach space*
$X \overset{\alpha}{\otimes} Y$ where $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are one parameter families of operators on $L(X), L(Y)$ respectively. Then we have

$$\sigma(T(s) \otimes S(t)) = \sigma(T(s))\sigma(S(t)).$$

Proof. *Let $T_1(s) = T(s) \otimes I_2$ and $T_2(t) = I_1 \otimes S(t)$ be the families of operators on $X \overset{\alpha}{\otimes} Y$. Then $T_1(s)$ and $T_2(t)$ commute and we have $\sigma(T_1(s)) = \sigma(T(s))$ and $\sigma(T_2(t)) = \sigma(S(t))$ for $s,t \geq 0$.*

Let $p(z_1,z_2) = z_1z_2$ be a polynomial in two variables. Then by [3] *we have*

$$\sigma(p(T_1(s), T_2(t))) = p(\sigma(T_1(s)), \sigma(T_2(t))$$

for every $s,t \geq 0$ and so,

$$\sigma(T_1(s)T_2(t)) = \sigma(T_1(s))\sigma(T_2(t)))$$

Thus,

$$\sigma(T(s) \otimes S(t)) = \sigma(T(s))\sigma(S(t)).$$

Theorem 1.6. [5] *For the generator $(A, D(A))$ of a C_0 - semigroup $(T(t))_{t\geq 0}$ on a Banach space X we have the inclusions*

$$\sigma(T(t)) \supset e^{t\sigma(A)} \text{ for } t \geq 0.$$

We now extend this theorem to TPS.

Theorem 1.7. *Let A_1 and A_2 be the infinitesimal generators of the C_0 - semigroups $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ on the Banach spaces X and Y respectively. Then we have the inclusions*

$$\sigma(T(s) \otimes S(t)) \supset e^{s\sigma(A_1)+t\sigma(A_2)} \text{ for } s,t \geq 0.$$

Proof. *By spectral inclusion theorem for* $(T(s))_{s\geq 0}$ *and* $(S(t))_{t\geq 0}$ *we have* $\sigma(T(s)) \supset e^{s\sigma(A_1)}$ *for* $s \geq 0$ *and* $\sigma(S(t)) \supset e^{t\sigma(A_2)}$ *for* $t \geq 0$. *Using Proposition 2.5, we have*

$$\sigma(T(s) \otimes S(t)) \supset e^{s\sigma(A_1)+t\sigma(A_2)} \text{ for } s,t \geq 0.$$

3. UNIFORM EXPONENTIAL STABILITY OF TPS

We begin with some basic definitions.

Definition 2.1. Let $A : D(A) \subset X \to X$ be a closed operator. Then $s(A) = sup\{Re\lambda/\lambda \in \sigma(A)\}$ is called the spectral bound of A.

If A_1 and A_2 are closed on $D(A_1)$ amd $D(A_2)$ respectively then $A_1 \otimes I + I \otimes A_2$ is also closed on $D(A_1) \otimes D(A_2)$ [6]. We now define the spectral bound for $A = A_1 \otimes I + I \otimes A_2$.

Definition 2.2. Spectral bound of $A = A_1 \otimes I + I \otimes A_2 : D(A_1) \otimes D(A_2) \to X \overset{\alpha}{\otimes} Y$ is given by

$$s(A) = sup\{Re\lambda/\lambda \in \ (\sigma(A_1) + \sigma(A_2))\}$$

As in the classical case,there exists constants $w \geq 0$ and $M \geq 1$ such that $||T(s) \overset{\alpha}{\otimes} S(t)|| \leq Me^{w(s+t)}$ for every $s,t \geq 0$.

Definition 2.3. Growth bound of the C_0 - TPS $(T(s) \otimes S(t))_{s,t\geq 0}$ is given by
$w_0 = w_0(A) = \inf\{w \in R : \exists M_w \geq 1$ such that $||T(s) \otimes S(t)|| \leq M_w e^{w(s+t)}$ for every $s,t \geq 0\}$.
$=\inf\{w \in R : lim_{s,t\to\infty} e^{-w(s+t)}||T(s) \otimes S(t)|| = 0\}$.

Definition 2.4. A C_0 - semigroup $(T(t))_{t\geq 0}$ is called uniformly exponentially stable if there exists $M \geq 1$ and $\varepsilon > 0$ such that $||T(t)|| \leq Me^{-\varepsilon t}$ for every $t \geq 0$ or equivalently if $w_0(T) < 0$.

Analogous to the charaterisation of uniform exponential stability for semigroups we state

Proposition 2.5. For the C_0 - TPS $(T(s) \otimes S(t))_{s,t\geq 0}$ the following are equivalent.
a) $w_0 = \alpha_0 + \beta_0 < 0$ i.e., $(T(s) \otimes S(t))_{s,t\geq 0}$ is uniformly exponentially stable.
b) $lim_{s,t\to\infty}||T(s) \otimes S(t)|| = 0$.
c) $||T(s_0) \otimes S(t_0)|| < 1$ for some $s_0,t_0 > 0$.
d) $r(T(s_1) \otimes S(t_1)) < 1$ for some $s_1,t_1 > 0$.
where α_0 and β_0 are growth bounds of $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ respectively.

Proposition 2.6. Let A_1 and A_2 be the generators of the C_0 - semigroups $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ on the Banach spaces X and Y with growth bounds α_0 and β_0 respectively. Also let A be the generator of the C_0 - TPS $(T(s)\otimes S(t))_{s,t\geq 0}$ on $X \overset{\alpha}{\otimes} Y$ with growth bound w_0. Then we have $-\infty \leq s(A) = s(A_1)s(A_2) \leq \alpha_0 + \beta_0 = w_0$.

Lemma 2.7. *Let $a,b \in R^+$. If $ab < 1$ then there exists $k > 0$ such that $ka < 1$ and $\frac{1}{k}b < 1$.*

Proof. Let $a > 1$ and $b < 1$. Choose $k = \frac{ab+1}{2a}$

Theorem 2.8. *The C_0 - TPS $(T(s)\otimes S(t))_{s,t\geq 0}$ is uniformly exponentially stable on $X \overset{\alpha}{\otimes} Y$ if and only if $(T\hat{(s)})_{s\geq 0}, (S\hat{(t)})_{t\geq 0}$ are uniformly exponentially stable on X and Y respectively where $(T\hat{(s)}) = \beta T(s)$ and $(S\hat{(t)}) = \frac{1}{\beta}S(t)$ for an unique $\beta > 0$.*

Proof. If both $(T\hat{(s)})_{s\geq 0}$ and$(S\hat{(t)})_{t\geq 0}$ are uniformly exponentially stable on X and Y respectively, then there exists constants $M_1, M_2 \geq 1$, $\varepsilon_1, \varepsilon_2 > 0$ so that $||(T\hat{(s)})|| \leq M_1 e^{-\varepsilon_1 s}$ and $||(S\hat{(t)})|| \leq M_2 e^{-\varepsilon_2 t}$.

Then $||(T(t)\otimes S(t))|| \leq Me^{-\varepsilon(s+t)}$, where $M = M_1M_2$ and $\varepsilon = min(\varepsilon_1, \varepsilon_2)$ showing that $T(s)\otimes S(t)$ is uniformly exponentially stable on $X \overset{\alpha}{\otimes} Y$.

Conversely let $(T(s)\otimes S(t))_{s,t\geq 0}$ be uniformly exponentially stable. Then by Proposition3.5, $r(T(s_1)\otimes S(t_1)) < 1$ and so $r(T(s_1).r(S(t_1) < 1$ for some
$s_1 > 0, t_1 > 0$. By Lemma 3.7, there exists $\beta > 0$ such that $\beta r(T(s_1) < 1$ and $\frac{1}{\beta}r(S(t_1) < 1$ showing that$(T\hat{(s)})_{s\geq 0}$ and$(S\hat{(t)})_{t\geq 0}$ are uniformly exponentially stable.

4. RESOLVENT OF C_0 -TPS

Recall the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ at $\lambda \in \rho(A)$ for a closed operator $(A, D(A))$ on a Banach space X. We extend the formulae relating the semigroup to the resolvent of its generators to tensor product case.

Theorem 3.1. *Let X and Y be Banach spaces and $(T(s)\otimes S(t))_{s,t\geq 0}$ be a C_0 - TPS on the Banach space $X \overset{\alpha}{\otimes} Y$ with infinitesimal generator $A = (\overline{A_1\otimes I}, \overline{I\otimes A_2})$ taking constants $w_1, w_2 \in R$, $M \geq 1$ such that $||T(s)\otimes S(t)|| \leq Me^{w_1 s + w_2 t}$ for $s,t \geq 0$. Then,*

i) If $\lambda \in \mathscr{C}$ such that $R(\lambda)(x\otimes y) = \int_0^\infty e^{-\lambda t}(T(at)\otimes S(bt))(x\otimes y)dt$ exists for all $x,y \geq 0$,

$a,b \in R^+$ then $\lambda \in \rho(\overline{A_1 \otimes I}, \overline{I \otimes A_2}) \begin{pmatrix} a \\ b \end{pmatrix}$ and

$R(\lambda, A \begin{pmatrix} a \\ b \end{pmatrix}) = R(\lambda)$.

ii) If $Re\lambda > aw_1 + bw_2$ then $\lambda \in \rho(\overline{A_1 \otimes I}, \overline{I \otimes A_2}) \begin{pmatrix} a \\ b \end{pmatrix}$ and the resolvent is given by the integral expression in (i).

iii) $||R(\lambda, A \begin{pmatrix} a \\ b \end{pmatrix})|| \leq \frac{M}{Re\lambda - (aw_1 + bw_2)}$ for all $Re\lambda > aw_1 + bw_2$.

Proof. (i) By assumption the integral exists as a Riemann integral i.e., $R(\lambda, A \begin{pmatrix} a \\ b \end{pmatrix}) = lim_{s \to \infty} \int_0^s e^{-\lambda t} (T(at) \otimes S(bt))(x \otimes y) dt$ exists and defines a bounded linear operator on $X \overset{\alpha}{\otimes} Y$.

Without loss of generality, assume $\lambda = 0$. Then for arbitrary $x \in X, y \in Y, h > 0$, we have

$$\begin{aligned}
&\frac{T(ah) \otimes S(bh) - I \otimes I}{h} R(0)(x \otimes y) \\
&= \frac{T(ah) \otimes S(bh) - I \otimes I}{h} \int_0^\infty [T(at) \otimes S(bt)](x \otimes y) dt \\
&= \frac{1}{h} \int_0^\infty [T(a(t+h))x \otimes S(b(t+h))y] dt - \frac{1}{h} \int_0^\infty [T(at)x \otimes S(bt)y] dt \\
&= \frac{1}{h} \int_h^\infty [T(at)x \otimes S(bt)y] dt - \frac{1}{h} \int_0^\infty [T(at)x \otimes S(bt)y] dt \\
&= -\frac{1}{h} \int_0^h (T(at) \otimes S(bt))(x \otimes y) dt
\end{aligned}$$

Take limit as $h \downarrow 0$ to obtain $(\overline{A_1 \otimes I}, \overline{I \otimes A_2}) \begin{pmatrix} a \\ b \end{pmatrix} R(0) = -I$ and $R(0)(x \otimes y) \in D(A \begin{pmatrix} a \\ b \end{pmatrix})$ for all $x \otimes y \in X \overset{\alpha}{\otimes} Y$.

On the other hand, for $x \otimes y \in D((A_1 \otimes I, I \otimes A_2) \begin{pmatrix} a \\ b \end{pmatrix}) \subseteq D((\overline{A_1 \otimes I}, \overline{I \otimes A_2}) \begin{pmatrix} a \\ b \end{pmatrix})$ we have

$$\lim_{s\to\infty}\int_0^s (T(at)\otimes S(bt))(x\otimes y)dt = R(0)(x\otimes y).$$

Then

$$\lim_{s\to\infty} A\begin{pmatrix} a \\ b \end{pmatrix}\int_0^s (T(at)\otimes S(bt))(x\otimes y)dt$$

$$= lim_{s\to\infty}\int_0^s (T(at)\otimes S(bt))A\begin{pmatrix} a \\ b \end{pmatrix}(x\otimes y)dt$$

$$= R(0)A\begin{pmatrix} a \\ b \end{pmatrix}(x\otimes y).$$

Since $A = (\overline{A_1\otimes I}, \overline{I\otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}$ is closed we get $R(0)A\begin{pmatrix} a \\ b \end{pmatrix}(x\otimes y) = A\begin{pmatrix} a \\ b \end{pmatrix}R(0)(x\otimes y) = -(x\otimes y)$ and so $R(0) = [-A\begin{pmatrix} a \\ b \end{pmatrix}]^{-1}$ as desired.

For (ii) and (iii) we observe

$||\int_0^s e^{-\lambda t}(T(at)\otimes S(bt))dt|| \leq M\int_0^s e^{(aw_1+bw_2-Re\lambda)t}dt$ and RHS converges to $\frac{M}{Re\lambda-(aw_1+bw_2)}$ as $s\to\infty$.

Corollary 3.2. $||R(\lambda, A\begin{pmatrix} a \\ b \end{pmatrix})^n|| \leq \frac{M}{[Re\lambda-(aw_1+bw_2)]^n}$ holds for all $n\in N$, $Re\lambda > aw_1+bw_2$.

5. HYPERBOLICITY OF C_0 -TPS

A semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called hyperbolic if X can be written as a direct sum $X = X_s\otimes X_u$ of two $(T(t))_{t\geq 0}$ - invariant, closed subspaces X_s, X_u such that the restricted semigroups $(T_s(t))_{t\geq 0}$ on X_s and $(T_u(t))_{t\geq 0}$ on X_u satisfy the following conditions

i) The semigroup $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable on X_s.

ii) The operators $T_u(t)$ are invertible on X_u and $(T_u(t)^{-1})_{t\geq 0}$ is uniformly exponentially stable on X_u.

Using the results obtained in Section 4, the next theorem discusses the hyperbolic structure of tensor product semigroups.

Theorem 4.1. Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be Banach spaces. Then $(T(as) \otimes S(bt))_{s,t\geq 0}$ for $a, b \in R^+$ is a hyperbolic C_0 - semigroup on $(X_1 \overset{\alpha}{\otimes} Y_1) \oplus (X_2 \overset{\alpha}{\otimes} Y_2)$ if and only if there exists a unique $\beta > 0$ so that $(\beta T(as))_{s\geq 0}$ and $(\frac{1}{\beta}S(bt))_{t\geq 0}$ are hyperbolic C_0 - semigroups on X and Y respectively.

Proof. Let $(T(as))_{s\geq 0}$ and $(S(bt))_{t\geq 0}$ be hyperbolic on X and Y respectively.. Then by Theorem 3.8, $(T_1(as) \otimes S_1(bt))_{s,t\geq 0}$ is uniformly exponentially stable on $X_1 \overset{\alpha}{\otimes} Y_1$ where $(T_1(as))_{s\geq 0}$ and $(S_1(bt))_{t\geq 0}$ are the restricted semigroups of $(T(as))_{s\geq 0}$ and $(S(bt))_{t\geq 0}$ on X_1 and Y_1 respectively.

Further let the operators $T_2(as)$ and $S_2(bt)$ denote the restrictions of $T(as)$ and $S(bt)$ on X_2 and Y_2 respectively. By assumption $T_2(as)$ and $S_2(bt)$ are invertible on X_2 and Y_2 respectively and so $T_2(as) \otimes S_2(bt)$ is invertible on $X_2 \overset{\alpha}{\otimes} Y_2$. Moreover $(T_2(as)^{-1})_{s\geq 0}$ and $(S_2(bt)^{-1})_{t\geq 0}$ are uniformly exponentially stable on X_2 and Y_2 respectively. Again by Theorem 3.8 $(T_2(as) \otimes S_2(bt))^{-1}_{s,t\geq 0}$ is uniformly exponentially stable on $X_2 \overset{\alpha}{\otimes} Y_2$.

Conversely let $(T(as) \otimes S(bt))_{s,t\geq 0}$ be a hyperbolic C_0 - semigroup on $(X_1 \overset{\alpha}{\otimes} Y_1) \oplus (X_2 \overset{\alpha}{\otimes} Y_2)$ so that $(T(as) \otimes S(bt))_{s,t\geq 0}$ is uniformly exponentially stable on $X_1 \overset{\alpha}{\otimes} Y_1$. Then by Theorem 3.8, $(\beta T(as))_{s\geq 0}$ and $(\frac{1}{\beta}S(bt))_{t\geq 0}$ are uniformly exponentially stable on X_1 and Y_1 respectively for unique $\beta > 0$.

Further invertibility of the operator $T(as) \otimes S(bt)$ on $X_2 \otimes Y_2$ shows the invertibility of $T(as)$ and $S(at)$ on X_2 and Y_2 respectively. Moreover $(T(as) \otimes S(bt))^{-1} = T(as)^{-1} \otimes S(bt)^{-1}$ is uniformly exponentially stable on $X_2 \overset{\alpha}{\otimes} Y_2$ shows that $(\beta T(as))^{-1}_{s\geq 0}$ and $(\frac{1}{\beta}S(bt))^{-1}_{t\geq 0}$ are uniformly exponentially stable on X_2 and Y_2 respectively. Thus $(\beta T(as))_{s\geq 0}$ and $(\frac{1}{\beta}S(bt))_{t\geq 0}$ are hyperbolic on $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ respectively.

To illustrate the theorem following example is given.

Example 4.2. Consider the C_0 - semigroup $(T(t))_{t\geq 0}$ on $X = R^2$ with infinitesimal genertaor $A = \begin{pmatrix} -ki & 0 \\ 0 & -ki \end{pmatrix}$ where $k > 1$. Then $(T(t))_{t\geq 0}$ is not hyperbolic.

But the C_0 - semigroup $(S(t))_{t\geq 0}$ on $Y = R^2$ with infinitesimal genertaor $B = \begin{pmatrix} -k+ki & 0 \\ 0 & -k+ki \end{pmatrix}$ where $k > 1$ is hyperbolic.

Then $T(t) \otimes S(t) = \begin{pmatrix} e^{-kt} & 0 & 0 & 0 \\ 0 & e^{-kt} & 0 & 0 \\ 0 & 0 & e^{-kt} & 0 \\ 0 & 0 & 0 & e^{-kt} \end{pmatrix}$

is a hyperbolic on $X \otimes Y = R^4$ with

$X_u = \{(x_1, x_2, x_3, 0)/x_1, x_2, x_3 \in R\}$

$X_v = \{(0,0,0,x_4)/x_4 \in R\}$.

Taking $\beta = \frac{e^{-kt}+1}{2}$ it can be observed that $(\beta T(t))_{t\geq 0}$ and $(\frac{1}{\beta}S(t))_{t\geq 0}$ are hyperbolic.

Conflict of Interests

The authors declare that there is no conflict of interests.

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